ELLIPTIC PROBLEMS WITH NONMONOTONE DISCONTINUITIES AT RESONANCE

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Using the critical point theory of Chang (1981) for locally Lipschitz functionals, we prove an existence theorem for some elliptic problems at resonance with no Carathéodory forcing term.

1. Introduction

In this paper, we consider elliptic problems with discontinuities at resonance. Recently, Bouchala and Drábek [2] using an extended type of Landesman-Lazer conditions proved existence theorems for both coercive and noncoercive cases. They assumed that the nonlinear right-hand side is of Carathéodory type. Here, we are interested in this problem but we do not assume that the right-hand side is Carathéodory and moreover we seek for nontrivial solutions.

For the noncoercive case we obtain a nontrivial solution using the mountain-pass theorem for locally Lipschitz functionals due to Chang [3]. The problem is an elliptic problem at resonance. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ,

$$\begin{gathered} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = f(z,x(z)) \quad \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{gathered} \tag{1.1}$$

In Section 2, we recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferential of Clarke.

2. Preliminaries

Let Y be a subset of X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, we have

$$|f(y) - f(x)| \leq K\|y - x\| \tag{2.1}$$

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for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x;v)$, is defined as follows:

$$f^o(x;v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+tv) - f(y)}{t}, \tag{2.2}$$

where y is a vector in X and t is a positive scalar. If f is Lipschitz of rank K near x, then the function $v \to f^o(x;v)$ is finite, positively homogeneous, subadditive, and satisfies $|f^o(x;v)| \leq K\|v\|$. In addition, f^o satisfies $f^o(x;-v) = (-f)^o(x;v)$. Now we are ready to introduce the generalized gradient denoted by $\partial f(x)$,

$$\partial f(x) = \{w \in X^* : f^o(x;v) \geq \langle w, v \rangle \ \forall v \in X\}. \tag{2.3}$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$;

(b) for every v in X, we have

$$f^o(x;v) = \max\{\langle w, v \rangle : w \in \partial f(x)\}. \tag{2.4}$$

If f_1, f_2 are locally Lipschitz functions, then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2. \tag{2.5}$$

We recall the Palais-Smale (PS) condition introduced by Chang [3].

Definition 2.1. A Lipschitz function f satisfies the Palais-Smale condition if any sequence $\{x_n\}$, along which $|f(x_n)|$ is bounded and

$$\lambda(x_n) = \min_{w \in \partial f(x_n)} \|w\|_{X^*} \longrightarrow 0, \tag{2.6}$$

possesses a convergent subsequence.

The PS-condition can also be formulated as follows (see Costa and Gonçalves [5]):

$(\mathrm{PS})^*_{c,+}$: whenever $(x_n) \subseteq X$, $(\varepsilon_n), (\delta_n) \subseteq \mathbb{R}_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that

$$f(x_n) \longrightarrow c, \quad f(x_n) \leq f(x) + \varepsilon_n \|x - x_n\| \quad \text{if } \|x - x_n\| \leq \delta_n, \tag{2.7}$$

then (x_n) possesses a convergent subsequence, $x_{n'} \to \hat{x}$.

Similarly, we define the $(\mathrm{PS})^*_c$ condition from below, $(\mathrm{PS})^*_{c,-}$, by interchanging x and x_n in (2.7). And finally, we say that f satisfies $(\mathrm{PS})^*_c$ provided that it satisfies $(\mathrm{PS})^*_{c,+}$ and $(\mathrm{PS})^*_{c,-}$.

Note that these two definitions are equivalent when f is locally Lipschitz functional.

We mention some facts about the first eigenvalue of the p-Laplacian. Consider the first eigenvalue λ_1 of $(-\Delta_p, W_o^{1,p}(Z))$. From Lindqvist [6] we know that $\lambda_1 > 0$ is isolated and simple, that is, any two solutions u, v of

$$-\Delta_p u = -\operatorname{div}\left(\|Du\|^{p-2}Du\right) = \lambda_1 |u|^{p-2}u \quad \text{a.e. on } Z, \qquad u|_\Gamma = 0, \quad 2 \le p < \infty \tag{2.8}$$

satisfy $u = cv$ for some $c \in \mathbb{R}$. In addition, the λ_1-eigenfunctions do not change sign in Z. Finally, we have the following variational characterization of λ_1 (Rayleigh quotient):

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_o^{1,p}(Z),\ x \neq 0\right]. \tag{2.9}$$

We are going to use the mountain-pass theorem of Chang [3].

