HYERS-ULAM STABILITY OF THE LINEAR RECURRENCE WITH CONSTANT COEFFICIENTS

DORIAN POPA

Received 5 November 2004 and in revised form 14 March 2005

Let X be a Banach space over the field $\mathbb{R}$ or $\mathbb{C}$, $a_1,\ldots,a_p \in \mathbb{C}$, and $(b_n)_{n\geq 0}$ a sequence in X. We investigate the Hyers-Ulam stability of the linear recurrence $x_{n+p} = a_1 x_{n+p-1} + \cdots + a_{p-1}x_{n+1} + a_p x_n + b_n$, $n \geq 0$, where $x_0, x_1, \ldots, x_{p-1} \in X$.

1. Introduction

In 1940, S. M. Ulam proposed the following problem.

Problem 1.1. *Given a metric group $(G,\cdot,d)$, a positive number ε, and a mapping $f : G \to G$ which satisfies the inequality $d(f(xy), f(x)f(y)) \leq \varepsilon$ for all $x, y \in G$, do there exist an automorphism a of G and a constant δ depending only on G such that $d(a(x), f(x)) \leq \delta$ for all $x \in G$?*

If the answer to this question is affirmative, we say that the equation $a(xy) = a(x)a(y)$ is stable. A first answer to this question was given by Hyers [5] in 1941 who proved that the Cauchy equation is stable in Banach spaces. This result represents the starting point theory of Hyers-Ulam stability of functional equations. Generally, we say that a functional equation is stable in Hyers-Ulam sense if for every solution of the perturbed equation, there exists a solution of the equation that differs from the solution of the perturbed equation with a small error. In the last 30 years, the stability theory of functional equations was strongly developed. Recall that very important contributions to this subject were brought by Forti [2], Găvruţa [3], Ger [4], Páles [6, 7], Székelyhidi [9], Rassias [8], and Trif [10]. As it is mentioned in [1], there are much less results on stability for functional equations in a single variable than in more variables, and no surveys on this subject. In our paper, we will investigate the discrete case for equations in single variable, namely, the Hyers-Ulam stability of linear recurrence with constant coefficients.

Let X be a Banach space over a field K and

$$x_{n+p} = f(x_{n+p-1},\ldots,x_n), \quad n \geq 0, \tag{1.1}$$

a recurrence in X, when p is a positive integer, $f : X^p \to X$ is a mapping, and $x_0, x_1, \ldots, x_{p-1} \in X$. We say that the recurrence (1.1) is stable in Hyers-Ulam sense if for every positive ε

Advances in Difference Equations 2005:2 (2005) 101–107
DOI: 10.1155/ADE.2005.101

and every sequence $(x_n)_{n\geq 0}$ that satisfies the inequality

$$||x_{n+p} - f(x_{n+p-1},\dots,x_n)|| < \varepsilon, \quad n \geq 0, \tag{1.2}$$

there exist a sequence $(y_n)_{n\geq 0}$ given by the recurrence (1.1) and a positive δ depending only on f such that

$$||x_n - y_n|| < \delta, \quad n \geq 0. \tag{1.3}$$

In [7], the author investigates the Hyers-Ulam-Rassias stability of the first-order linear recurrence in a Banach space. Using some ideas from [7] in this paper, one obtains a result concerning the stability of the n-order linear recurrence with constant coefficients in a Banach space, namely,

$$x_{n+p} = a_1 x_{n+p-1} + \cdots + a_{p-1} x_{n+1} a + a_p x_n + b_n, \quad n \geq 0, \tag{1.4}$$

where $a_1, a_2, \dots, a_p \in K$, $(b_n)_{n\geq 0}$ is a given sequence in X, and $x_0, x_1, \dots, x_{p-1} \in X$. Many new and interesting results concerning difference equations can be found in [1].

2. Main results

In what follows, we denote by K the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. Our stability result is based on the following lemma.

Lemma 2.1. *Let X be a Banach space over K, ε a positive number, $a \in K \setminus \{-1,0,1\}$, and $(a_n)_{n\geq 0}$ a sequence in X. Suppose that $(x_n)_{n\geq 0}$ is a sequence in X with the following property:*

$$||x_{n+1} - a x_n - a_n|| \leq \varepsilon, \quad n \geq 0. \tag{2.1}$$

Then there exists a sequence $(y_n)_{n\geq 0}$ in X satisfying the relations

$$y_{n+1} = a y_n + a_n, \quad n \geq 0, \tag{2.2}$$

$$||x_n - y_n|| \leq \frac{\varepsilon}{\big||a| - 1\big|}, \quad n \geq 0. \tag{2.3}$$

Proof. Denote $x_{n+1} - a x_n - a_n := b_n$, $n \geq 0$. By induction, one obtains

$$x_n = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1}(a_k + b_k), \quad n \geq 1. \tag{2.4}$$

(1) Suppose that $|a| < 1$. Define the sequence $(y_n)_{n\geq 0}$ by the relation (2.2) with $y_0 = x_0$. Then it follows by induction that

$$y_n = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1} b_k, \quad n \geq 1. \tag{2.5}$$

By the relation (2.4) and (2.5), one gets

$$
\begin{aligned}
\|x_n - y_n\| &\le \Big\| \sum_{k=0}^{n-1} b_k a^{n-k-1} \Big\| \le \sum_{k=0}^{n-1} \|b_k\| |a|^{n-k-1} \\
&\le \varepsilon \frac{1-|a|^n}{1-|a|} < \frac{\varepsilon}{1-|a|}, \quad n \ge 1.
\end{aligned}
\tag{2.6}
$$

(2) If $|a| > 1$, by using the comparison test, it follows that the series $\sum_{n=1}^{\infty}(b_{n-1}/a^n)$ is absolutely convergent, since

$$
\begin{gathered}
\Big\| \frac{b_{n-1}}{a^n} \Big\| \le \frac{\varepsilon}{|a|^n}, \quad n \ge 1, \\
\sum_{n=1}^{\infty} \frac{\varepsilon}{|a|^n} = \frac{\varepsilon}{|a|-1}.
\end{gathered}
\tag{2.7}
$$

Denoting

$$
s := \sum_{n=1}^{\infty} \frac{b_{n-1}}{a^n}, \tag{2.8}
$$

we define the sequence $(y_n)_{n\ge 0}$ by the relation (2.2) with $y_0 = x_0 + s$.

Then one obtains

$$
\begin{aligned}
\|x_n - y_n\| \Big\| -a^n s + \sum_{k=0}^{n-1} b_k a^{n-k-1} \Big\| &= |a|^n \Big\| -s + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \Big\| \\
&= |a|^n \Big\| \sum_{k=n}^{\infty} \frac{b_k}{a^{k+1}} \Big\| \\
&\le \varepsilon \sum_{n=1}^{\infty} \frac{1}{|a|^n} = \frac{\varepsilon}{|a|-1}, \quad n \ge 0.
\end{aligned}
\tag{2.9}
$$

The lemma is proved. □

Remark 2.2. (1) If $|a| > 1$, then the sequence $(y_n)_{n\ge 0}$ from Lemma 2.1 is uniquely determined.

(2) If $|a| < 1$, then there exists an infinite number of sequences $(y_n)_{n\ge 0}$ in Lemma 2.1 that satisfy (2.2) and (2.3).

Proof. (1) Suppose that there exists another sequence $(y_n)_{n\ge 0}$ defined by (2.2), $y_0 \ne x_0 + s$, that satisfies (2.3). Hence,

$$
\|x_n - y_n\| \Big\| a^n (x_0 - y_0) + \sum_{k=0}^{n-1} b_k a^{n-k-1} \Big\| = |a|^n \Big\| x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \Big\|, \quad n \ge 1. \tag{2.10}
$$

Since

$$
\lim_{n\to\infty} \Big\| x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \Big\| = \|x_0 + s - y_0\| \ne 0, \tag{2.11}
$$

it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = \infty. \tag{2.12}$$

(2) If $|a| < 1$, one can choose $y_0 = x_0 + u$, $\|u\| \le \varepsilon$. Then

$$\begin{aligned} \|x_n - y_n\| &= \left\| -a^n u + \sum_{k=0}^{n-1} b_k a^{n-k-1} \right\| \le \varepsilon \sum_{k=0}^{n} |a|^k \\ &= \varepsilon \frac{1-|a|^{n+1}}{1-|a|} \le \frac{\varepsilon}{1-|a|}, \quad n \ge 1. \end{aligned} \tag{2.13}$$

□

The stability result for the p-order linear recurrence with constant coefficients is contained in the next theorem.

