MULTIPLE POSITIVE SOLUTIONS OF SINGULAR DISCRETE p-LAPLACIAN PROBLEMS VIA VARIATIONAL METHODS

RAVI P. AGARWAL, KANISHKA PERERA, AND DONAL O'REGAN

Received 31 March 2005

We obtain multiple positive solutions of singular discrete p-Laplacian problems using variational methods.

1. Introduction

We consider the boundary value problem

$$\begin{gathered} -\Delta\big(\varphi_p(\Delta u(k-1))\big) = f(k,u(k)), \quad k \in [1,n], \\ u(k) > 0, \quad k \in [1,n], \\ u(0) = 0 = u(n+1), \end{gathered} \tag{1.1}$$

where n is an integer greater than or equal to 1, $[1,n]$ is the discrete interval $\{1,\ldots,n\}$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$, and we only assume that $f \in C([1,n] \times (0,\infty))$ satisfies

$$a_0(k) \le f(k,t) \le a_1(k)t^{-\gamma}, \quad (k,t) \in [1,n] \times (0,t_0) \tag{1.2}$$

for some nontrivial functions $a_0, a_1 \ge 0$ and $\gamma, t_0 > 0$, so that it may be singular at $t = 0$ and may change sign.

Let $\lambda_1, \varphi_1 > 0$ be the first eigenvalue and eigenfunction of

$$\begin{gathered} -\Delta\big(\varphi_p(\Delta u(k-1))\big) = \lambda\varphi_p(u(k)), \quad k \in [1,n], \\ u(0) = 0 = u(n+1). \end{gathered} \tag{1.3}$$

Theorem 1.1. *If (1.2) holds and*

$$\limsup_{t\to\infty} \frac{f(k,t)}{t^{p-1}} < \lambda_1, \quad k \in [1,n], \tag{1.4}$$

then (1.1) has a solution.

Advances in Difference Equations 2005:2 (2005) 93–99
DOI: 10.1155/ADE.2005.93

Theorem 1.2. *If (1.2) holds and*

$$f(k,t_1) \le 0, \quad k \in [1,n], \tag{1.5}$$

for some $t_1 > t_0$, then (1.1) has a solution $u_1 < t_1$. If, in addition,

$$\liminf_{t\to\infty} \frac{f(k,t)}{t^{p-1}} > \lambda_1, \quad k \in [1,n], \tag{1.6}$$

then there is a second solution $u_2 > u_1$.

Example 1.3. Problem (1.1) with $f(k,t) = t^{-\gamma} + \lambda t^{\beta}$ has a solution for all $\gamma > 0$ and λ (resp., $\lambda < \lambda_1$, $\lambda \le 0$) if $\beta < p-1$ (resp., $\beta = p-1$, $\beta > p-1$) by Theorem 1.1.

Example 1.4. Problem (1.1) with $f(k,t) = t^{-\gamma} + e^t - \lambda$ has two solutions for all $\gamma > 0$ and sufficiently large $\lambda > 0$ by Theorem 1.2.

Our results seem new even for $p = 2$. Other results on discrete p-Laplacian problems can be found in [1, 2] in the nonsingular case and in [3, 4, 5, 6] in the singular case.

2. Preliminaries

First we recall the *weak comparison principle* (see, e.g., Jiang et al. [2]).

Lemma 2.1. *If*

$$\begin{gathered} -\Delta\big(\varphi_p(\Delta u(k-1))\big) \ge -\Delta\big(\varphi_p(\Delta v(k-1))\big), \quad k \in [1,n], \\ u(0) \ge v(0), \qquad u(n+1) \ge v(n+1), \end{gathered} \tag{2.1}$$

then $u \ge v$.

Next we prove a local comparison result.

Lemma 2.2. *If*

$$\begin{gathered} -\Delta\big(\varphi_p(\Delta u(k-1))\big) \ge -\Delta\big(\varphi_p(\Delta v(k-1))\big), \\ u(k) = v(k), \qquad u(k\pm 1) \ge v(k\pm 1), \end{gathered} \tag{2.2}$$

then $u(k\pm 1) = v(k\pm 1)$.

Proof. We have

$$-\varphi_p(\Delta u(k)) + \varphi_p(\Delta u(k-1)) \ge -\varphi_p(\Delta v(k)) + \varphi_p(\Delta v(k-1)), \tag{2.3}$$

$$\Delta u(k) \ge \Delta v(k), \qquad \Delta u(k-1) \le \Delta v(k-1). \tag{2.4}$$

Combining with the strict monotonicity of φ_p shows that

$$0 \le \varphi_p(\Delta u(k)) - \varphi_p(\Delta v(k)) \le \varphi_p(\Delta u(k-1)) - \varphi_p(\Delta v(k-1)) \le 0, \tag{2.5}$$

and hence, the equalities hold in (2.4). □

The following *strong comparison principle* is now immediate.

LEMMA 2.3. *If*

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) \geq -\Delta(\varphi_p(\Delta v(k-1))), \quad k \in [1,n], \\ u(0) \geq v(0), \qquad u(n+1) \geq v(n+1), \end{gathered} \tag{2.6}$$

then either $u > v$ in $[1,n]$, or $u \equiv v$. In particular, if

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) \geq 0, \quad k \in [1,n], \\ u(0) \geq 0, \qquad u(n+1) \geq 0, \end{gathered} \tag{2.7}$$

then either $u > 0$ in $[1,n]$ or $u \equiv 0$.

Consider the problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) = g(k,u(k)), \quad k \in [1,n], \\ u(0) = 0 = u(n+1), \end{gathered} \tag{2.8}$$

where $g \in C([1,n] \times \mathbb{R})$. The class W of functions $u : [0,n+1] \to \mathbb{R}$ such that $u(0) = 0 = u(n+1)$ is an n-dimensional Banach space under the norm

$$\|u\| = \left(\sum_{k=1}^{n+1} |\Delta u(k-1)|^p \right)^{1/p}. \tag{2.9}$$

Define

$$\Phi_g(u) = \sum_{k=1}^{n+1} \left[\frac{1}{p} |\Delta u(k-1)|^p - G(k,u(k)) \right], \quad u \in W, \tag{2.10}$$

where $G(k,t) = \int_0^t g(k,s)ds$. Then the functional Φ_g is C^1 with

$$\begin{aligned} (\Phi_g'(u),v) &= \sum_{k=1}^{n+1} [\varphi_p(\Delta u(k-1))\Delta v(k-1) - g(k,u(k))\, v(k)] \\ &= -\sum_{k=1}^{n} [\Delta(\varphi_p(\Delta u(k-1))) + g(k,u(k))] v(k) \end{aligned} \tag{2.11}$$

(summing by parts), so solutions of (2.8) are precisely the critical points of Φ_g.

LEMMA 2.4. *If*

$$\limsup_{|t| \to \infty} \frac{g(k,t)}{|t|^{p-2}t} < \lambda_1, \quad k \in [1,n], \tag{2.12}$$

then Φ_g has a global minimizer.

Proof. By (2.12), there is a $\lambda \in [0,\lambda_1)$ such that

$$G(k,t) \leq \frac{\lambda}{p}|t|^p + C, \tag{2.13}$$

where C denotes a generic positive constant. Since

$$\lambda_1 = \min_{u \in W\setminus\{0\}} \frac{\sum_{k=1}^{n+1} |\Delta u(k-1)|^p}{\sum_{k=1}^{n} |u(k)|^p}, \tag{2.14}$$

then

$$\Phi_g(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - C\|u\|, \tag{2.15}$$

so Φ_g is bounded from below and coercive. □

Lemma 2.5. *If*

$$\liminf_{t\to+\infty} \frac{g(k,t)}{t^{p-1}} > \lambda_1, \quad \lim_{t\to-\infty} \frac{g(k,t)}{|t|^{p-1}} = 0, \quad k \in [1,n], \tag{2.16}$$

then Φ_g satisfies the Palais-Smale compactness condition (PS): every sequence (u_j) in W such that $\Phi_g(u_j)$ is bounded and $\Phi_g'(u_j) \to 0$ has a convergent subsequence.

Proof. It suffices to show that (u_j) is bounded since W is finite dimensional, so suppose that $\rho_j := \|u_j\| \to \infty$ for some subsequence. We have

$$o(1)\|u_j^-\| = (\Phi_g'(u_j), u_j^-) \leq -\|u_j^-\|^p - \sum_{k=1}^{n+1} g(k, -u_j^-(k))u_j^-(k), \tag{2.17}$$

where $u_j^- = \max\{-u_j, 0\}$ is the negative part of u_j, so it follows from (2.16) that (u_j^-) is bounded. So, for a further subsequence, $\tilde{u}_j := u_j/\rho_j$ converges to some $\tilde{u} \geq 0$ in W with $\|\tilde{u}\| = 1$.

We may assume that for each k, either $(u_j(k))$ is bounded or $u_j(k) \to \infty$. In the former case, $\tilde{u}(k) = 0$ and $g(k,u_j(k))/\rho_j^{p-1} \to 0$, and in the latter case, $g(k,u_j(k)) \geq 0$ for large j by (2.16). So it follows from

$$o(1) = \frac{(\Phi_g'(u_j), v)}{\rho_j^{p-1}} = \sum_{k=1}^{n+1}\left[\varphi_p(\Delta\tilde{u}_j(k-1))\Delta v(k-1) - \frac{g(k,u_j(k))}{\rho_j^{p-1}}v(k)\right] \tag{2.18}$$

that

$$\sum_{k=1}^{n+1} \varphi_p(\Delta\tilde{u}(k-1))\Delta v(k-1) \geq 0 \quad \forall v \geq 0, \tag{2.19}$$

and hence, $\widetilde{u} > 0$ in $[1,n]$ by Lemma 2.3. Then $u_j(k) \to \infty$ for each k, and hence, (2.18) can be written as

$$\sum_{k=1}^{n+1} \left[\varphi_p(\Delta\widetilde{u}_j(k-1))\Delta v(k-1) - \alpha_j(k)\widetilde{u}_j(k)^{p-1}v(k)\right] = o(1), \tag{2.20}$$

where

$$\alpha_j(k) = \frac{g(k,u_j(k))}{u_j(k)^{p-1}} \geq \lambda, \quad j \text{ large}, \tag{2.21}$$

for some $\lambda > \lambda_1$ by (2.16).

Choosing v appropriately and passing to the limit shows that each $\alpha_j(k)$ converges to some $\alpha(k) \geq \lambda$ and

$$\begin{gathered} -\Delta(\varphi_p(\Delta\widetilde{u}(k-1))) = \alpha(k)\widetilde{u}(k)^{p-1}, \quad k \in [1,n], \\ \widetilde{u}(0) = 0 = \widetilde{u}(n+1). \end{gathered} \tag{2.22}$$

This implies that the first eigenvalue of the corresponding weighted eigenvalue problem is given by

$$\min_{u \in W \setminus \{0\}} \frac{\sum_{k=1}^{n+1} |\Delta u(k-1)|^p}{\sum_{k=1}^{n} \alpha(k)|u(k)|^p} = 1. \tag{2.23}$$

Then

$$1 \leq \frac{\sum_{k=1}^{n+1} |\Delta\varphi_1(k-1)|^p}{\sum_{k=1}^{n} \alpha(k)\varphi_1(k)^p} \leq \frac{\lambda_1}{\lambda} < 1, \tag{2.24}$$

a contradiction. □

3. Proofs

The problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) = a_0(k), \quad k \in [1,n], \\ u(0) = 0 = u(n+1), \end{gathered} \tag{3.1}$$

has a unique solution $u_0 > 0$ by Lemmas 2.3 and 2.4. Fix $\varepsilon \in (0,1]$ so small that $\underline{u} := \varepsilon^{1/(p-1)}u_0 < t_0$. Then

$$-\Delta(\varphi_p(\Delta\underline{u}(k-1))) - f(k,\underline{u}(k)) \leq -(1-\varepsilon)a_0(k) \leq 0 \tag{3.2}$$

by (1.2), so $\underline{u}$ is a subsolution of (1.1). Let

$$f_{\underline{u}}(k,t) = \begin{cases} f(k,t), & t \geq \underline{u}(k), \\ f(k,\underline{u}(k)), & t < \underline{u}(k). \end{cases} \tag{3.3}$$

Proof of Theorem 1.1. By (1.4), there are $\lambda \in [0,\lambda_1)$ and $T > t_0$ such that

$$f(k,t) \le \lambda t^{p-1}, \quad (k,t) \in [1,n] \times (T,\infty). \tag{3.4}$$

Then

$$f_{\underline{u}}(k,t) \begin{cases} \le a_1(k)\underline{u}(k)^{-\gamma} + \max f([1,n] \times [t_0,T]) + \lambda t^{p-1}, & t \ge 0, \\ \ge a_0(k), & t < 0, \end{cases} \tag{3.5}$$

by (1.2), so the modified problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) = f_{\underline{u}}(k,u(k)), \quad k \in [1,n], \\ u(0) = 0 = u(n+1), \end{gathered} \tag{3.6}$$

has a solution u by Lemma 2.4. By Lemma 2.1, $u \ge \underline{u}$, and hence, also a solution of (1.1). □

Proof of Theorem 1.2. Noting that t_1 is a supersolution of (3.6), let

$$\tilde{f}_{\underline{u}}(k,t) = \begin{cases} f_{\underline{u}}(k,t_1), & t > t_1, \\ f_{\underline{u}}(k,t), & t \le t_1. \end{cases} \tag{3.7}$$

By (1.2),

$$\tilde{f}_{\underline{u}}(k,t) \begin{cases} \le a_1(k)\underline{u}(k)^{-\gamma} + \max f([1,n] \times [t_0,t_1]), & t \ge 0, \\ \ge a_0(k), & t < 0, \end{cases} \tag{3.8}$$

so $\Phi_{\tilde{f}_{\underline{u}}}$ has a global minimizer u_1 by Lemma 2.4. By Lemmas 2.1 and 2.2, $\underline{u} \le u_1 < t_1$, so $\Phi_{\tilde{f}_{\underline{u}}} = \Phi_{f_{\underline{u}}}$ near u_1 and hence, u_1 is a local minimizer of $\Phi_{f_{\underline{u}}}$. Let

$$f_{u_1}(k,t) = \begin{cases} f(k,t), & t \ge u_1(k), \\ f(k,u_1(k)), & t < u_1(k). \end{cases} \tag{3.9}$$

Since u_1 is also a subsolution of (1.1), repeating the above argument with u_1 in place of $\underline{u}$, we see that $\Phi_{f_{u_1}}$ also has a local minimizer, which we assume is u_1 itself, for otherwise we are done. By (1.6), there are $\lambda > \lambda_1$ and $T > t_1$ such that

$$f(k,t) \ge \lambda t^{p-1}, \quad (k,t) \in [1,n] \times (T,\infty), \tag{3.10}$$

so

$$\Phi_{f_{u_1}}(t\varphi_1) \le -\frac{t^p}{p}\left(\frac{\lambda}{\lambda_1} - 1\right) + Ct < \Phi_{f_{u_1}}(u_1), \quad t > 0 \text{ large}. \tag{3.11}$$

Since $\Phi_{f_{u_1}}$ satisfies (PS) by Lemma 2.5, the mountain-pass lemma now gives a second critical point u_2, which is greater than u_1 by Lemmas 2.1 and 2.2. □

References

[1] R. Avery and J. Henderson, *Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian*, J. Difference Equ. Appl. **10** (2004), no. 6, 529–539.

[2] D. Jiang, J. Chu, D. O'Regan, and R. P. Agarwal, *Positive solutions for continuous and discrete boundary value problems to the one-dimension p-Laplacian*, Math. Inequal. Appl. **7** (2004), no. 4, 523–534.

[3] D. Jiang, D. O'Regan, and R. P. Agarwal, *A generalized upper and lower solution method for singular discrete boundary value problems for the one-dimensional p-Laplacian*, to appear in J. Appl. Anal.

[4] ———, *Existence theory for single and multiple solutions to singular boundary value problems for the one-dimension p-Laplacian*, Adv. Math. Sci. Appl. **13** (2003), no. 1, 179–199.

[5] D. Jiang, L. Zhang, D. O'Regan, and R. P. Agarwal, *Existence theory for single and multiple solutions to singular positone discrete Dirichlet boundary value problems to the one-dimension p-Laplacian*, Archivum Mathematicum (Brno) **40** (2004), no. 4, 367–381.

[6] D. Q. Jiang, P. Y. H. Pang, and R. P. Agarwal, *Upper and lower solutions method and a superlinear singular discrete boundary value problem*, to appear in Dynam. Systems Appl.

Ravi P. Agarwal: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
E-mail address: agarwal@fit.edu

Kanishka Perera: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
E-mail address: kperera@fit.edu

Donal O'Regan: Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie