CONSTRUCTION OF UPPER AND LOWER SOLUTIONS FOR SINGULAR DISCRETE INITIAL AND BOUNDARY VALUE PROBLEMS VIA INEQUALITY THEORY

HAISHEN LÜ AND DONAL O'REGAN

We present new existence results for singular discrete initial and boundary value problems. In particular our nonlinearity may be singular in its dependent variable and is allowed to change sign.

1. Introduction

An upper- and lower-solution theory is presented for the singular discrete boundary value problem

$$-\Delta\big(\varphi_p(\Delta u(k-1))\big) = q(k)f(k,u(k)), \quad k \in N = \{1,\dots,T\},$$
$$u(0) = u(T+1) = 0, \tag{1.1}$$

and the singular discrete initial value problem

$$\Delta u(k-1) = q(k)f(k,u(k)), \quad k \in N = \{1,\dots,T\},$$
$$u(0) = 0, \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\Delta u(k-1) = u(k) - u(k-1)$, $T \in \{1,2,\dots\}$, $N^+ = \{0,1,\dots,T\}$, and $u : N^+ \to \mathbb{R}$. Throughout this paper, we will assume $f : N \times (0,\infty) \to \mathbb{R}$ is continuous. As a result, our nonlinearity $f(k,u)$ may be singular at $u = 0$ and may change sign.

Remark 1.1. Recall a map $f : N \times (0,\infty) \to \mathbb{R}$ is continuous if it is continuous as a map of the topological space $N \times (0,\infty)$ into the topological space $\mathbb{R}$. Throughout this paper, the topology on N will be the discrete topology.

We will let $C(N^+,\mathbb{R})$ denote the class of map u continuous on N^+ (discrete topology) with norm $\|u\| = \max_{k \in N^+} \|u(k)\|$. By a solution to (1.1) (resp., (1.2)) we mean a $u \in C(N^+,\mathbb{R})$ such that u satisfies (1.1) (resp., (1.2)) for $i \in N$ and u satisfies the boundary (resp., initial) condition.

It is interesting to note here that the existence of solutions to singular initial and boundary value problems in the continuous case have been studied in great detail in

the literature (see [2, 4, 5, 6, 7, 9, 10, 11] and the references therein). However, only a few papers have discussed the discrete singular case (see [1, 3, 8] and the references therein).

In [7], the following result has been proved.

THEOREM 1.2. *Let $n_0 \in \{1,2,\dots\}$ be fixed and suppose the following conditions are satisfied:*

$$f : N \times (0,\infty) \longrightarrow \mathbb{R} \text{ is continuous,} \tag{1.3}$$

$$q \in C(N,(0,\infty)), \tag{1.4}$$

there exists a function $\alpha \in C(N^+,\mathbb{R})$ with
$$\alpha(0) = \alpha(T+1) = 0, \quad \alpha > 0 \text{ on } N \text{ such that} \tag{1.5}$$
$$q(k)f(k,\alpha(k)) \geq -\Delta(\varphi_p(\alpha(k-1))) \quad \text{for } k \in N,$$

there exists a function $\beta \in C(N^+,\mathbb{R})$ with
$$\beta(k) \geq \alpha(k), \quad \beta(k) \geq \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \tag{1.6}$$
$$q(k)f(k,\beta(k)) \leq -\Delta(\varphi_p(\beta(k-1))) \quad \text{for } k \in N.$$

Then (1.1) has a solution $u \in C(N^+,\mathbb{R})$ with $u(k) \geq \alpha(k)$ for $k \in N^+$.

In [1], the following result has been proved.