Theorem 2.3. *Let X be a Banach space over the field K, $\varepsilon > 0$, and $a_1, a_2, \dots, a_p \in K$ such that the equation*

$$r^p - a_1 r^{p-1} - \cdots - a_{p-1} r - a_p = 0 \tag{2.14}$$

admits the roots $r_1, r_2, \dots, r_p$, $|r_k| \ne 1$, $1 \le k \le p$, and $(b_n)_{n\ge 0}$ is a sequence in X. Suppose that $(x_n)_{n\ge 0}$ is a sequence in X with the property

$$\|x_{n+p} - a_1 x_{n+p-1} - \cdots - a_{p-1} x_{n+1} - a_p x_n - b_n\| \le \varepsilon, \quad n \ge 0. \tag{2.15}$$

Then there exists a sequence $(y_n)_{n\ge 0}$ in X given by the recurrence

$$y_{n+p} = a_1 y_{n+p-1} + \cdots + a_{p-1} y_{n+1} + a_p y_n + b_n, \quad n \ge 0, \tag{2.16}$$

such that

$$\|x_n - y_n\| \le \frac{\varepsilon}{\left| (|r_1| - 1) \cdots (|r_p| - 1) \right|}, \quad n \ge 0. \tag{2.17}$$

Proof. We prove Theorem 2.3 by induction on p.

For $p = 1$, the conclusion of Theorem 2.3 is true in virtue of Lemma 2.1. Suppose now that Theorem 2.3 holds for a fixed $p \ge 1$. We have to prove the following assertion.

Assertion 2.4. *Let ε be a positive number and $a_1, a_2, \dots, a_{p+1} \in K$ such that the equation*

$$r^{p+1} - a_1 r^p - \cdots - a_p r - a_{p+1} = 0 \tag{2.18}$$

admits the roots $r_1, r_2, \dots, r_{p+1}$, $|r_k| \ne 1$, $1 \le k \le p+1$, and $(b_n)_{n\ge 0}$ is a sequence in X. If $(x_n)_{n\ge 0}$ is a sequence in X satisfying the relation

$$\|x_{n+p+1} - a_1 x_{n+p} - \cdots - a_p x_{n+1} - a_{p+1} x_n - b_n\| \le \varepsilon, \quad n \ge 0, \tag{2.19}$$

then there exists a sequence $(y_n)_{n\ge 0}$ in X, given by the recurrence

$$y_{n+p+1} = a_1 y_{n+p} + \cdots + a_p y_{n+1} + a_{p+1} y_n + b_n, \quad n \ge 0, \tag{2.20}$$

such that

$$\|x_n - y_n\| \le \frac{\varepsilon}{|(|r_1| - 1)\cdots(|r_{p+1}| - 1)|}, \quad n \ge 0. \tag{2.21}$$

The relation (2.19) can be written in the form

$$\|x_{n+p+1} - (r_1 + \cdots + r_{p+1})x_{n+p} - \cdots + (-1)^{p+1} r_1 \cdots r_{p+1} x_n - b_n\| \le \varepsilon, \quad n \ge 0. \tag{2.22}$$

Denoting $x_{n+1} - r_{p+1}x_n = u_n$, $n \ge 0$, one gets by (2.22)

$$\|u_{n+p} - (r_1 + \cdots + r_p)u_{n+p-1} + \cdots + (-1)^p r_1 r_2 \cdots r_p u_n - b_n\| \le \varepsilon, \quad n \ge 0. \tag{2.23}$$

By using the induction hypothesis, it follows that there exists a sequence $(z_n)_{n\ge 0}$ in X, satisfying the relations

$$z_{n+p} = a_1 z_{n+p-1} + \cdots + a_p z_n + b_n, \quad n \ge 0, \tag{2.24}$$

$$\|u_n - z_n\| \le \frac{\varepsilon}{|(|r_1| - 1)\cdots(|r_p| - 1)|}, \quad n \ge 0. \tag{2.25}$$

Hence

$$\|x_{n+1} - r_{p+1}x_n - z_n\| \le \frac{\varepsilon}{|(|r_1| - 1)\cdots(|r_p| - 1)|}, \quad n \ge 0, \tag{2.26}$$

and taking account of Lemma 2.1, it follows from (2.26) that there exists a sequence $(y_n)_{n\ge 0}$ in X, given by the recurrence

$$y_{n+1} = r_{p+1} y_n + z_n, \quad n \ge 0, \tag{2.27}$$

that satisfies the relation

$$\|x_n - y_n\| \le \frac{\varepsilon}{|(|r_1| - 1)\cdots(|r_{p+1}| - 1)|}, \quad n \ge 0. \tag{2.28}$$

By (2.24) and (2.27), one gets

$$y_{n+p+1} = a_1 y_{n+p} + \cdots + a_{p+1} y_n + b_n, \quad n \ge 0. \tag{2.29}$$

The theorem is proved. □

Remark 2.5. If $|r_k| > 1$, $1 \le k \le p$, in Theorem 2.3, then the sequence $(y_n)_{n\ge 0}$ is uniquely determined.

Proof. The proof follows from Remark 2.2. □

Remark 2.6. If there exists an integer s, $1 \le s \le p$, such that $|r_s| = 1$, then the conclusion of Theorem 2.3 is not generally true.

Proof. Let $\varepsilon > 0$, and consider the sequence $(x_n)_{n\geq 0}$, given by the recurrence

$$x_{n+2} + x_{n+1} - 2x_n = \varepsilon, \quad n \geq 0,\ x_0, x_1 \in K. \tag{2.30}$$

A particular solution of this recurrence is

$$x_n = \frac{\varepsilon}{3} n, \quad n \geq 0, \tag{2.31}$$

hence the general solution of the recurrence is

$$x_n = \alpha + \beta(-2)^n + \frac{\varepsilon}{3} n, \quad n \geq 0,\ \alpha, \beta \in K. \tag{2.32}$$

Let $(y_n)_{n\geq 0}$ be a sequence satisfying the recurrence

$$y_{n+2} + y_{n+1} - 2y_n = 0, \quad n \geq 0,\ y_0, y_1 \in K. \tag{2.33}$$

Then $y_n = \gamma + \delta(-2)^n$, $n \geq 0$, $\gamma, \delta \in K$, and

$$\sup_{n\in\mathbb{N}} |x_n - y_n| = \infty. \tag{2.34}$$

□

Example 2.7. Let X be a Banach space and ε a positive number. Suppose that $(x_n)_{n\geq 0}$ is a sequence in X satisfying the inequality

$$||x_{n+2} - x_{n+1} - x_n|| \leq \varepsilon, \quad n \geq 0. \tag{2.35}$$

Then there exists a sequence $(f_n)_{n\geq 0}$ in X given by the recurrence

$$f_{n+2} - f_{n+1} - f_n = 0, \quad n \geq 0, \tag{2.36}$$

such that

$$||x_n - f_n|| \leq (2 + \sqrt{5})\varepsilon, \quad n \geq 0. \tag{2.37}$$

Proof. The equation $r^2 - r - 1 = 0$ has the roots $r_1 = (1 + \sqrt{5})/2$, $r_2 = (1 - \sqrt{5})/2$. By the Theorem 2.3, it follows that there exists a sequence $(f_n)_{n\geq 0}$ in X such that

$$||x_n - f_n|| \leq \frac{\varepsilon}{|(|r_1| - 1)(|r_2| - 1)|} = (2 + \sqrt{5})\varepsilon, \quad n \geq 0. \tag{2.38}$$

□

References

[1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.

[2] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.

[3] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[4] R. Ger, *A survey of recent results on stability of functional equations*, Proc. of the 4th International Conference on Functional Equations and Inequalities (Cracow), Pedagogical University of Cracow, Poland, 1994, pp. 5–36.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.

[6] Z. Páles, *Generalized stability of the Cauchy functional equation*, Aequationes Math. **56** (1998), no. 3, 222–232.

[7] ______, *Hyers-Ulam stability of the Cauchy functional equation on square-symmetric groupoids*, Publ. Math. Debrecen **58** (2001), no. 4, 651–666.

[8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[9] L. Székelyhidi, *Note on Hyers's theorem*, C. R. Math. Rep. Acad. Sci. Canada **8** (1986), no. 2, 127–129.

[10] T. Trif, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), no. 1-2, 47–61.

Dorian Popa: Department of Mathematics, Faculty of Automation and Computer Science, Technical University of Cluj-Napoca, 25-38 Gh. Baritiu Street, 3400 Cluj-Napoca, Romania

E-mail address: popa.dorian@math.utcluj.ro