Theorem 2.2. *If a locally Lipschitz functional $f : X \to \mathbb{R}$ on the reflexive Banach space X satisfies the PS-condition and the hypotheses,*

(i) *there exist positive constants ρ and a such that*

$$f(u) \ge a \quad \forall x \in X \text{ with } \|x\| = \rho; \tag{2.10}$$

(ii) *$f(0) = 0$ and there is a point $e \in X$ such that*

$$\|e\| > \rho, \quad f(e) \le 0, \tag{2.11}$$

then there exists a critical value $c \ge a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f(g(t)), \tag{2.12}$$

where

$$G = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}. \tag{2.13}$$

Motreanu and Panagiotopoulos [7, Theorem 1 and Corollary 1] provide a proof for the generalized mountain-pass theorem for locally Lipschitz functionals.

3. Existence theorems

Here, we give the hypotheses that we need for our existence theorem.

Let

$$f_1(z,x) = \liminf_{x' \to x} f(z,x'), \qquad f_2(z,x) = \limsup_{x' \to x} f(z,x'). \tag{3.1}$$

Hypothesis 3.1. The function $f : Z \times \mathbb{R} \to \mathbb{R}$ is an N measurable function (i.e., if $x(z)$ is measurable so is $f_1(x(z))$, $f_2(x(z))$), and moreover,

(i) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|f(z,x)| \le c_1|x|^{p-1} + c|x|^{p^*-1}$, with $p^* = Np/(N-p)$,

(ii) there exist $\theta > p$ and $r_o > 0$ such that for all $|x| \ge r_o$, and all $v \in [f_1(z,x), f_2(z,x)]$ we have $0 < \theta F(z,x) \le vx$, and moreover there exists some $a_1 \in L^1(Z)$ such that $F(z,x) \ge c_3|x|^\theta - a_1(z)$ for every $x \in \mathbb{R}$, with $F(z,x) = \int_o^r f(z,r)\,dr$,

(iii) uniformly, for all $z \in Z$ we have $\limsup_{x\to 0}(pF(z,x)/|x|^p) \le \theta(z) \le 0$ with $\theta(z) \in L^\infty(Z)$ and $\theta(z) < 0$ on a set of positive measure.

Remark 3.2. Hypothesis (iii) is the crucial one in order to have a nontrivial solution. Many authors have used such kind of hypothesis but this form is more general, so to our knowledge Theorem 3.5 below is new even when the right-hand side is Carathéodory.

Definition 3.3. We say that $x \in W_o^{1,p}(Z)$ is a solution of type I if there exists some $w \in L^{q^*}(Z)$ with $w(z) \in [f_1(z,x(z)), f_2(z,x(z))]$ such that

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = w(z) \quad \text{a.e. on } Z. \tag{3.2}$$

Definition 3.4. We say that $x \in W_o^{1,p}(Z)$ is a solution of type II if x satisfies

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = f(z,x(z)) \quad \text{a.e. on } Z. \tag{3.3}$$

It is well known that the existence of a solution of type I does not imply the existence of type II.

First, we derive an existence result of type I and then, using a stronger set of hypotheses, we obtain an existence result of type II.

Theorem 3.5. *If Hypothesis 3.1 holds, then problem (1.1) has a nontrivial solution of type I.*

Proof. Let $R_1 : W_o^{1,p}(Z) \to \mathbb{R}$ such that $R_1(x) = (1/p)\|Dx_n\|_p^p - (\lambda_1/p)\|x_n\|_p^p$, and $R_2 : W_o^{1,p}(Z) \to \mathbb{R}$ such that $R_2(x) = -\int_Z F(z,x(z))\,dz$ with $F(z,x) = \int_o^x f(z,r)\,dr$. So our energy functional is $R = R_1 + R_2$. It is well known that R is locally Lipschitz (see Chang [3]).

Claim 3.6. *The functional $R(\cdot)$ satisfies the $(PS)_{c,+}$-condition in the sense of Costa and Gonçalves [5].*

Indeed, let $\{x_n\}_{n\ge 1} \subseteq W_o^{1,p}(Z)$ such that $R(x_n) \to c$ and

$$R(x_n) \le R(x) + \varepsilon_n\|x - x_n\|, \quad \|x - x_n\| \le \delta_n, \tag{3.4}$$

with $\varepsilon_n, \delta_n \to 0$.