THEOREM 1.3. *Let $n_0 \in \{1,2,\dots\}$ be fixed and suppose the following conditions are satisfied:*

$$f : N \times (0,\infty) \longrightarrow \mathbb{R} \text{ is continuous,} \tag{1.7}$$

$$q \in C(N,(0,\infty)), \tag{1.8}$$

there exists a function $\alpha \in C(N^+,\mathbb{R})$ with
$$\alpha(0) = 0, \quad \alpha > 0 \text{ on } N \text{ such that} \tag{1.9}$$
$$q(k)f(k,\alpha(k)) \geq \Delta\alpha(k-1) \quad \text{for } k \in N,$$

there exists a function $\beta \in C(N^+,\mathbb{R})$ with
$$\beta(k) \geq \alpha(k), \quad \beta(k) > \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \tag{1.10}$$
$$q(k)f(k,\beta(k)) \leq \Delta\beta(k-1) \quad \text{for } k \in N.$$

Then (1.2) has a solution $u \in C(N^+,\mathbb{R})$ with $u(k) \geq \alpha(k)$ for $k \in N^+$.

Also some results from the literature, which will be needed in Section 2 are presented.

LEMMA 1.4 [8]. *Let $u \in C(N^+,\mathbb{R})$ satisfy $u(k) \geq 0$ for $k \in N^+$. If $u \in C(N^+,\mathbb{R})$ satisfies*

$$-\Delta^2 u(k-1) = u(k), \quad k \in N = \{1,2,\dots,T\},$$
$$u(0) = u(T+1) = 0, \tag{1.11}$$

then

$$u(k) \geq \mu(k)\|u\| \quad \text{for } k \in N^+; \tag{1.12}$$

here

$$\mu(k) = \min\left\{\frac{T+1-k}{T+1}, \frac{k}{T}\right\}. \tag{1.13}$$

LEMMA 1.5 [8]. *Let* $[a,b] = \{a, a+1, \ldots, b\} \subset N$. *If* $u \in C(N^+, \mathbb{R})$ *satisfies*

$$\begin{aligned}
\Delta\left(\varphi_p(\Delta u(k-1))\right) &\leq 0, \quad k \in [a,b], \\
u(a-1) &\geq 0, \quad u(b+1) \geq 0,
\end{aligned} \tag{1.14}$$

then $u(k) \geq 0$ *for* $k \in [a-1, b+1] = \{a-1, a, \ldots, b+1\} \subset N^+$.

In Theorems 1.2 and 1.3 the construction of a lower solution α and an upper solution β is critical. We present an easily verifiable condition in Section 2.

2. Main results

We begin with a result for boundary value problems.

THEOREM 2.1. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose (1.3), (1.4) hold. Also assume the following conditions are satisfied:*

there exists a constant $c_0 > 0$ *such that*

$$q(k)f(k,u) \geq c_0 \quad \text{for } k \in N, \ 0 < u \leq \frac{1}{n_0}, \tag{2.1}$$

there exist $h > 0$ *continuous and nondecreasing on* $[0, \infty)$ *such that*

$$|f(k,u)| \leq h(u) \quad \text{for } (k,u) \in N \times \left[\frac{1}{n_0}, \infty\right), \tag{2.2}$$

there exist $M > \frac{1}{n_0}$ *such that*

$$M - \frac{1}{n_0} > \varphi_p^{-1}(h(M))b_0; \tag{2.3}$$

here

$$b_0 = \max_{k \in N}\left\{\sum_{i=1}^{k} \varphi_p^{-1}\left(\sum_{j=i}^{k} q(j)\right), \sum_{i=1}^{k} \varphi_p^{-1}\left(\sum_{j=i}^{k} q(j)\right)\right\}. \tag{2.4}$$

Then (1.1) has a solution $u \in C(N^+, \mathbb{R})$ *with* $u(k) > 0$ *for* $k \in N$.

Proof. First we construct the lower solution α in (1.5). Let $\alpha(k) = cv(k)$, $k \in N^+$, where $v \in C(N^+, [0, \infty))$ is the solution of

$$-\Delta(\varphi_p(\Delta v(k-1))) = 1, \quad k \in N,$$
$$v(0) = v(T+1) = 0, \tag{2.5}$$

$$0 < c < \min\left\{c_0^{1/(p-1)}, \frac{1}{n_0\|v\|}\right\}. \tag{2.6}$$

Since $-\Delta(\varphi_p(\Delta v(k-1))) > 0$ implies $\Delta^2 v(k-1) < 0$ for $k \in N$, it follows from Lemma 1.4 that $v(k) \geq \mu(k)\|v\|$ for $k \in N^+$. Thus,

$$0 < \alpha(k) \leq \frac{1}{n_0} \quad \text{for } k \in N, \tag{2.7}$$

$$-\Delta(\varphi_p(\Delta\alpha(k-1))) = c^{p-1} \leq c_0 \quad \text{for } k \in N,$$
$$\alpha(0) = \alpha(T+1) = 0. \tag{2.8}$$

As a result (1.5) holds, since

$$q(k)f(k, \alpha(k)) \geq c_0 \geq -\Delta(\varphi_p(\Delta\alpha(k-1))) \quad \text{for } k \in N. \tag{2.9}$$

Next we discuss the boundary value problem

$$-\Delta(\varphi_p(\Delta u(k-1))) = q(k)h(M), \quad k \in N,$$
$$u(0) = u(T+1) = \frac{1}{n_0}. \tag{2.10}$$

It follows from [8] that (2.10) has a solution $u \in C(N^+, \mathbb{R})$. Let $v(k) = u(k) - 1/n_0$ for $k \in N^+$. Then $\Delta(\varphi_p(\Delta u(k-1))) = -\Delta(\varphi_p(\Delta v(k-1))) \leq 0$ for $k \in N$, and $v(0) = v(T+1) = 0$. Lemma 1.5 guarantees that $v(k) \geq 0$ and so $u(k) \geq 1/n_0$ for $k \in N^+$. Next we prove $u(k) \leq M$ for $k \in N^+$. Now since $\Delta(\varphi_p(\Delta u(k-1))) \leq 0$ on N implies $\Delta^2 u(k-1) \leq 0$ on N, then there exists $k_0 \in N$ with $\Delta u(k) \geq 0$ on $[0, k_0) = \{0, 1, \ldots, k_0 - 1\}$ and $\Delta u(k) \leq 0$ on $[k_0, T+1) = \{k_0, k_0 + 1, \ldots, T\}$, and $u(k_0) = \|u\|$. Suppose $u(k_0) > M$.

Also notice that for $k \in N$, we have

$$-\Delta(\varphi_p(\Delta u(k-1))) = q(k)h(M). \tag{2.11}$$

We sum (2.11) from $j + 1$ ($0 \leq j < k_0$) to k_0 to obtain

$$\varphi_p(\Delta u(j)) = \varphi_p(\Delta u(k_0)) + h(M)\sum_{k=j+1}^{k_0} q(k). \tag{2.12}$$

Now since $\Delta u(k_0) \leq 0$, we have

$$\varphi_p(\Delta u(j)) \leq h(M)\sum_{k=j+1}^{k_0} q(k) \quad \text{for } 0 \leq j < k_0, \tag{2.13}$$

that is,

$$\Delta u(j) \le \varphi_p^{-1}(h(M)) \varphi_p^{-1}\left(\sum_{k=j+1}^{k_0} q(k)\right) \quad \text{for } 0 \le j < k_0. \tag{2.14}$$

Then we sum the above from 0 to $k_0 - 1$ to obtain

$$u(k_0) - u(0) \le \varphi_p^{-1}(h(M)) \sum_{j=0}^{k_0-1} \varphi_p^{-1}\left(\sum_{k=j+1}^{k_0} q(k)\right)$$

$$\le \varphi_p^{-1}(h(M)) \sum_{j=1}^{k_0} \varphi_p^{-1}\left(\sum_{k=j}^{k_0} q(k)\right). \tag{2.15}$$

Similarly, we sum (2.11) from k_0 to j ($k_0 \le j \le T+1$) to obtain

$$-\varphi_p(\Delta u(j)) = -\varphi_p(\Delta u(k_0 - 1)) + h(M) \sum_{k=k_0}^{j} q(k) \quad \text{for } j \ge k_0. \tag{2.16}$$

Now since $\Delta u(k_0 - 1) \ge 0$, we have

$$-\Delta u(j) = \varphi_p^{-1}(h(M)) \varphi_p^{-1}\left(\sum_{k=k_0}^{j} q(k)\right) \quad \text{for } j \ge k_0. \tag{2.17}$$

We sum the above from k_0 to T to obtain

$$u(k_0) - u(T+1) \le \varphi_p^{-1}(h(M)) \sum_{j=k_0}^{T} \varphi_p^{-1}\left(\sum_{k=k_0}^{j} q(k)\right). \tag{2.18}$$

Now (2.15) and (2.18) imply

$$M - \frac{1}{n_0} \le b_0 \varphi_p^{-1}(h(M)). \tag{2.19}$$

This contradicts (2.3). Thus

$$\frac{1}{n_0} \le u(k) \le M \quad \text{for } k \in N^+. \tag{2.20}$$

Let $\beta(k) \equiv u(k)$ for $k \in N^+$. Now (2.7) and (2.20) guarantee

$$\alpha(k) \le \beta(k) \quad \text{for } k \in N^+. \tag{2.21}$$

Now (2.2) and (2.20) imply $f(k, \beta(k)) \le h(\beta(k)) \le h(M)$ so

$$\beta \in C(N^+, \mathbb{R}) \text{ with}$$

$$\beta(k) \ge \alpha(k), \quad \beta(k) \ge \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \tag{2.22}$$

$$q(k) f(k, \beta(k)) \le -\Delta(\varphi_p(\beta(k-1))) \quad \text{for } k \in N.$$

Now Theorem 1.2 guarantees that (1.1) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) \ge \alpha(k) > 0$ for $k \in N$. $\qquad \square$

Example 2.2. Consider the boundary value problem

$$\Delta^2 u(k-1) = \frac{k}{[u(k)]^\alpha} + [u(k)]^\beta - A, \quad k \in N,$$

$$u(0) = u(T+1) = 0 \tag{2.23}$$

with $p = 2$, $\alpha > 0$, $0 \le \beta < 1$, and $A > 0$. Then (2.23) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

To see this, we will apply Theorem 2.1 with

$$q(k) = 1, \qquad f(k, u) = \frac{k}{u^\alpha} + u^\beta - A. \tag{2.24}$$

Let $n_0 > (2A)^{1/\alpha}$ and $c_0 = A$. Then for $k \in N$ and $0 < u \le 1/n_0$,

$$q(k) f(k, u) = \frac{k}{u^\alpha} + u^\beta - A \ge \frac{k}{u^\alpha} - A \ge \frac{1}{u^\alpha} - A \ge 2A - A = A = c_0, \tag{2.25}$$

so (2.1) is satisfied. Let $h(u) = u^\beta + n_0^\alpha T + A$. Then (2.2) is immediate. Also since $0 \le \beta < 1$, we see that

there exist $M > \dfrac{1}{n_0}$ such that $M - \dfrac{1}{n_0} > b_0 (M^\beta + n_0^\alpha T + A)$; $\tag{2.26}$

here

$$b_0 = \max_{k \in N} \left(\sum_{j=1}^{k} (k - j + 1), \sum_{j=k}^{T} (j - k + 1) \right). \tag{2.27}$$

Thus (2.3) holds. Theorem 2.1 guarantees that (2.23) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

Next we present a result for initial value problems.

THEOREM 2.3. *Let $n_0 \in \{1, 2, \ldots\}$ be fixed and suppose (1.2), (1.3) hold. Also assume the following conditions are satisfied:*

there exists a constant $c_0 > 0$ such that

$$q(k) f(k, u) \ge c_0 \quad for\ k \in N,\ 0 < u \le \frac{1}{n_0}, \tag{2.28}$$

there exist $h > 0$ continuous and nondecreasing on $[0, \infty)$ such that

$$|f(k, u)| \le h(u) \quad for\ (k, u) \in N \times \left[\frac{1}{n_0}, \infty \right), \tag{2.29}$$

there exist $M > \dfrac{1}{n_0}$ such that

$$M - \frac{1}{n_0} > h(M) \sum_{k=1}^{T} q(k). \tag{2.30}$$

Then (1.2) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

Proof. First we construct the lower solution α in (1.9). Let

$$\alpha(k) = \begin{cases} c\sum_{i=1}^{k} q(i), & k \in N, \\ 0, & k = 0, \end{cases} \tag{2.31}$$

where

$$0 < c < \frac{1}{n_0 \sum_{i=1}^{T} q(i)}, \quad c\max_{k \in N} q(k) \leq c_0. \tag{2.32}$$

Then (2.7) holds, and $\alpha(0) = 0$, $\Delta\alpha(k-1) = \alpha(k) - \alpha(k-1) = cq(k) \leq c_0$ for $k \in N$ with (1.9) holding, since

$$q(k)f(k,\alpha(k)) \geq c_0 \geq \Delta\alpha(k-1) \quad \text{for } k \in N. \tag{2.33}$$

Next we discuss the initial value problem

$$\Delta u(k-1) = q(k)f^*(k,u(k)), \quad k \in N,$$
$$u(0) = \frac{1}{n_0}; \tag{2.34}$$

here

$$f^*(k,u) = \begin{cases} f\left(k, \dfrac{1}{n_0}\right), & u \leq \dfrac{1}{n_0}, \\ f(k,u), & \dfrac{1}{n_0} \leq u \leq M, \\ f(k,M), & u \geq M. \end{cases} \tag{2.35}$$

Then (2.34) is equivalent to

$$u(k) = \begin{cases} \dfrac{1}{n_0} + \displaystyle\sum_{i=1}^{k} q(i)f^*(i,u(i)), & k \in N, \\ \dfrac{1}{n_0}, & k = 0. \end{cases} \tag{2.36}$$

From Brouwer's fixed point theorem, we know that (2.34) has a solution $u \in C(N^+, \mathbb{R})$. We first show

$$u(k) \geq \frac{1}{n_0} \quad \text{for } k \in N^+. \tag{2.37}$$

Suppose (2.37) is not true. Then there exists a $\tau \in N$ such that

$$u(\tau) < \frac{1}{n_0}, \quad u(\tau-1) \geq \frac{1}{n_0} \tag{2.38}$$

since $u(0) = 1/n_0$. Thus we have, from (2.28)

$$\Delta u(\tau - 1) = q(\tau) f^*(\tau, u(\tau)) = q(\tau) f\left(\tau, \frac{1}{n_0}\right) > 0, \tag{2.39}$$

so

$$u(\tau) - \frac{1}{n_0} > u(\tau - 1) - \frac{1}{n_0} \geq 0, \tag{2.40}$$

a contradiction. Thus (2.37) is satisfied. Next we show

$$u(k) \leq M \quad \text{for } k \in N^+. \tag{2.41}$$

Suppose (2.41) is false. Then since $u(0) = 1/n_0$, there exists $\tau \in N$ such that

$$u(\tau) > M, \qquad u(k) \leq M \quad \text{for } k \in \{0, 1, \ldots, \tau - 1\}. \tag{2.42}$$

Thus, we have

$$\begin{aligned}
\Delta u(\tau - 1) &= u(\tau) - u(\tau - 1) \leq q(\tau) h(M), \\
\Delta u(\tau - 2) &= u(\tau - 1) - u(\tau - 2) \leq q(\tau - 1) h(M), \\
&\vdots \\
\Delta u(0) &= u(1) - u(0) \leq q(1) h(M).
\end{aligned} \tag{2.43}$$

