OPTIMIZATION OF DISCRETE AND DIFFERENTIAL INCLUSIONS OF GOURSAT-DARBOUX TYPE WITH STATE CONSTRAINTS

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Necessary and sufficient conditions of optimality under the most general assumptions are deduced for the considered and for discrete approximation problems. Formulation of sufficient conditions for differential inclusions is based on proved theorems of equivalence of locally conjugate mappings.

1. Introduction

In the last decade, discrete and continuous time processes with lumped and distributed parameters found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games [1–19].

The present article is devoted to an investigation of problems of this kind with distributed parameters, where the treatment is in finite-dimensional Euclidean spaces. It can be divided conditionally into four parts.

In the first part (Section 2), a certain extremal problem is formulated for discrete inclusions of Goursat-Darboux type. For such problems we use constructions of convex and nonsmooth analysis in terms of convex upper approximations, local tents, and locally conjugate mappings for both convex and for nonconvex problems to get necessary and sufficient conditions for optimality.

In the third part (Section 4), we use difference approximations of derivatives and grid functions on a uniform grid to approximate the problem with differential inclusions of Goursat-Darboux type and to formulate a necessary and sufficient condition for optimality for the discrete approximation problem. It is obvious that such difference problems can play an important role also in computational procedures.

In the fourth part (Section 5), we are able to use results in Section 4 to get sufficient conditions for optimality for differential inclusions of Goursat-Darboux type. The derivation of this condition is implemented by passing to the formal limit as the discrete

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steps tend to zero. At the end of Section 5, the considered example shows that in known problems, the conjugate inclusion coincides with the conjugate equation which is traditionally obtained with the help of the Hamiltonian function.

Since the discrete and continuous problems posed are described by multivalued mappings, it is obvious that many problems involving optimal control of chemical engineering, sorbtion, and dissorbtion of gases can be reduced to this formulation.

2. Needed facts and problem statement

Let R^n be n-dimensional Euclidean space and let $P(R^n)$ be the set of all nonempty subsets of R^n. If $x, y \in R^n$, then (x, y) is a pair of elements x and y, and $\langle x, y\rangle$ is their scalar product. The multivalued mapping $a : R^{3n} \to P(R^n)$ is convex closed if its graph $gfa = \{(x,y,z,v) : v \in a(x,y,z)\}$ is a convex closed set in R^{4n}. It is convex-valued if $a(x,y,z)$ is a convex set for each $(x,y,z) \in \operatorname{dom} a = \{(x,y,z) : a(x,y,z) \neq \emptyset\}$.

For convex-valued mappings, the following designations are valid:

$$W_a(x,y,z,v^*) = \inf_v \{\langle v, v^*\rangle : v \in a(x,y,z)\}, \quad v \in R^n,$$
$$b(x,y,z,v^*) = \{v \in a(x,y,z) : \langle v, v^*\rangle = W_a(x,y,z,v^*)\}. \tag{2.1}$$

For convex a, we let $W_a(x,y,z,v^*) = +\infty$ if $a(x,y,z) = \emptyset$. Let $\operatorname{int} A$ be the interior of the set $A \subset R^n$ and let $\operatorname{ri} A$ be the relative interior of the set A, that is, the set of interior points of A with respect to its affine hull $\operatorname{Aff} A$.

A convex cone $K_A(x_0) := \{\overline{x} : x_0 + \lambda\overline{x} + \varphi(\lambda) \in A \text{ and } \lambda^{-1}\varphi(\lambda) \to 0 \text{ as } \lambda \downarrow 0\}$ is the cone of tangent vectors to A at $x_0 \in A$ if there exists such function $\varphi(\lambda) \in R^n$ satisfying $\lambda^{-1}\varphi(\lambda) \to 0$ as $\lambda \downarrow 0$.

A cone $K_A(x_0)$ is a local tent if for any $\overline{x}_0 \in \operatorname{ri} K_A(x_0)$ there exists a convex cone $K \subseteq K_A(x_0)$ and the continuous mapping $\Psi(\overline{x})$ defined in the neighbourhood of the origin of coordinates such that

(1) $\overline{x}_0 \in \operatorname{ri} K$, $\operatorname{Lin} K = \operatorname{Lin} K_A(x_0)$,

(2) $\psi(\overline{x}) = \overline{x} + r(\overline{x})$ and $\|\overline{x}\|^{-1} r(\overline{x}) \to 0$, as $\overline{x} \to 0$,

(3) $x_0 + \psi(\overline{x}) \in A$ if $\overline{x} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0)$ is the ball of radius ε.

For convex mapping a at point $(x,y,z,v) \in gfa$,

$$K_{gfa}(x,y,z,v) = \{(\overline{x},\overline{y},\overline{z},\overline{v}) : \overline{x} = \lambda(x_1 - x), \overline{y} = \lambda(y_1 - y), \overline{z} = \lambda(z_1 - z),$$
$$\overline{v} = \lambda(v_1 - v), \lambda > 0, (x_1,y_1,z_1,v_1) \in gfa\}. \tag{2.2}$$

Later, the cone $K_{gfa}(x,y,z,v)$ will be denoted by $K_a(x,y,z,v)$. The multivalued mapping

$$a^*(v^*;(x,y,z,v)) = \{(x^*,y^*,z^*,v^*) : (-x^*,-y^*,-z^*,v^*) \in K_a^*(x,y,z,v)\} \tag{2.3}$$

is a locally conjugate mapping (LCM) to a at point $(x,y,z,v) \in gfa$, if $K_a^*(x,y,z,v)$ is the cone dual to the cone $K_a(x,y,z,v)$,

$$K_a^*(x,y,z,v) := \{(x^*,y^*,z^*,v^*) : \langle\overline{x},x^*\rangle + \langle\overline{y},y^*\rangle + \langle\overline{z},z^*\rangle + \langle\overline{v},v^*\rangle \geq 0$$
$$\forall (\overline{x},\overline{y},\overline{z},\overline{v}) \in K_a(x,y,z,v)\}. \tag{2.4}$$

For convex mappings a [13, Theorem 2.1], it holds

$$a^*(v^*;(x,y,z,v)) = \begin{cases} \partial_{(x,y,z)} W_a(x,y,z,v^*), & v \in b(x,y,z,v^*), \\ \varnothing, & v \notin b(x,y,z,v^*), \end{cases} \tag{2.5}$$

where $\partial_{(x,y,z)} W_a(x,y,z,v^*)$ is a subdifferential of convex function $W_a(\cdot,\cdot,\cdot,v^*)$ at a given point.

According to [13], $h(\overline{x},x)$ is called a convex upper approximation (CUA) of the function $g(\cdot): R^n \to R^1 \cup \{\pm\infty\}$ at a point $x \in \operatorname{dom} g = \{x : |g(x)| < +\infty\}$ if

(1) $h(\overline{x},x) \geq F(\overline{x},x)$ for all $\overline{x} \neq 0$,

(2) $h(\overline{x},x)$ is a convex closed (or lower semicontinuous) positive homogeneous function on $\overline{x}$, and

$$F(\overline{x},x) = \sup_{r(\cdot)} \limsup_{\lambda \downarrow 0} \frac{g(x+\lambda\overline{x}+r(\lambda)) - g(x)}{\lambda}, \quad \lambda^{-1} r(\lambda) \longrightarrow 0, \text{ as } \lambda \downarrow 0. \tag{2.6}$$

Here the set

$$\partial h(0,x) = \{x^* \in R^n : h(\overline{x},x) \geq \langle \overline{x},x\rangle,\ \forall \overline{x} \in R^n\} \tag{2.7}$$

is called a subdifferential of the function g at point x and is denoted by $\partial g(x)$. For a function g, for which $F(\cdot,x)$ is a convex closed positive homogeneous function, the inclusion $\partial g(x) \supseteq \partial F(0,x)$ is fulfilled. [18, Theorem 2.2] and in case of convexity of g, the main subdifferential corresponding to the main CUA coincides with the usual definition of a subdifferential [18, Theorem 2.10]. It should be noted that for various classes of functions the notion of subdifferential can be defined in different ways [8, 18].

A function g is a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$.

Section 2 deals with the following discrete model of Goursat-Darboux type:

$$\sum_{(t,\tau) \in H_1 \times L_1} g_{t-1,\tau-1}(x_{t-1,\tau-1}) \longrightarrow \inf, \tag{2.8}$$

$$x_{t,\tau} \in a(x_{t,\tau-1}, x_{t-1,\tau}, x_{t-1,\tau-1}), \quad (t,\tau) \in H_1 \times L_1, \tag{2.9}$$

$$x_{t,\tau} \in F_{t,\tau}, \quad (t,\tau) \in H_0 \times L_0, \tag{2.10}$$

$$\begin{gathered} x_{t,0} = \alpha_t, \quad t \in H_0, \qquad x_{0,\tau} = \beta_\tau, \quad \tau \in L_0 (\alpha_0 = \beta_0), \\ H_i = \{t : t = i,\dots,T\}, \quad L_i = \{\tau : \tau = i,\dots,L\}, \quad i = 0,1, \end{gathered} \tag{2.11}$$

where $x_{t,\tau} \in R^n, F_{t,\tau} \subseteq R^n$ are some sets, $g_{t,\tau}$ are real-valued functions, $g_{t,\tau}$: $R^n \to R^1 \cup \{\pm\infty\}$, a is multivalued mapping: $a : R^{3n} \to P(R^n)$, T and L are fixed natural numbers.

Condition (2.10) is simply state constraint and (2.11) are boundary conditions. A sequence

$$\{x_{t,\tau}\}_{H_0 \times L_0} = \{x_{t,\tau} : (t,\tau) \in H_0 \times L_0\} \tag{2.12}$$

is called the admissible solution for the stated problem (2.8)–(2.11). It is evident that this sequence consists of $(T+1)(L+1)$ points of the space R^n.

The problem (2.8)–(2.11) is said to be convex if the a and $F_{t,\tau}$ are convex and the $g_{t,\tau}$ are convex proper functions.

Definition 2.1. Say that for the convex problem (2.8)–(2.11) *the nondegeneracy condition is satisfied if for points* $x^0_{t,\tau} \in R^n, (t,\tau) \in H_0 \times L_0$ *one of the following cases is fulfilled*:

$$\begin{aligned}
&\text{(i) } (x^0_{t,\tau-1}, x^0_{t-1,\tau}, x^0_{t-1,\tau-1}, x^0_{t,\tau}) \in \operatorname{ri} gf a,\\
&\qquad (t,\tau) \in H_1 \times L_1, x^0_{t,\tau-1} \in \operatorname{ri}(F_{t,\tau} \cap \operatorname{dom} g_{t,\tau}), \quad (t,\tau) \in H_0 \times L_0,\\
&\text{(ii) } (x^0_{t,\tau-1}, x^0_{t-1,\tau}, x^0_{t-1,\tau-1}, x^0_{t,\tau}) \in \operatorname{int}(gf F_{t,\tau} \cap \operatorname{dom} g_{t,\tau}),\\
&\qquad\qquad (t,\tau) \in H_0 \times L_0,\ x^0_{t,\tau-1} \in \operatorname{int} x^*
\end{aligned} \tag{2.13}$$

and $g_{t,\tau}$ *are continuous at points* $x^0_{t,\tau}$, where (t_0, τ_0) is a fixed pair.

Condition 2.2. Suppose that in the problem (2.8)–(2.11) the mapping a and the sets $F_{t,\tau}$, $(t,\tau) \in H_0 \times L_0$ are such that the cones of tangent directions $K_{gfa}(\widetilde{x}_{t,\tau-1}, \widetilde{x}_{t-1,\tau}, \widetilde{x}_{t-1,\tau-1}, \widetilde{x}_{t,\tau})$ and $K_{F_{t,\tau}}(\widetilde{x}_{t,\tau})$ are local tents, where $\widetilde{x}_{t,\tau}$ are the points of the optimal solution $\{\widetilde{x}_{t,\tau}\}_{H_0 \times L_0}$. Suppose, moreover, that the functions $g_{t,\tau}$ admit a CUA $h_{t,\tau}(\overline{x}, \widetilde{x}_{t,\tau})$ at the points $\widetilde{x}_{t,\tau}$ that are continuous with respect to $\overline{x}$. The latter means that the subdifferentials $\partial g_{t,\tau}(\widetilde{x}_{t,\tau}) = \partial h_{t,\tau}(0, \widetilde{x}_{t,\tau})$ are defined.

In Section 4, we study the convex problem for differential indusions of Goursat-Darboux type:

$$I(x(\cdot,\cdot)) = \iint_Q g(x(t,\tau),t,\tau)\,dt\,d\tau \longrightarrow \inf, \tag{2.14}$$

$$x''_{t\tau}(t,\tau) \in a(x(t,\tau)), \qquad (t,\tau) \in Q = [0,1] \times [0,1], \tag{2.15}$$

$$x(t,\tau) \in F(t,\tau), \tag{2.16}$$

$$x(t,0) = \alpha(t), \qquad x(0,\tau) = \beta(\tau), \qquad \alpha(0) = \beta(0). \tag{2.17}$$

Here $a: R^n \to P(R^n)$ is a convex multivalued mapping, F is convex-valued mapping, $F: Q \to P(R^n)$, g is continuous and convex with respect to $x, g: R^n \times Q \to R^1$, and α, β are absolutely continuous functions, $\alpha: [0,1] \to R^n$, $\beta: [0,1] \to R^n$. The problem is to find a solution $\widetilde{x}(t,\tau)$ of the boundary value problem (2.15)–(2.17) that minimizes $I(x(\cdot,\cdot))$.

Here an admissible solution is understood to be an absolutely continuous function defined on Q with an integrable derivative $x''_{t\tau}(\cdot,\cdot)$ satisfying (2.15) almost everywhere (a.e.) on Q and satisfying the state constraints (2.16) on Q, and boundary conditions (2.17) on [0,1].

It is known that system (2.15) is often regarded as a continuous analog of the discrete Fornosini-Marchesini [7] model which plays an essential role in the theory of automatic control of systems with two independent variables [9].

3. Necessary and sufficient conditions for discrete inclusions

At first we consider the convex problem (2.8)–(2.11). We have the following.

THEOREM 3.1. *Let a and $F_{t,\tau},(t,\tau)\in H_0\times L_0$ be convex and convex-valued mappings, respectively, $g_{t,\tau}$ continuous at the points of some admissible solution $\{x^0_{t,\tau}\}_{H_0\times L_0}$. Then in order for the function (2.8) to attain the least possible value on the solution $\{\widetilde{x}_{t,\tau}\}_{H_0\times L_0}$ with boundary conditions (2.11) among all admissible solutions it is necessary that there exist a number $\lambda=0$ or 1 and vectors $\{x^*_{t,\tau}\},\{\varphi^*_{t,\tau+1}\},\{\eta^*_{t+1,\tau}\}(x^*_{0,0}=\eta^*_{T+1,L}=\varphi^*_{T,L+1}=0)\,(t,\tau)\in H_0\times L_0$ simultaneously, not all zero, such that*

$$\begin{aligned}
&(1)\ (\varphi^*_{t,\tau},\eta^*_{t,\tau},x^*_{t-1,\tau-1})\in a^*\left(x_{t,\tau};(\widetilde{x}_{t,\tau-1},\widetilde{x}_{t-1,\tau},\widetilde{x}_{t-1,\tau-1},\widetilde{x}_{t,\tau})\right)+\{0\}\times\{0\}\\
&\qquad\qquad\times\left\{\lambda\partial g_{t-1,\tau-1}(\widetilde{x}_{t-1,\tau-1})-K^*_{F_{t-1,\tau-1}}(\widetilde{x}_{t-1,\tau-1})+\varphi^*_{t-1,\tau}+\eta^*_{t,\tau-1}\right\},\\
&(2)\ \varphi^*_{T,\tau+1}-x^*_{T,\tau}\in K^*_{F_{T,\tau}}(\widetilde{x}_{T,\tau}),\quad \tau\in L_0,\quad \eta^*_{t+1,L}-x^*_{t,L}\in K^*_{F_{t,L}}(\widetilde{x}_{t,L}),\quad t\in H_0.
\end{aligned}\tag{3.1}$$

And if the condition of nondegeneracy is satisfied these conditions are sufficient for the optimality of the solution $\{\widetilde{x}_{t,\tau}\}_{H_0\times L_0}$.

Proof. We construct for each $t\in H_0$ an $m=n(L+1)$ dimensional vector $x_t=(x_{t,0},\dots,x_{t,L})\in R^m$. We assume that $w=(x_0,\dots,x_T)\in R^{m(T+1)}$. Define in the space $R^{m(T+1)}$ the following convex sets:

$$\begin{aligned}
M_{t,\tau}&=\left\{w=(x_0,\dots,x_T):(x_{t,\tau-1},x_{t-1,\tau},x_{t-1,\tau-1},x_{t,\tau})\in gfa\right\},\quad (t,\tau)\in H_1\times L_1,\\
Q_{t,\tau}&=\left\{w=(x_0,\dots,x_T):x_{t,\tau}\in F_{t,\tau}\right\},\quad (t,\tau)\in H_0\times L_0,\\
N_1&=\left\{w=(x_0,\dots,x_T):x_{t,0}=\alpha_t,\ t\in H_0\right\},\\
N_2&=\left\{w=(x_0,\dots,x_T):x_{0,\tau}=\beta_\tau,\ T\in L_0\right\}.
\end{aligned}\tag{3.2}$$

Let

$$g(w)=\sum_{\substack{t=0,\dots,T-1\\ \tau=1,\dots,L-1}}g_{t,\tau}(x_{t,\tau}).\tag{3.3}$$

It can easily be seen that our basic problem (2.8)–(2.11) is equivalent to the following one:

$$g(w)\longrightarrow\inf,\quad w\in P,\tag{3.4}$$

where

$$P=\left(\bigcap_{(t,\tau)\in H_1\times L_1}M_{t,\tau}\right)\cap\left(\bigcap_{(t,\tau)\in H_0\times L_0}Q_{t,\tau}\right)\cap N_1\cap N_2\tag{3.5}$$

is a convex set.

Further, by the hypothesis of the theorem, $\{\widetilde{x}_{t,\tau}\}_{H_0\times L_0}$ is an optimal solution, consequently, $\widetilde{w}=(\widetilde{x}_0,\dots,\widetilde{x}_T)$ is a solution of the problem (3.4). Apply [18, Theorem 2.4] to the problem (3.4). By this theorem there exist such vectors

$$\begin{gathered} w^*(t,\tau) = \big(x_0^*(t,\tau),\dots,x_T^*(t,\tau)\big), \\ w^*(t,\tau) \in K^*_{M_{t,\tau}}, \quad (t,\tau) \in H_1 \times L_1, \quad x_t^*(t,\tau) = \big(x_{t,0}^*(t,\tau),\dots,x_{t,L}^*(t,\tau)\big), \quad t \in H_0, \\ w^* \in K^*_{N_1}(\widetilde{w}), \quad \widetilde{w}^* \in K^*_{N_2}(\widetilde{w}); \quad \overline{w}^*(t,\tau) \in K^*_{F_{t,\tau}}(\widetilde{w}), \quad (t,\tau) \in H_0 \times L_0, \end{gathered} \tag{3.6}$$

$w^{0*} \in \partial_w g(\widetilde{w})$, and the number $\lambda = 0$ or 1, such that

$$\lambda w^{0*} = \sum_{(t,\tau)\in H_1\times L_1} w^*(t,\tau) + \sum_{(t,\tau)\in H_0\times L_0} \overline{w}^*(t,\tau) + w^* + \widetilde{w}^*, \tag{3.7}$$

where the given vectors and the number λ are not simultaneously equal to zero.

Here the indicated dual cones can be calculated easily; by elementary computations we find that

$$\begin{gathered} K^*_{M_{t,\tau}}(w) = \{w^* = (x_0^*,\dots,x_T^*) : (x^*_{t,\tau-1}, x^*_{t-1,\tau}, x^*_{t-1,\tau-1}, x^*_{t,\tau}) \in K^*_{a_{t,\tau}}(x_{t,\tau-1}, x_{t-1,\tau}, x_{t-1,\tau-1}, x_{t,\tau}), \\ x^*_{i,j} = 0,\, i \neq t, t-1,\, j \neq \tau, \tau-1\}, \quad (t,\tau) \in H_1 \times L_1. \end{gathered} \tag{3.8}$$

Then, using the definition of an LCM, new notations

$$x^*_{t,\tau}(t+1,\tau) = -\eta^*_{t+1,\tau}, x^*_{t,\tau}(t,\tau+1) = -\varphi^*_{t,\tau+1}, x^*_{t,\tau}(t,\tau) = x^*_{t,\tau}, \tag{3.9}$$

and componentwise representation of (3.7) we can obtain the required first part of the theorem [13]. As for the sufficiency of the conditions obtained, it is clear that by [18, Theorem 3.10] under the nondegeneracy condition, the representation (3.7) holds with parameter $\lambda = 1$ for the point $w^{0*} \in \partial_w g(\widetilde{w}) \cap K_P^*(\widetilde{w})$. □

Theorem 3.2. *Assume that Condition 2.2 for the problem (2.8)–(2.11) holds. Then for $\{\widetilde{x}_{t,\tau}\}_{H_0\times L_0}$ to be a solution of this nonconvex problem it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{x^*_{t,\tau}\}$, $\{\varphi^*_{t,\tau}\}$, $\{\eta^*_{t,\tau}\}$, not all zero, satisfying conditions (1) and (2) of Theorem 3.1.*

Proof. In this case Condition 2.2 ensures the conditions of [18, Theorem 4.2, page 243] for the problem (3.4). Therefore, according to this theorem, we get the necessary condition as in Theorem 3.1 by starting from the relation (3.7), written out for the nonconvex problem. □

Remark 3.3. Let $g_{t,\tau}$ and $W_a(\cdot,\cdot,\cdot,v^*)$ be continuously differentiable functions. Then by virtue of [18, Theorem 2.1] the component-by-component representation of the inclusions (1) and (2) makes it possible to obtain a support principle from the conditions of the theorem.

Remark 3.4. It is seen from the proof of the theorem that if the consideration is carried out in a separable locally convex topological space and the designation $\langle w^*, w\rangle$ is understood as the action of a linear continuous functional w^* on the element w, then from the item (ii) of the condition of nondegeneracy and from the assertion (ii) of the condition of nondegeneracy, and from the assertion (ii) of Section 1 it is easy to conclude that the theorem is valid in this general case too.

4. Approximation of the continuous problem and sufficient conditions for optimality for differential inclusions of Goursat-Darboux type

Let δ and h be steps on the t-and τ-axes, respectively, and $x(t,\tau) = x_{\delta h}(t,\tau)$ are grid functions on a uniform grid on Q. We introduce the following difference operator, defined on the four-point models [20]:

$$Ax(t+\delta,\tau+h) = \frac{1}{\delta h}\left[x(t+\delta,\tau+h) - x(t+\delta,\tau) - x(t,\tau+h) + x(t,\tau)\right], \\ t = 0,\delta,\dots,1-\delta,\ \tau = 0,h,\dots,1-h. \tag{4.1}$$

With the problem (2.15)–(2.17) we now associate the following difference boundary value problem approximating it:

$$I_{\delta h}(x(\cdot,\cdot)) = \sum_{\substack{t=0,\dots,1-\delta \\ \tau=0,\dots,1-h}} \delta h g(x(t,\tau),t,\tau) \longrightarrow \inf, \tag{4.2}$$

$$Ax(t+\delta,\tau+h) \in a(x(t,\tau)), \quad t = 0,\dots,1-\delta,\ \tau = 0,\dots,1-h, \\ x(t,\tau) \in F(t,\tau), \quad x(t,0) = \alpha(t), \quad x(0,\tau) = \beta(\tau), \quad t = 0,\delta,\dots,1,\ \tau = 0,h,\dots,1. \tag{4.3}$$

We reduce the problem (4.2) and (4.3) to a problem of the form (2.8)–(2.11). To do this we introduce a new mapping

$$\widetilde{a}(x,y,z) = x + y - z + \delta h a(z) \tag{4.4}$$

and we rewrite the problem (4.2), (4.3) as follows:

$$I_{\delta h}(x(\cdot,\cdot)) \longrightarrow \inf, \\ x(t+\delta,\tau+h) \in \widetilde{a}(x(t+\delta,\tau),x(t,\tau+h),x(t,\tau)), \quad t = 0,\delta,\dots,1,\ \tau = 0,h,\dots,1. \tag{4.5}$$

By Theorem 3.1 for optimality of the solution $\{\widetilde{x}(t,\tau)\}$, $t = 0,\delta,\dots,1$, $\tau = 0,h,\dots,1$, in problem (4.5) it is necessary that there exist vectors $\{\eta^*(t,\tau)\}$, $\{\varphi^*(t,\tau)\}$, $\{x^*(t,\tau)\}$, and a number $\lambda = \lambda_{\delta h} \in \{0,1\}$, not all zero, such that

$$\left(\varphi^*(t+\delta,\tau+h),\eta^*(t+\delta,\tau+h),x^*(t,\tau) - \varphi^*(t,\tau+h) - \eta^*(t+\delta,\tau)\right) \\ \in \widetilde{a}^*\left(x^*(t+\delta,\tau+h);\left(\widetilde{x}(t+\delta,\tau),\widetilde{x}(t,\tau+h),\widetilde{x}(t+\delta,\tau+h)\right)\right) \\ + \{0\} \times \{0\} \times \left\{\delta h \lambda_{\delta h} \partial g(\widetilde{x}(t,\tau),t,\tau) - K^*_{F(t,\tau)}(\widetilde{x}(t,\tau))\right\}, \tag{4.6}$$

$$\varphi^*(1,\tau+h) - x^*(1,\tau) \in K^*_{F(1,T)}(\widetilde{x}(1,\tau)), \qquad \eta^*(t+\delta,1) - x^*(t,1) \in K^*_{F(t,1)}(\widetilde{x}(t,1)), \\ x^*(0,0) = \eta^*(1+\delta,1) = \varphi^*(1,1+h) = 0, \quad t = 0,\delta,\dots,1-\delta, \quad \tau = 0,h,\dots,1-h. \tag{4.7}$$

In (4.6) $\widetilde{a}^*$ must be expressed in terms of a^*.

Theorem 4.1. *If a is a convex multivalued mapping, then the following inclusions are equivalent:*

$$\begin{aligned}&(1)\ (x^*,y^*,z^*)\in\widetilde{a}^*(v^*;(x,y,z,v)),\quad v\in\widetilde{b}(x,y,z,v^*),\\&(2)\ \frac{z^*+v^*}{\delta h}\in a^*(v^*;(z,v)),\quad \frac{v-x-y+z}{\delta h}\in b(z,v^*),\quad v^*\in R^n,\end{aligned}\tag{4.8}$$

where $x^*=y^*=v^*$.

Proof. It is easy to see that

$$W_{\widetilde{a}}(x,y,z,v^*)=\delta h W_a(z,v^*)+\langle x+y-z,v^*\rangle.\tag{4.9}$$

Then using the Moreau-Rockafellar theorem [5, 8, 18, 19] we get from (4.9),

$$\partial W_{\widetilde{a}}(x,y,z,v^*)=(v^*,v^*)\times\{\delta h\partial W_a(z,v^*)-v^*\}.\tag{4.10}$$

And by formula (2.5),

$$\begin{aligned}&\widetilde{a}^*(v^*(x,y,z,v))\\&\quad=(v^*,v^*)\times\{\delta h\partial W_a(z,v^*)-v^*\},\quad v\in\widetilde{b}(x,y,z,v^*),\quad \frac{v-x-y+z}{\delta h}\in a(z,v^*).\end{aligned}\tag{4.11}$$

Thus, the inclusions $(z^*+v^*)/\delta h\in a^*(v^*;(z,v))$, and $(x^*,y^*,z^*)\in\widetilde{a}^*(v^*;(x,y,z,v))$, and $(x^*,y^*,z^*)\in\widetilde{a}^*(v^*;(x,y,z,v))$, $x^*=y^*=v^*$ are equivalent.

If the problem (2.14)–(2.17) is nonconvex and consequently the mapping a is nonconvex we can establish the equivalence of the inclusions in Theorem 4.1 by using the definition of a local tent. □

Theorem 4.2. *Suppose that the convex-valued mapping $\widetilde{a}:R^{3n}\to P(R^n)$ is such that the cones $K_{\widetilde{a}}(x,y,z,v)$, $(x,y,z,v)\in gf\widetilde{a}$ of tangent directions determine a local tent. Then the inclusions (1), (2) of Theorem 4.1 are equivalent.*

Proof. By the definition of a local tent, there exist functions $r_i(\overline{u}),\overline{u}=(\overline{x},\overline{y},\overline{z},\overline{v})$ such that $r_i(\overline{u})\|\overline{u}\|^{-1}\to 0\,(i=1,2,3)$ and $r(\overline{u})\|\overline{u}\|^{-1}\to 0$ as $\overline{u}\to 0$, and

$$v+\overline{v}+r(\overline{u})\in x+\overline{x}+r_1(\overline{u})+y+\overline{y}+r_2(\overline{u})-z-\overline{z}-r_3(\overline{u})+\delta h a(z+\overline{z}+r_3(\overline{u}))\tag{4.12}$$

for sufficiently small $\overline{u}\in K$, where $K\subseteq \mathrm{ri}\,K_{\widetilde{a}}(x,y,z,v)$ is a convex cone.

Transforming this inclusion we can write

$$\frac{v-x-y+z}{\delta h}+\frac{\overline{v}-\overline{x}-\overline{y}+\overline{z}}{\delta h}+\frac{r(\overline{u})-r_1(\overline{u})-r_2(\overline{u})+r_3(\overline{u})}{\delta h}\in a(z+\overline{z}+r_3(\overline{u})).\tag{4.13}$$

Here it is not hard to see that the cone $K_a(z,(v-x-y+z)/\delta h)$ is a local tent of $gf\,a$, and

$$\left(\overline{z},\frac{\overline{v}-\overline{x}-\overline{y}+\overline{z}}{\delta h}\right)\in K_a\left(\overline{z},\frac{v-x-y+z}{\delta h}\right).\tag{4.14}$$

By going in the reverse direction, it is clear to see from (4.14) that

$$(\overline{x},\overline{y},\overline{z},\overline{v}) \in K_{\widetilde{a}}(x,y,z,v). \tag{4.15}$$

This means that (4.14) and (4.15) are equivalent. Suppose now that

$$(x^*,y^*,z^*) \in \widetilde{a}^*(v^*;(x,y,z,v)) \tag{4.16}$$

or, what is the same,

$$\begin{gathered} -\langle\overline{x},x^*\rangle - \langle\overline{y},y^*\rangle - \langle\overline{z},z^*\rangle + \langle\overline{v},v^*\rangle \geq 0, \\ (\overline{x},\overline{y},\overline{z},\overline{v}) \in K_{\widetilde{a}}(x,y,z,v). \end{gathered} \tag{4.17}$$

Let us consider the relation

$$-\langle\overline{z},\psi_0^*\rangle + \left\langle\frac{\overline{v}-\overline{x}-\overline{y}+z}{\delta h},\psi^*\right\rangle \geq 0, \qquad \left(\overline{z},\frac{\overline{v}-\overline{x}-\overline{y}+\overline{z}}{\delta h}\right) \in K_a\left(\overline{z},\frac{v-x-y+z}{\delta h}\right). \tag{4.18}$$

By the definition of LAM it means that $\psi_0^* \in a^*(\psi^*;(z,v))$, where ψ_0^*, ψ^* are to be determined.

Carrying out the necessary transformations in (4.18) we have

$$-\langle\overline{x},\psi^*\rangle - \langle\overline{y},\psi^*\rangle - \langle\overline{z},\delta h\psi_0^* - \psi^*\rangle + \langle\overline{v},\psi^*\rangle \geq 0. \tag{4.19}$$

Then comparing this inequality with (4.17) we observe that

$$\psi_0^* = \frac{x^*+v^*}{\delta h}, \qquad \psi^* = x^* = y^* = v^*. \tag{4.20}$$

Then it follows from the equivalence of (4.14) and (4.15) that

$$\frac{z^*+v^*}{\delta h} \in a^*(v^*;(z,v)). \tag{4.21}$$

On the other hand it is not hard to see that

$$\widetilde{a}^*(v^*;(x,y,z,v)) \neq \varnothing, \quad v \in \widetilde{b}(x,y,z,v^*), \quad a^*(v^*;(z,v)) \neq \varnothing, \quad \frac{v-x+y+z}{\delta h} \in b(z,v^*). \tag{4.22}$$

The theorem is proved. □

Let us return to conditions (4.6), (4.7). By Theorem 4.1 condition (4.6) for convex problem takes the form

$$\begin{aligned} &\frac{x^*(t+\delta,\tau+h)+x^*(t,\tau)-\varphi^*(t,\tau+h)-\eta^*(t+\delta,\tau)}{\delta h} \\ &\quad \in a^*(x^*(t+\delta,\tau+h);(\widetilde{x}(t,\tau),A\widetilde{x}(t+\delta,\tau+h))) + \lambda_{\delta h}\partial g(\widetilde{x}(t,\tau),t,\tau) - K^*_{F_{(t,\tau)}}(\widetilde{x}(t,\tau)), \end{aligned} \tag{4.23}$$

and condition (4.7) can be rewritten as follows:

$$\begin{aligned} \frac{\varphi^*(1,\tau+h)-x^*(1,\tau)}{h} &\in K^*_{F(1,\tau)}(\widetilde{x}(1,\tau)), \\ \frac{\eta^*(t+\delta,1)-x^*(t,1)}{\delta} &\in K^*_{F(t,1)}(\widetilde{x}(t,1)), \end{aligned} \tag{4.24}$$

$$\begin{aligned} Ax^*(t+\delta,\tau+h) \in a^*\left(x^*(t+\delta,\tau+h);(\widetilde{x}(t,\tau),A\widetilde{x}(t+\delta,\tau+h))\right) \\ +\lambda_{\delta h}\partial g(\widetilde{x}(t,\tau),t,\tau) - K^*_{F_{(t,\tau)}}(\widetilde{x}(t,\tau)), \end{aligned} \tag{4.25}$$

$$\begin{aligned} \frac{x^*(1,\tau+h)-x^*(1,\tau)}{h} &\in K^*_{F(1,\tau)}(\widetilde{x}(1,\tau)), \\ \frac{x^*(t+\delta,1)-x^*(t,1)}{\delta} &\in K^*_{F(t,1)}(\widetilde{x}(t,1)), \\ x^*(0,0)=x^*(1+\delta,1)&=x^*(1,1+h)=0. \end{aligned} \tag{4.26}$$

Remark 4.3. In (4.24) it is taken into account that for real number $\mu>0$ $K^*_{F(1,\tau)}=\mu K^*_{F(1,\tau)}$ and $K^*_{F(t,1)}=\mu K^*_{F(t,1)}$.

We formulate the result just obtained as the following theorem.

THEOREM 4.4. *Suppose that a is convex, and g is a proper function convex with respect to x and continuous at the points of some admissible solution $\{x^0(t,\tau)\}$, $t=0,\delta,\dots,1$, $\tau=0,h,\dots,1$. Then for the optimality of the solution $\{\widetilde{x}(t,\tau)\}$ in the discrete approximation problem (4.2), (4.3) with state constraints it is necessary that there exist a number $\lambda=\lambda_{\delta h}=0$ or 1 and vectors $\{x^*(t,\tau)\}$, not all zero, satisfying (4.25), (4.26). And under the nondegeneracy condition, (4.25)-(4.26) are also sufficient for the optimality of $\{\widetilde{x}(t,\tau)\}$.*

Analogously, using Theorem 4.2 we have the following theorem.

THEOREM 4.5. *Suppose that Condition 2.2 is satisfied for the nonconvex problem. Then for $\{\widetilde{x}(t,\tau)\}$ to be a solution of this problem it is necessary that there exist a number $\lambda=0$ or 1 and vectors $\{x^*(t,\tau)\}$, not all zero, satisfying (4.23), (4.26) for nonconvex case.*

5. Sufficient conditions for optimality for differential inclusions of Goursat-Darboux type

Using results in Section 3, we formulate a sufficient condition for optimality for the continuous problem (2.14)–(2.17). Setting $\lambda_{\delta h}=1$ and passing to the formal limit in (4.23), (4.24) as δ and h tend to 0, we find that

$$\begin{aligned} &\text{(i) } x^{*''}_{t\tau}(t,\tau)\in a^*\left(x^*(t,\tau);(\widetilde{x}(t,\tau),\widetilde{x}''_{t\tau}(t,\tau)\right)+\partial g(\widetilde{x}(t,\tau)),t,\tau)-K^*_{F(t,\tau)}(\widetilde{x}(t,\tau)), \\ &\text{(ii) } x^{*'}_t(1,\tau)\in K^*_{F(1,\tau)}(\widetilde{x}(1,\tau)), \quad x^{*'}_t(t,1)\in K^*_{F(t,1)}(\widetilde{x}(t,1)), \quad x^*(0,0)=x^*(1,1)=0. \end{aligned} \tag{5.1}$$

Along with this we get one more condition ensuring that the LCM a^* is nonempty (see (2.5)),

$$\text{(iii) } \widetilde{x}''_{t\tau}(t,\tau)\in b(\widetilde{x}(t,\tau),x^*(t,\tau)). \tag{5.2}$$

The arguments in Section 3 suggest the sufficiency of conditions (i)–(iii) for optimality. It turns out that the following assertion is true.

Theorem 5.1. *Suppose that $g : R^n \times Q \to R^1$ is continuous and convex with respect to x, and a is a convex mapping. Moreover $F : Q \to P(R^n)$ is a convex-valued mapping. Then for the optimality of the solution $\widetilde{x}(t,\tau)$ among all admissible solutions of the problem (2.14)–(2.17) it is sufficient that there exists an absolutely continuous function $x^*(t,\tau)$ with an integrable mixed partial derivative and satisfying a.e. conditions (i)–(iii).*

Proof. By formula (2.5),

$$a^*(v^*;(z,v)) = \partial_z W_a(z,v^*), \quad v \in b(z,v^*). \tag{5.3}$$

Then by using the Moreau-Rockafellar theorem [5, 8, 18, 19] from condition (i) we obtain the differential inclusion

$$\begin{gathered} x^{*''}_{t\tau}(t,\tau) + u^*(t,\tau) \in \partial_x\left[W_a(\widetilde{x}(t,\tau),x^*(t,\tau)) + g(\widetilde{x}(t,\tau),t,\tau)\right], \\ u^*(t,\tau) \in K^*_{F(t,\tau)}(\widetilde{x}(t,\tau)), \quad (t,\tau) \in Q. \end{gathered} \tag{5.4}$$

Using the definition of W_a we rewrite the last relation in the form:

$$\begin{aligned} &\langle x''_{t\tau}(t,\tau) - \widetilde{x}''_{t\tau}(t,\tau), x^*(t,\tau)\rangle + g(x(t,\tau),t,\tau) - g(\widetilde{x}(t,\tau),t,\tau) \\ &\quad \geq \langle x^{*''}_{t\tau}(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle + \langle u^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle. \end{aligned} \tag{5.5}$$

On the other hand by the definition of a dual cone from $u^*(t,\tau) \in K^*_{F(t,\tau)}(\widetilde{x}(t,\tau))$ it follows that $\langle u^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle \geq 0$ for all admissible solutions $x(t,\tau) \in F(t,\tau)$ or

$$\langle u^*(t,\tau), x(t,\tau)\rangle = \inf_{x(t,\tau)\in F(t,\tau)} \langle u^*(t,\tau), x(t,\tau)\rangle, \quad (t,\tau) \in Q. \tag{5.6}$$

Therefore

$$\begin{aligned} &g(x(t,\tau),t,\tau) - g(\widetilde{x}(t,\tau),t,\tau) \\ &\quad \geq \langle x^{*''}_{t\tau}(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle + \langle \widetilde{x}''_{t\tau}(t,\tau) - x''_{t\tau}(t,\tau), x^*(t,\tau)\rangle, \quad (t,\tau) \in Q. \end{aligned} \tag{5.7}$$

Integrating this relation we get

$$\begin{aligned} &\iint_Q \left[g(x(t,\tau),t,\tau) - g(\widetilde{x}(t,\tau),t,\tau)\right] dt\, d\tau \\ &\quad \geq \iint_Q \langle x^{*''}_{t\tau}(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau)\rangle\, dt\, d\tau + \iint_Q \langle \widetilde{x}''_{t\tau}(t,\tau) - x''_{t\tau}(t,\tau), x^*(t,\tau)\rangle\, dt\, d\tau. \end{aligned} \tag{5.8}$$

Now, after simple transformations of first double integral over Q of right-hand side of inequality (5.8) using equality of mixed partial derivatives of $x^*(t,\tau)$ we can write

$$\begin{aligned}\iint_Q &\left\langle \frac{\partial^2 x^*(t,\tau)}{\partial t \partial \tau}, x(t,\tau) - \tilde{x}(t,\tau) \right\rangle dt\, d\tau \\ &= \int_0^1 \left\langle \frac{\partial x^*(t,1)}{\partial t}, x(t,1) - \tilde{x}(t,1) \right\rangle dt - \int_0^1 \left\langle \frac{\partial x^*(t,0)}{\partial t}, x(t,0) - \tilde{x}(t,0) \right\rangle dt \\ &\quad + \iint_Q \left\langle \frac{\partial x^*(t,\tau)}{\partial t}, \frac{\partial}{\partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)) \right\rangle dt\, d\tau\end{aligned} \tag{5.9}$$

and similarly

$$\begin{aligned}\iint_Q &\left\langle \frac{\partial^2}{\partial t \partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)), x^*(t,\tau) \right\rangle dt\, d\tau \\ &= \int_0^1 \left\langle \frac{\partial}{\partial \tau}(\tilde{x}(1,\tau) - x(1,\tau)), x^*(1,\tau) \right\rangle d\tau - \int_0^1 \left\langle \frac{\partial}{\partial \tau}(\tilde{x}(0,\tau) - x(0,\tau)), x^*(0,\tau) \right\rangle d\tau \\ &\quad - \iint_Q \left\langle \frac{\partial}{\partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)), \frac{\partial x^*(t,\tau)}{\partial t} \right\rangle dt\, d\tau.\end{aligned} \tag{5.10}$$

So with use of the boundary conditions (2.17) the relations (5.9), (5.10) can be written in the form

$$\begin{aligned}\iint_Q &\left\langle \frac{\partial^2 x^*(t,\tau)}{\partial t \partial \tau}, x(t,\tau) - \tilde{x}(t,\tau) \right\rangle dt\, d\tau \\ &= \int_0^1 \left\langle \frac{\partial x^*(t,1)}{\partial t}, x(t,1) - \tilde{x}(t,1) \right\rangle dt + \iint_Q \left\langle \frac{\partial x^*(t,\tau)}{\partial t}, \frac{\partial}{\partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)) \right\rangle d\tau \\ &\quad - \int_0^1 \left\langle \frac{\partial x^*(t,0)}{\partial t}, x(t,0) - \tilde{x}(t,0) \right\rangle dt - \int_0^1 \left\langle x^*(t,0), \frac{\partial}{\partial t}(x(t,0) - \tilde{x}(t,0)) \right\rangle dt,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\iint_Q &\left\langle \frac{\partial^2}{\partial t \partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)), x^*(t,\tau) \right\rangle dt\, d\tau \\ &= \int_0^1 \left\langle \frac{\partial x^*(1,\tau)}{\partial \tau}, x(1,\tau) - \tilde{x}(1,\tau) \right\rangle d\tau - \iint_Q \left\langle \frac{\partial}{\partial \tau}(\tilde{x}(t,\tau) - x(t,\tau)), \frac{\partial x^*(t,\tau)}{\partial t} \right\rangle dt\, d\tau \\ &\quad + \int_0^1 \left\langle \frac{\partial x^*(1,\tau)}{\partial \tau}, \tilde{x}(1,\tau) - x(1,\tau) \right\rangle d\tau + \int_0^1 \left\langle x^*(1,\tau), \frac{\partial}{\partial \tau}(\tilde{x}(1,\tau) - x(1,\tau)) \right\rangle d\tau,\end{aligned} \tag{5.12}$$

respectively.

Adding the equalities (5.11) and (5.12) and using the conditions (ii) of Theorem 5.1 we get

$$
\begin{aligned}
&\iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial t\partial\tau}, x(t,\tau)-\widetilde{x}(t,\tau)\right\rangle dt\,d\tau + \iint_Q \left\langle \frac{\partial^2}{\partial t\partial\tau}(\widetilde{x}(t,\tau)-x(t,\tau)), x^*(t,\tau)\right\rangle dt\,d\tau \\
&\quad = \int_0^1 \left\langle \frac{\partial x^*(1,\tau)}{\partial\tau}, x(1,\tau)-\widetilde{x}(1,\tau)\right\rangle d\tau + \int_0^1 d_\tau\left[\langle x^*(1,\tau),\widetilde{x}(1,\tau)-x(1,\tau)\rangle\right] \\
&\qquad + \int_0^1 \left\langle \frac{\partial x^*(t,1)}{\partial t}, x(t,1)-\widetilde{x}(t,1)\right\rangle dt - \int_0^1 d_t\left[\langle x^*(t,0),x(t,0)-\widetilde{x}(t,0)\rangle\right] \\
&\quad \geq \int_0^1 d_\tau\left[\langle x^*(1,\tau),\widetilde{x}(1,\tau)-x(1,\tau)\rangle\right] - \int_0^1 d_t\left[\langle x^*(t,0),x(t,0)-\widetilde{x}(t,0)\rangle\right] \\
&\quad = \langle x^*(1,1),\widetilde{x}(1,1)-x(1,1)\rangle - \langle x^*(1,0),\widetilde{x}(1,0)-x(1,0)\rangle \\
&\qquad - \langle x^*(1,0),x(1,0)-\widetilde{x}(1,0)\rangle + \langle x^*(0,0),x(0,0)-\widetilde{x}(0,0)\rangle = 0.
\end{aligned} \tag{5.13}
$$

Thus, we conclude that for all admissible solutions $x(t,\tau)$, $(t,\tau)\in Q$, the right-hand side of inequality (5.8) is nonnegative and we have finally

$$x^*(t,1)=0, \qquad x^*(1,t)=0. \tag{5.14}$$

□

Remark 5.2. If $F(t,\tau)=R^n$, then $K^*_{F(t,\tau)}(\widetilde{x}(t,\tau))=\{0\}$ and condition (ii) of Theorem 5.1 implies that $x^*(t,1)=0$, $x^*(1,t)=0$.

In the conclusion of this section let us consider an example. At first we study the linear discrete problem (2.8)–(2.11), where

$$
\begin{gathered}
x_{t,\tau} = A_1x_{t,\tau-1}+A_2x_{t-1,\tau}+A_3x_{t-1,\tau-1}+Bu_{t-1,\tau-1}, \quad u_{t-1,\tau-1}\in U, \\
(t,\tau)\in H_1\times L_1, \quad F_{t,\tau}=R^n, \quad (t,\tau)\in H_0\times L_0.
\end{gathered} \tag{5.15}
$$

And $A_i=1,2,3$ are $n\times n$ matrices, B is $n\times r$ matrix, $U\subset R^n$ is a convex closed set, g is continuously differentiable function of x. It is required to find controlling parameters $\widetilde{u}_{t,\tau}\in U$ such that the solution $\{\widetilde{x}_{t,\tau}\}_{H_0\times L_0}$ corresponding to them minimizes (2.8). In the consideration case,

$$a(x,y,z)=A_1x+A_2y+A_3z+BU. \tag{5.16}$$

Then by elementary computations we find that

$$
a^*(v^*;(x,y,z,v)) = \begin{cases} (A_1^*v^*,A_2^*v^*,A_3^*v^*), & B^*v^*\in K_U^*(\widetilde{u}), \\ \varnothing, & B^*v^*\notin K_U^*(\widetilde{u}), \end{cases} \tag{5.17}
$$

where $v=A_1x+A_2y+A_3z+B\widetilde{u}$, $\widetilde{u}\in U$, $A_i^*(i=1,2,3)$ and B^* are adjoint matrices, and $K^*_{F_{t,\tau}}=\{0\}$.

So using Theorem 3.1 and formula (5.17) we get the relations

$$\varphi^*_{t,\tau} = A_1^* x^*_{t,\tau}, \qquad \eta^*_{t,\tau} = A_2^* x^*_{t,\tau}, \tag{5.18}$$

$$x^*_{t-1,\tau-1} = A_3^* x^*_{t,\tau} + \lambda g'_{t-1,\tau-1}(\widetilde{x}_{t-1,\tau-1}) + \varphi^*_{t-1,\tau} + \eta^*_{t,\tau-1}, \quad (t,\tau) \in H_1 \times L_1, \tag{5.19}$$

$$\begin{gathered} \langle u - \widetilde{u}_{t-1,\tau-1}, B^* x^*_{t,\tau} \rangle \geq 0, \quad u \in U, \\ \varphi^*_{T,\tau+1} - x^*_{T,\tau} = 0, \quad \tau \in L_0, \end{gathered} \tag{5.20}$$

$$\begin{gathered} \eta^*_{t+1,L} - x^*_{t,L} = 0, \quad t \in H_0, \\ x^*_{0,0} = \eta^*_{T+1,L} = \varphi^*_{T,L+1} = 0. \end{gathered} \tag{5.21}$$

Substituting (5.18) in (5.19) and (5.21) we have

$$\begin{gathered} x^*_{t-1,\tau-1} = A_1^* x^*_{t-1,\tau} + A_2^* x^*_{t,\tau-1} + A_3^* x^*_{t,\tau} + \lambda g'_{t-1,\tau-1}(\widetilde{x}_{t-1,\tau-1}), \\ x^*_{T,\tau} = A_1^* x^*_{T,\tau+1}, \quad \tau \in L_0, \quad x^*_{t,L} = A_2^* x^*_{t+1,L}, \quad t \in H_0, \end{gathered} \tag{5.22}$$

$$A_1^* x^*_{T,L+1} = 0, \qquad A_2^* x^*_{T+1,L} = 0. \tag{5.23}$$

Now, it is not hard to see that (5.20) and (5.23) can be written as follows, respectively,

$$\langle B\widetilde{u}_{t-1,\tau-1}, x^*_{t,\tau} \rangle = \inf_{u \in U} \langle Bu, x^*_{t,\tau} \rangle, \quad (t,\tau) \in H_1 \times L_1, \tag{5.24}$$

$$x^*_{T,\tau} = 0, \quad \tau \in L_0; \quad x^*_{t,L} = 0, \quad t \in H_0. \tag{5.25}$$

It is noteworthy that the number λ in (5.22) is nonzero, that is, $\lambda = 1$. In fact if $\lambda = 0$, then it follows immediately from the boundary value conjugate problem (5.22), (5.23) that $x^*_{t,\tau} = 0$, $(t,\tau) \in H_0 \times L_0$. But on Theorem 3.1 $x^*_{t,\tau}$ and λ do not equal zero for all $(t,\tau) \in H_0 \times L_0$. Thus the nondegeneracy condition in Theorem 5.1 is superfluous for linear problem and we conclude the validity of the following theorem.

Theorem 5.3. *The existence of $\{x^*_{t,\tau}\}_{H_0 \times L_0}$ of the boundary value problem (5.22), (5.24), (5.25) is sufficient for the optimality of the solution $\{x^*_{t,\tau}\}_{H_0 \times L_0}$ of problem (2.8), (2.11), (5.15).*

Suppose now we have the so-called linear continuous problem of Goursat-Darboux type (see (2.14)–(2.17)),

$$\begin{gathered} I(x(\cdot,\cdot)) = \iint_Q g(x(t,\tau),t,\tau)\,dt\,d\tau \longrightarrow \inf, \\ x''_{t\tau}(t,\tau) = Ax(t,\tau) + Bu(t,\tau), \qquad u(t,\tau) \in U, \\ (t,\tau) \in Q = [0,1] \times [0,1], \\ x(t,0) = \alpha(t), \qquad x(0,\tau) = \beta(\tau), \end{gathered} \tag{5.26}$$

where g is convex and continuously differentiable function on x, A and B are $n \times n$ and $n \times r$ matrices, respectively, U is a convex closed subset of R^r. The problem is to find an absolutely continuous controlling parameter $\widetilde{u}(t,\tau) \in U$such that the solution $\widetilde{x}(t,\tau)$

corresponding to it minimizes $I(x(\cdot,\cdot))$. For problem (5.26) $a(z) = Az + BU$ and

$$a^*(v^*;(z,v)) = \begin{cases} A^*v^*, & B^*v^* \in K_U^*(\widetilde{u}), \\ \varnothing, & B^*v^* \notin K_U^*(\widetilde{u}), \end{cases} \qquad v = Az + B\widetilde{u}, \quad \widetilde{u} \in U. \tag{5.27}$$

In this problem we are proceeding on the basic of Theorem 5.1. Thus using Theorem 5.1 and similarly computations of Theorem 5.3 we can establish the following result.

THEOREM 5.4. *The solution $\widetilde{x}(t,\tau)$ corresponding to the controlling parameter $\widetilde{u}(t,\tau)$ minimizes $I(x(\cdot,\cdot))$ in the problem (5.26) if there exists an absolutely continuous function $x^*(t,\tau)$ satisfying the following conditions:*

$$\begin{gathered} x_{t\tau}^{*''}(t,\tau) = A^*x^*(t,\tau) + g'(\widetilde{x}(t,\tau),t,\tau), \quad a.e., \\ x^*(t,1) = 0, \quad x^*(1,\tau) = 0, \quad (t,\tau) \in Q, \\ \langle B\widetilde{u}(t,\tau), x^*(t,\tau)\rangle = \inf_{u\in U} \langle Bu, x^*(t,\tau)\rangle. \end{gathered} \tag{5.28}$$

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