Let $x = x_n + \delta x_n$ with $\delta\|x_n\| \le \delta_n$. Divide with δ.

It is easy to see that

$$\lim_{\delta\downarrow 0} \frac{R_1(x_n+\delta x_n) - R_1(x_n)}{\delta} = ||Dx_n||_p^p - \lambda_1||x_n||_p^p. \tag{3.5}$$

Moreover, we have

$$\lim_{\delta\downarrow 0} \frac{R_2(x_n+\delta x_n) - R_2(x_n)}{\delta} \le R_2^o(x_n;x_n). \tag{3.6}$$

Thus,

$$R_2^o(x_n;x_n) + ||Dx_n||_p^p - \lambda_1||x_n||_p^p \ge -\varepsilon_n||x_n||. \tag{3.7}$$

On the other hand, for the $(\mathrm{PS})_{c,-}$ we have

$$\mathbb{R}(x) \le \mathbb{R}(x_n) + \varepsilon_n||x - x_n||, \quad ||x - x_n|| \le \delta_n, \tag{3.8}$$

with $\varepsilon_n, \delta_n \to 0$. Equation (3.8) is equivalent to

$$(-R)(x) - (-R)(x_n) \ge -\varepsilon_n||x - x_n||, \quad ||x - x_n|| \le \delta_n, \tag{3.9}$$

with $\varepsilon_n, \delta_n \to 0$. Note that $(-R)$ is locally Lipschitz too.

Choose here $x = x_n - \delta x_n$. Then as before we have that

$$\begin{aligned} &\lim_{\delta\downarrow 0} \frac{(-R_1)(x_n-\delta x_n) - (-R_1)(x_n)}{\delta} = ||Dx_n||_p^p - \lambda_1||x_n||_p^p, \\ &\lim_{\delta\downarrow 0} \frac{(-R_2)(x_n-\delta x_n) - (-R_2)(x_n)}{\delta} \le (-R_2)^o(x_n;-x_n) = R_2^o(x_n;x_n). \end{aligned} \tag{3.10}$$

Thus, finally we obtain again (3.7).

Note that there exists some $w_n' \in \partial(R_2(x_n))$ such that $\langle w_n', x_n\rangle = R_2^o(x_n;x_n)$. This means that

$$\langle w_n, x_n\rangle - ||Dx_n||_p^p + \lambda_1||x_n||_p^p \le \varepsilon_n||x_n||, \tag{3.11}$$

for some $w_n \in \partial(-R_2(x_n))$. Note that $w_n(z) \in [f_1(z,x_n(z)), f_2(z,x_n(z))]$.

From the choice of the sequence $\{x_n\} \subseteq W_o^{1,p}(Z)$, we have

$$\theta R(x_n) \leq M_1 \quad \text{for some } M_1 > 0. \tag{3.12}$$

Adding (3.11) and (3.12), we have

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|Dx_n\|_p^p + \lambda_1\left(1-\frac{\theta}{p}\right)\|x_n\|_p^p + \int_Z \left(w_n(z)x_n(z) - \theta F(z,x_n(z))\right)dz \\ \leq \varepsilon_n\|x_n\| + M_1.\end{aligned} \tag{3.13}$$

From Hypothesis 3.1(ii) we know that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have $vx - \theta F(z,x) + a(z) \geq 0$ for some $a \in L^{q^*}(Z)$ and for every $v \in \partial(F(z,x))$.

Suppose now that $\|x_n\| \to \infty$. Inequality (3.13) becomes then

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|Dx_n\|_p^p + \lambda_1\left(1-\frac{\theta}{p}\right)\|x_n\|_p^p \\ + \int_Z \left(w_n(z)x_n(z) - \theta F(z,x_n(z))\right)dz + \int_Z a(z)\,dz \\ \leq \varepsilon_n\|x_n\| + \int_Z a(z)\,dz + M_1.\end{aligned} \tag{3.14}$$

Divide this inequality with $\|Dx_n\|_p^p$, then we have in the limit

$$\frac{\theta}{p} - 1 \leq 0, \tag{3.15}$$

recall that $\|Dx_n\|$ is an equivalent norm in $W_o^{1,p}(Z)$ and

$$-\lambda_1\left(1-\frac{\theta}{p}\right)\|x_n\|_p^p \geq -\left(\frac{\theta}{p}-1\right)\|Dx_n\|_p^p. \tag{3.16}$$

Since $\theta > p$, we have a contradiction. So $\|x_n\|$ is bounded.

From the properties of the subdifferential of Clarke, we have

$$\begin{aligned}\partial R(x_n) &\subseteq \partial(R_1(x_n)) + \partial(R_2(x_n)) \\ &\subseteq \partial(R_2(x_n)) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p\right)\end{aligned} \tag{3.17}$$

(see Clarke [4, page 83]). So, we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle - \int_Z v_n(z)y(z)\,dz, \tag{3.18}$$

with w_n the element with minimal norm of the subdifferential of R (recall that $\|w_n\|_* \to 0$), $v_n \in [f_1(z,x_n(z)), f_2(z,x_n(z))]$, and $A : W_o^{1,p}(Z) \to W^{-1,q}(Z)$ such that

$$\langle Ax, y\rangle = \int_Z ||Dx(z)||^{p-2}(Dx(z), Dy(z))_{R^N}\, dz - \lambda_1 \int_Z ||x_n||_p^{p-2} x_n y_n\, dz, \quad (3.19)$$

for all $y \in W_o^{1,p}(Z)$. But $x_n \xrightarrow{w} x$ in $W_o^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W_o^{1,p}(Z) \subseteq L^p(Z)$. Note that v_n is bounded. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$. Recall the following inequality:

$$\sum_{j=1}^{N} (a_j(\eta) - a_j(\eta'))(\eta_j - \eta_j') \geq C|\eta - \eta'|^p, \quad (3.20)$$

for $\eta, \eta' \in R^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

By virtue of this inequality we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W_o^{1,p}(Z)$. The claim is proved. Thus R satisfies $(\mathrm{PS})_c$.

We will show now that there exists $\rho > 0$ such that $R(x) \geq \eta > 0$ with $\|x\| = \rho$. To this end, we show that for every sequence $\{x_n\}_{n\geq 1} \subseteq W_o^{1,p}(Z)$ with $\|x_n\| = \rho_n \to 0$, we have $R(x_n) \downarrow 0$. Suppose that it is not true. Then there exists a sequence as above such that $R(x_n) \leq 0$. Since $\|x_n\| \to 0$ we have $x_n(z) \to 0$ a.e. on Z.

So we have

$$||Dx_n||_p^p - \lambda_1 ||x_n||_p^p \leq \int_Z pF(z, x_n(z))\, dz. \quad (3.21)$$

Let $y_n(z) = x_n(z)/\|x_n\|_{1,p}$. Also, from Hypothesis 3.1(iii) we have uniformly, for all $z \in Z$, that for all $\varepsilon > 0$ we can find $\delta > 0$ such that for $|x| \leq \delta$ we have

$$pF(z, x(z)) \leq \theta(z)|x(z)|^p + \varepsilon|x(z)|^p. \quad (3.22)$$

On the other hand, from hypothesis (i) we have that there exist some c_1, c_2 such that $pF(z,x) \leq c_1|x|^p + c_2|x|^{p^*} + p|x|$ for almost all $z \in Z$ and all $x \in \mathbb{R}$. Thus we can always find some $\gamma > 0$ such that $pF(z,x) \leq (\theta(z)+\varepsilon)|x|^p + \gamma|x|^{p^*}$. Indeed, choose $\gamma \geq |c_1 - \theta(z) - \varepsilon||\delta|^{p-p^*} + c_2 + p|\delta|^{1-p^*}$.

Then we obtain,

$$||Dx_n||_p^p - \lambda_1 ||x_n||_p^p \leq \int_Z (\theta(z)+\varepsilon)|x_n(z)|^p\, dz + \gamma \int_Z |x_n(z)|^{p^*}\, dz. \quad (3.23)$$

Dividing the last inequality, by $\|x_n\|_{1,p}^p$, we have

$$||Dy_n||^p - \lambda_1||y_n||_p^p \leq \int_Z (\theta(z)+\varepsilon)\,|\,y_n(z)\,|^p\,dz + \gamma\frac{\int_Z |\,x_n(z)\,|^{p^*}\,dz}{||x_n||_{1,p}^p} \\ \leq \varepsilon||y_n||_p^p + \gamma_1||x_n||_{1,p}^{p^*-p}, \tag{3.24}$$

recall that $W_o^{1,p}(Z)$ is continuously embedded on $L^{p^*}(Z)$.

Using the variational characterization of the first eigenvalue we have that $0 \leq \|Dy_n\|_p^p - \lambda_1\|y_n\|_p^p \leq \varepsilon\|y_n\|_p^p + \gamma_1\|x_n\|_{1,p}^{p^*-p}$.

Recall that $\|y_n\| = 1$ so $y_n \to y$ weakly in $W_o^{1,p}(Z)$, $y_n(z) \to y(z)$ a.e. on Z. Thus, from (3.24) we have that $\|Dy_n\| \to \lambda_1\|y\|$. Also, from the weak lower semicontinuity of the norm we have that $\|Dy\| \leq \liminf \|Dy_n\| \to \lambda_1\|y\|$. Using the Rayleigh quotient we have that $\|Dy\| = \lambda_1\|y\|$. Recall that $y_n \to y$ weakly in $W_o^{1,p}(Z)$ and $\|Dy_n\| \to \|Dy\|$. So, from a well-known argument we obtain $y_n \to y$ in $W_o^{1,p}(Z)$, and since $\|y_n\| = 1$ we have that $\|y\| = 1$. That is, $y \neq 0$ and from the equality $\|Dy\| = \lambda_1\|y\|$ we have that $y(z) = \pm u_1(z)$. Suppose that $y(z) = u_1(z)$.

Dividing now (3.23) by $\|x_n\|_{1,p}^p$ and using the variational characterization of the first eigenvalue, there exists for every $\varepsilon > 0$ some n_o such that for $n \geq n_o$ we have

$$0 \leq \int_Z (\theta(z)+\varepsilon)\,|\,y_n(z)\,|^p\,dz + \gamma_1||x_n||_{1,p}^{p^*-p}. \tag{3.25}$$

So in the limit we obtain

$$0 \leq \int_Z (\theta(z)+\varepsilon)u_1^p(z)\,dz \leq \varepsilon||u_1||_p^p \quad \forall \varepsilon > 0. \tag{3.26}$$

Thus, $\int_Z \theta(z)u_1^p(z)\,dz = 0$. Recall that $u_1(z) > 0$ a.e. on Z. This is a contradiction. So there exists $\rho > 0$ such that $R(x) \geq \eta > 0$ for all $x \in W_o^{1,p}(Z)$ with $\|x\| = \rho$.

Next, it is easy to see that

$$R(su_1) = -\int_Z F(z, su_1(z))\,dz, \tag{3.27}$$

(here we used again the Rayleigh quotient).

But from hypothesis (ii) we have that $-F(z, su_1(z)) \leq -c_3|su_1(z)|^\theta + a_1(z)$ a.e. on Z. So for s large enough, we obtain that $R(su_1) \leq 0$. Then we can use Theorem 2.2 to obtain $x \in W_o^{1,p}(Z)$ such that $x \neq 0$ and $0 \in \partial R(x)$. It follows that

$$Ax = \lambda_1|x|^{p-2}x + v, \tag{3.28}$$

with $v \in \partial(\int_Z F(z,x(z))\,dz)$. So for every $\phi \in C_o^\infty(Z)$ we have

$$\langle Ax, \phi\rangle = \lambda_1 \langle |x|^{p-2}x, \phi\rangle_{pq} + (v,\phi)_{pq}. \tag{3.29}$$

By $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$. Thus,

$$\int_Z ||Dx(z)||^{p-2}(Dx(z), D\phi(z))_{R^N}\,dz = \int_Z \left(\lambda_1 |x(z)|^{p-2}x(z) + v(z)\right)\phi(z)\,dz. \tag{3.30}$$

From the definition of the distributional derivative,

$$-\operatorname{div}\left(||Dx(z)||^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = v(z) \quad \text{a.e. on } Z. \tag{3.31}$$

So $x \in W_o^{1,p}(Z)$ is a nontrivial solution of type I. □

In order to have an existence result of type II, we have to impose stronger hypotheses on f. Our hypotheses are the following.

Hypothesis 3.7. The function $f : Z \times \mathbb{R} \to \mathbb{R}$ satisfies Hypothesis 3.1. Moreover, we suppose that $f_1(z,a)dz + \lambda_1|a|^{p-2}a > 0$ or that $f_2(z,a) + \lambda_1|a|^{p-2}a < 0$ a.e. on Z, for any $a \in D(f) = \{x \in \mathbb{R} : f_1(z,x) \neq f_2(z,x) \text{ a.e. on } S_x \subseteq Z\}$ (i.e., the set of the discontinuity points of f). Finally, we suppose that $f(z,\cdot)$ has countable number of discontinuities.

Theorem 3.8. *If Hypothesis 3.7 holds, then problem (1.1) has a nontrivial solution of type II.*

Proof. From Theorem 3.5 we know that there exists a nontrivial solution of type I. That is, there exists some $w \in L^q(Z)$ with $w(z) \in [f_1(z,x(z)), f_2(z,x(z))]$ such that

$$\begin{gathered}-\operatorname{div}\left(||Dx(z)||^{p-2}Dx(z)\right) - \lambda_1 |x(z)|^{p-2}x(z) = w(z) \quad \text{a.e. on } Z,\\ x|_\Gamma = 0.\end{gathered} \tag{3.32}$$

We suppose that there exists some $A \subseteq Z$ with $|A| > 0$ such that $x(z) = a_1 \in D(f)$ a.e. on A, and that $|A \cap S_{a_1}| \neq 0$. Take now the closure of that set, that is, $\overline{A \cap S_{a_1}}$. It is clear that the interior of that set is nonempty (recall that $\overline{A \cap S_{a_1}} = (A \cap S_{a_1})^o \cup \partial(A \cap S_{a_1})$) because we have supposed that $|A \cap S_{a_1}| \neq 0$. So, there exist some $z \in (A \cap S_{a_1})^o$ and some $r > 0$ such that $B(z,r) \subseteq A \cap S_{a_1}$. Take now $r' = r/2$, then it is clear that $B(z,r') \subseteq B(z,r) \subseteq A \cap S_{a_1}$ (here by $B(z,r)$ we denote the open ball centered at z with radius r).

We know that there exists a test function which is equal to 1 on $\overline{B(z,r')}$, equal to 0 outside $B(z,r)$, and assumes values in $[0,1]$ in $B(z,r) \setminus \overline{B(z,r')}$. Multiply (3.32) with this function and then integrate over $B(z,r)$. Using the definition of the distributional derivative and finally the well-known theorem of Stampacchia, which states that if $x(z) \in W^{1,p}(Z)$ and $x(z) = a$ a.e. on A then $D_k x(z) = 0$ a.e. on A, we have

$$\int_{B(z,r)} w(z)\phi(z)\,dz = \int_{B(z,r)} \left(-\lambda_1 |a_1|^{p-2} a_1\right)\phi(z)\,dz. \tag{3.33}$$

But we know that $w(z) \in [f_1(z,x(z)), f_2(z,x(z))]$ a.e. on Z.

If $f_1(z,a_1) + \lambda_1 |a_1|^{p-2} a_1 > 0$ a.e. on Z, we obtain

$$\begin{aligned} 0 &< \int_{B(z,r)} \left(f_1(z,a_1) + \lambda_1 |a_1|^{p-2} a_1\right)\phi(z)\,dz \\ &\le \int_{B(z,r)} \left(w(z) + \lambda_1 |a_1|^{p-2} a_1\right)\phi(z)\,dz = 0. \end{aligned} \tag{3.34}$$

Thus we have a contradiction. The same holds if $f_2(z,a_1)dz + \lambda_1 |a_1|^{p-2} a_1 < 0$ a.e. on Z. So $|A \cap S_{a_1}| = 0$. Set now $B \subseteq Z$ such that

$$B = \bigcup_{n=1}^{\infty} B_n, \tag{3.35}$$

where $B_n = A_n \cap S_{a_n} \subseteq Z$ is such that $x(z) = a_n$ on A_n with $a_n \in D(f)$ (recall that f has countable number of discontinuities). Then from the above arguments we have that $|B| = 0$. That is, x is a solution of type II. □

Remark 3.9. As far as we know, this is the first existence result of type II for the p-Laplacian with nonmonotone discontinuities and without using the method of upper and lower solution. All the known results need the solution to be in $W_o^{2,p}(Z)$ (cf. [1]), but here we do not have such a regularity result, so the arguments that we have used are more complicated.

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