Adding both sides of the above formula gives

$$u(\tau) - u(0) \leq h(M) \sum_{k=1}^{\tau} q(k) \leq h(M) \sum_{k=1}^{T} q(k), \tag{2.44}$$

that is,

$$M - \frac{1}{n_0} \leq h(M) \sum_{k=1}^{T} q(k). \tag{2.45}$$

This contradicts (2.30). Thus, we have (2.20). Let $\beta(k) \equiv u(k)$ for $k \in N^+$. By (2.7) and (2.37), we have $\alpha(k) \leq \beta(k)$ for $k \in N^+$. Then

$$\beta \in C(N^+, \mathbb{R}) \text{ with}$$

$$\beta(k) \geq \alpha(k), \quad \beta(k) > \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \tag{2.46}$$

$$q(k) f(k, \beta(k)) = \Delta \beta(k - 1) \quad \text{for } k \in N.$$

Now Theorem 1.3 guarantees that (1.2) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) \geq \alpha(k) > 0$ for $k \in N$.   □

Example 2.4. Consider the initial value problem

$$\Delta u(k-1) = k[u(k)]^{-\alpha} + [u(k)]^{\beta} - A, \quad k \in N,$$
$$u(0) = 0 \tag{2.47}$$

with $\alpha > 0$, $0 \leq \beta < 1$, and $A > 0$. Now (2.47) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

To see this we will apply Theorem 2.3 with (2.24). Let $n_0 > (2A)^{1/\alpha}$ and $c_0 = A$. Then for $k \in N$ and $0 < u \leq 1/n_0$, (2.25) holds and so (2.28) is satisfied. Let $h(u) = u^\beta + n_0^\alpha T + A$. Then (2.29) is immediate. Also since $0 \leq \beta < 1$, we see

$$\text{there exist } M > \frac{1}{n_0} \quad \text{such that } M - \frac{1}{n_0} > T(M^\beta + n_0^\alpha T + A), \tag{2.48}$$

so (2.30) holds. Theorem 2.3 guarantees that (2.47) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

Acknowledgment

References

[1] R. P. Agarwal, D. Jiang, and D. O'Regan, *A generalized upper and lower solution method for singular discrete initial value problems*, Demonstratio Math. **37** (2004), no. 1, 115–122.

[2] R. P. Agarwal, H. Lü, and D. O'Regan, *Existence theorems for the one-dimensional singular p-Laplacian equation with sign changing nonlinearities*, Appl. Math. Comput. **143** (2003), no. 1, 15–38.

[3] R. P. Agarwal and D. O'Regan, *Nonpositone discrete boundary value problems*, Nonlinear Anal. **39** (2000), no. 2, 207–215.

[4] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, *Existence criteria for singular initial value problems with sign changing nonlinearities*, Math. Probl. Eng. **7** (2001), no. 6, 503–524.

[5] R. P. Agarwal, D. O'Regan, V. Lakshmikantham, and S. Leela, *Existence of positive solutions for singular initial and boundary value problems via the classical upper and lower solution approach*, Nonlinear Anal. **50** (2002), no. 2, 215–222.

[6] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[7] P. Habets and F. Zanolin, *Upper and lower solutions for a generalized Emden-Fowler equation*, J. Math. Anal. Appl. **181** (1994), no. 3, 684–700.

[8] D. Jiang, D. O'Regan, and R. P. Agarwal, *A generalized upper and lower solution method for singular discrete boundary value problems for the one-dimensional p-Laplacian*, to appear in J. Appl. Anal.

[9] H. Lü and C. Zhong, *A note on singular nonlinear boundary value problems for the one-dimensional p-Laplacian*, Appl. Math. Lett. **14** (2001), no. 2, 189–194.

[10]   R. Manásevich and F. Zanolin, *Time-mappings and multiplicity of solutions for the one-dimensional p-Laplacian*, Nonlinear Anal. **21** (1993), no. 4, 269–291.

[11]   M. N. Nkashama, *A generalized upper and lower solutions method and multiplicity results for nonlinear first-order ordinary differential equations*, J. Math. Anal. Appl. **140** (1989), no. 2, 381–395.

Haishen Lü: Department of Applied Mathematics, Hohai University, Nanjing 210098, China
E-mail address: haishen2001@yahoo.com.cn

Donal O'Regan: Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie