POSITIVE SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS WITH p-LAPLACIAN OPERATOR

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The author studies the boundary value problems with p-Laplacian functional difference equation $\triangle\phi_p(\triangle x(t))+r(t)f(x_t)=0$, $t\in[0,N]$, $x_0=\psi\in C^+$, $x(0)-B_0(\triangle x(0))=0$, $\triangle x(N+1)=0$. By using a fixed point theorem in cones, sufficient conditions are established for the existence of twin positive solutions.

1. Introduction

For notation, given $a<b$ in $\mathbb{Z}$, we employ intervals to denote discrete sets such as $[a,b]=\{a,a+1,\dots,b\}$, $[a,b)=\{a,a+1,\dots,b-1\}$, $[a,\infty)=\{a,a+1,\dots\}$, and so forth. Let $\tau,N\in\mathbb{Z}$ and let $0\le\tau\le N$. In this paper, we are concerned with the p-Laplacian functional difference equation

$$\begin{gathered}\triangle\phi_p(\triangle x(t))+r(t)f(x_t)=0,\quad t\in[0,N],\\ x_0=\psi\in C^+,\qquad x(0)-B_0(\triangle x(0))=0,\qquad \triangle x(N+1)=0,\end{gathered}\tag{1.1}$$

where $\phi_p(u)$ is the p-Laplacian operator, that is, $\phi_p(u)=|u|^{p-2}u$, $p>1$, $(\phi_p)^{-1}(u)=\phi_q(u)$, $1/p+1/q=1$. For all $t\in\mathbb{Z}$, let $x_t=x_t(k)=x(t+k)$, $k\in[-\tau,-1]$; then $x_t\in C$, where $C=C([-\tau,-1],\mathbb{R})$ is a Banach space with the norm $\|\varphi\|_C=\max_{k\in[-\tau,-1]}|\varphi|$. Let $C^+=\{\varphi\in C:\varphi(k)\ge 0,\ k\in[-\tau,-1]\}$ and let $d=\max_{k\in[-\tau,-1]}\psi(k)$, $\psi\in C^+$. As usual, $\triangle$ denotes the forward difference operator defined by $\triangle x(t)=x(t+1)-x(t)$.

We will assume that

(H_1) $f(\varphi)$ is a nonnegative continuous functional defined on C^+;

(H_2) $r(t)$ is a nonnegative function defined on $[0,N]$;

(H_3) $B_0:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies that there are $\beta\ge\alpha\ge 0$ such that $\alpha s\le B_0(s)\le\beta s$ for $s\in\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

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Recently, the existence of positive solutions of finite difference equations with different boundary value conditions is investigated in [1–5] and references therein. In this paper, we consider the functional difference equation (1.1) and apply the twin fixed point theorem to obtain at least two positive solutions of the boundary value problem (BVP) (1.1) when growth conditions are imposed on f. Finally, we present two corollaries that show that under the assumptions that f is superlinear or sublinear, BVP (1.1) has at least two positive solutions. An example to illustrate our results in this paper is included.

We note that $x(t)$ is a solution of (1.1) if and only if

$$x(t)=\begin{cases} B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n)f(x_n)\right)\right)+\sum_{m=0}^{t-1}\phi_q\left(\sum_{n=m}^{N} r(n)f(x_n)\right), & t\in[0,N+2],\\ \psi, & t\in[-\tau,-1]. \end{cases} \tag{1.2}$$

We assume that $\overline{x}(t)$ is the solution of BVP (1.1) with $f\equiv 0$. Clearly, it can be expressed as

$$\overline{x}(t)=\begin{cases} 0, & t\in[0,N+2],\\ \psi, & t\in[-\tau,-1]. \end{cases} \tag{1.3}$$

It is obvious that $\overline{x}_n\equiv 0$ for $n\in[\tau,N]$.

Let $x(t)$ be a solution of BVP (1.1) and $y(t)=x(t)-\overline{x}(t)$. Noting that $y(t)=x(t)$ for $t\in[0,N+2]$, then we have from (1.2) that

$$y(t)=\begin{cases} B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n)f(y_n+\overline{x}_n)\right)\right)+\sum_{m=0}^{t-1}\phi_q\left(\sum_{n=m}^{N} r(n)f(y_n+\overline{x}_n)\right), & t\in[0,N+2],\\ 0, & t\in[-\tau,-1]. \end{cases} \tag{1.4}$$

Let $E=\{y:[-\tau,N+2]\to\mathbb{R}\}$ with norm $\|y\|=\max_{t\in[-\tau,N+2]}|y(t)|$, then $(E,\|\cdot\|)$ is a Banach space.

Define a cone P by

$$\begin{aligned} P=\{&y\in E: y(t)=0 \text{ for } t\in[-\tau,-1];\ y(t)\geq 0 \text{ for } t\in[0,N+2],\\ &\text{and } \triangle^2 y(t)\leq 0,\ \triangle y(t)\geq 0 \text{ for } t\in[0,N+2],\ \triangle y(N+1)=0\}. \end{aligned} \tag{1.5}$$

Clearly, $\|y\|=\|y\|_{[0,N+2]}=y(N+2)$ for $y(t)\in P$, where $\|y\|_{[0,N+2]}=\max_{t\in[0,N+2]}|y(t)|$. Define $T:P\to E$ by

$$Ty(t)=\begin{cases} B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n)f(y_n+\overline{x}_n)\right)\right)+\sum_{m=0}^{t-1}\phi_q\left(\sum_{n=m}^{N} r(n)f(y_n+\overline{x}_n)\right), & t\in[0,N+2],\\ 0, & t\in[-\tau,-1]. \end{cases} \tag{1.6}$$

The following lemma will play an important role in the proof of our results and can be found in [2]. Let

$$\begin{aligned} P(\delta,e) &= \{x \in P : \delta(x) < e\}, \\ \partial P(\delta,e) &= \{x \in P : \delta(x) = e\}, \\ \overline{P(\delta,e)} &= \{x \in P : \delta(x) \le e\}. \end{aligned} \tag{1.7}$$

Lemma 1.1. *Let X be a real Banach space, P a cone of X, γ and α two nonnegative increasing continuous maps, θ a nonnegative continuous map, and $\theta(0) = 0$. There are two positive numbers c and M such that*

$$\gamma(x) \le \theta(x) \le \alpha(x), \quad \|x\| \le M\gamma(x) \quad \text{for } x \in \overline{P(\gamma,c)}. \tag{1.8}$$

In addition, assume that $T : \overline{P(\gamma,c)} \to P$ is completely continuous. There are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x) \quad \forall \lambda \in [0,1],\ x \in \partial P(\theta,b), \tag{1.9}$$

and

(i) *$\gamma(Tx) > c$ for $x \in \partial P(\gamma,c)$;*
(ii) *$\theta(Tx) < b$ for $x \in \partial P(\theta,b)$;*
(iii) *$\alpha(Tx) > a$ and $P(\alpha,a) \ne \emptyset$ for $x \in \partial P(\alpha,a)$.*

Then T has at least two fixed points x_1 and $x_2 \in \overline{P(\gamma,c)}$ satisfying

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c. \tag{1.10}$$

The following lemma is similar to Lemma 1.1; the proof is omitted.

Lemma 1.2. *Let X be a real Banach space, P a cone of X, γ and α two nonnegative increasing continuous maps, θ a nonnegative continuous map, and $\theta(0) = 0$. There are two positive numbers c and M such that*

$$\gamma(x) \le \theta(x) \le \alpha(x), \quad \|x\| \le M\gamma(x) \quad \text{for } x \in \overline{P(\gamma,c)}. \tag{1.11}$$

In addition, assume that $T : \overline{P(\gamma,c)} \to P$ is completely continuous. There are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x) \quad \forall \lambda \in [0,1],\ x \in \partial P(\theta,b), \tag{1.12}$$

and

(i) *$\gamma(Tx) < c$ for $x \in \partial P(\gamma,c)$;*
(ii) *$\theta(Tx) > b$ for $x \in \partial P(\theta,b)$;*
(iii) *$\alpha(Tx) < a$ and $P(\alpha,a) \ne \emptyset$ for $x \in \partial P(\alpha,a)$.*

Then T has at least two fixed points x_1 and $x_2 \in \overline{P(\gamma,c)}$ satisfying

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c. \tag{1.13}$$

2. Main results

Choose $h = [(N+2)/2]$, where $[x]$ is the greatest integer not greater than x.

Lemma 2.1. *Let T be defined by* (1.4). *If $y \in P$, then*

(i) *$T(P) \subset P$;*
(ii) *$T : P \to P$ is completely continuous;*
(iii) *finding positive solutions of BVP (1.1) is equivalent to finding fixed points of the operator T on P;*
(iv) *if $y \in P$, then*

$$y(t) \geq \frac{1}{2}\|y\| = \frac{1}{2}y(N+2), \quad t \in [h, N+2]. \tag{2.1}$$

The proof is simple and is omitted.

Define the nonnegative, increasing, continuous functionals γ, θ, and α on P by

$$\begin{gathered}\gamma(y) = y(h),\\ \theta(y) = \max_{t\in[0,h]} y(t) = y(h),\\ \alpha(y) = \max_{t\in[0,h]} y(t) = y(h).\end{gathered} \tag{2.2}$$

We have

$$\begin{gathered}\gamma(y) = \theta(y) = \alpha(y), \quad y \in P,\\ \theta(y) = \gamma(y) = y(h) \geq \left(\frac{1}{2}\right)y(N+2) = \left(\frac{1}{2}\right)\|y\| \text{ for each } y \in P.\end{gathered} \tag{2.3}$$

Then

$$\begin{gathered}\|y\| \leq 2\gamma(y), \quad \text{for each } y \in P,\\ \theta(\lambda y) = \lambda\theta(y), \quad \forall\, \lambda \in [0,1],\ y \in \partial P(\theta, b).\end{gathered} \tag{2.4}$$

For the notational convenience, we denote σ and ρ by

$$\begin{gathered}\sigma = (\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)\right);\\ \rho = (\beta+h)\phi_q\left(\sum_{n=0}^{N} r(n)\right).\end{gathered} \tag{2.5}$$

Throughout the paper, we assume that $h+\tau \leq N$ and $\sum_{n=h+\tau}^{N} r(n) > 0$.

Theorem 2.2. *Suppose that there are positive numbers $a < b < c$ such that*

$$0 < a < \frac{\sigma}{\rho}b < \frac{\sigma}{2\rho}(c-d). \tag{2.6}$$

Assume that $f(\varphi)$ satisfies the following conditions:

(A) $f(\varphi) > \phi_p(c/\sigma)$ *for* $c \le \|\varphi\|_C \le 2c$,

(B) $f(\varphi) < \phi_p(b/\rho)$ *for* $0 \le \|\varphi\|_C \le 2b + d$,

(C) $f(\varphi) > \phi_p(a/\sigma)$ *for* $a \le \|\varphi\|_C \le 2a$.

Then BVP (1.1) has at least two positive solutions x_1 and x_2 such that

$$a < \max_{t\in[0,h]} x_1(t) < b < \max_{t\in[0,h]} x_2(t) < c. \tag{2.7}$$

Proof. Firstly, we verify that $y \in \partial P(\gamma, c)$ implies that $\gamma(Ty) > c$.

Since $\gamma(y) = c = y(h)$, one gets $y(t) \ge c$ for $t \in [h, N+2]$.

Recalling that $\|y\| \le 2\gamma(y) = 2c$, we know that $c \le \|y_n\|_C \le 2c$ for $n \in [h+\tau, N]$. Then, we get

$$\begin{aligned}
\gamma(Ty) &= B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n) f(y_n + \overline{x}_n)\right)\right) + \sum_{m=0}^{h-1} \phi_q\left(\sum_{n=m}^{N} r(n) f(y_n + \overline{x}_n)\right) \\
&\ge \alpha\phi_q\left(\sum_{n=0}^{N} r(n) f(y_n + \overline{x}_n)\right) + \sum_{m=0}^{h-1} \phi_q\left(\sum_{n=m}^{N} r(n) f(y_n + \overline{x}_n)\right) \\
&\ge \alpha\phi_q\left(\sum_{n=h+\tau}^{N} r(n) f(y_n)\right) + \phi_q\left(\sum_{n=h+\tau}^{N} r(n) f(y_n)\right) \\
&= (\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n) f(y_n)\right) > (\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)\phi_p\left(\frac{c}{\sigma}\right)\right) \\
&= \frac{c}{\sigma}(\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)\right) = c.
\end{aligned} \tag{2.8}$$

Secondly, we prove that $y \in \partial P(\theta, b)$ implies that $\theta(Ty) < b$.

Since $\theta(y) = b$ implies that $y(h) = b$, it follows that $0 \le y(t) \le b$ for $t \in [0, h]$ and

$$b \le y(t) \le \|y\| \le 2\theta(y) = 2b, \quad \text{for } t \in [h+1, N],\ y \in P. \tag{2.9}$$

So

$$||y_n + \overline{x}_n||_C \le ||y_n||_C + ||\overline{x}_n||_C \le 2b + d. \tag{2.10}$$

Then, we have

$$\begin{aligned}
\theta(Ty) &= B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n) f(y_n + \overline{x}_n)\right)\right) + \sum_{m=0}^{h-1} \phi_q\left(\sum_{n=m}^{N} r(n) f(y_n + \overline{x}_n)\right) \\
&< \beta\phi_q\left(\sum_{n=0}^{N} r(n) f(y_n + \overline{x}_n)\right) + \sum_{m=0}^{h-1} \phi_q\left(\sum_{n=0}^{N} r(n) f(y_n + \overline{x}_n)\right) \\
&= \frac{b}{\rho}(\beta+h)\phi_q\left(\sum_{n=0}^{N} r(n)\right) = b.
\end{aligned} \tag{2.11}$$

Finally, we show that

$$P(\alpha,a) \neq \emptyset, \quad \alpha(Ty) > a \quad \forall y \in \partial P(\alpha,a). \tag{2.12}$$

It is obvious that $P(\alpha,a) \neq \emptyset$. On the other hand, $\alpha(y) = y(h) = a$ implies that

$$\begin{aligned} a \le \|y\| \le 2a \quad &\text{for } t \in [h,N], \\ a \le \|y_n\|_C \le 2a \quad &\text{for } n \in [h+\tau,N]. \end{aligned} \tag{2.13}$$

Thus,

$$\begin{aligned} \alpha(Ty) &= B_0\left(\phi_q\left(\sum_{n=0}^{N} r(n)f(y_n+\overline{x}_n)\right)\right) + \sum_{m=0}^{h-1}\phi_q\left(\sum_{n=m}^{N} r(n)f(y_n+\overline{x}_n)\right) \\ &\ge \alpha\phi_q\left(\sum_{n=0}^{N} r(n)f(y_n+\overline{x}_n)\right) + \sum_{m=0}^{h-1}\phi_q\left(\sum_{n=m}^{N} r(n)f(y_n+\overline{x}_n)\right) \\ &\ge \alpha\phi_q\left(\sum_{n=h+\tau}^{N} r(n)f(y_n)\right) + \phi_q\left(\sum_{n=h+\tau}^{N} r(n)f(y_n)\right) \\ &= (\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)f(y_n)\right) > (\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)\phi_p\left(\frac{a}{\sigma}\right)\right) \\ &= \frac{a}{\sigma}(\alpha+1)\phi_q\left(\sum_{n=h+\tau}^{N} r(n)\right) = a. \end{aligned} \tag{2.14}$$

Hence by Lemma 1.1, T has at least two different fixed points y_1 and y_2. Let $x_i = y_i + \overline{x}$ $(i = 1,2)$, which are twin positive solutions of BVP (1.1) such that (2.7) holds. The proof is complete. □

Theorem 2.3. *Suppose that there are positive numbers* $0 < a < b < c$ *such that*

$$0 < 2a + d < b < \frac{\sigma}{\rho}c. \tag{2.15}$$

Assume that $f(\varphi)$ *satisfies the following conditions:*
(A′) $f(\varphi) < \phi_p(c/\rho)$ *for* $0 \le \|\varphi\|_C \le 2c+d$,
(B′) $f(\varphi) > \phi_p(b/\sigma)$ *for* $b \le \|\varphi\|_C \le 2b$,
(C′) $f(\varphi) < \phi_p(a/\rho)$ *for* $0 \le \|\varphi\|_C \le 2a+d$.
Then BVP (1.1) has at least two positive solutions x_1 *and* x_2 *such that*

$$a < \max_{t\in[0,h]} x_1(t) < b < \max_{t\in[0,h]} x_2(t) < c. \tag{2.16}$$

The proof is omitted since it is similar to that of Theorem 2.2.

Now, we give theorems which may be considered as the corollaries of Theorems 2.2 and 2.3.

Let

$$f_0 = \lim_{\|\varphi\|_C \to 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}}; \qquad f_\infty = \lim_{\|\varphi\|_C \to \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}}, \tag{2.17}$$

and choose k_1, k_2, k_3 such that

$$k_i\sigma > 1, \quad i = 1,2,\ 0 < k_3\rho < 1. \tag{2.18}$$

Theorem 2.4. *Let the following conditions be satisfied:*

(D) $f_0 > {k_1}^{p-1}$, $f_\infty > {k_2}^{p-1}$;

(E) *there exists a* $p_1 > 0$ *such that for* $0 \le \|\varphi\|_C \le 2p_1 + d$, *one has* $f(\varphi) < (p_1/\rho)^{p-1}$.

Then BVP (1.1) has at least two positive solutions.

Proof. Firstly, choose $b = p_1$, then

$$f(\varphi) < \left(\frac{2p_1}{\rho}\right)^{p-1} = \phi_p\left(\frac{b}{\rho}\right) \quad \text{for } 0 \le \|\varphi\|_C \le 2b + d. \tag{2.19}$$

Secondly, since $f_0 > k_1^{p-1}$, there is $R_1 > 0$ sufficiently small such that

$$f(\varphi) > (k_1\|\varphi\|_C)^{p-1} \quad \text{for } 0 \le \|\varphi\|_C \le R_1. \tag{2.20}$$

Without loss of generality, suppose that

$$R_1 \le \frac{2\sigma}{\rho} b. \tag{2.21}$$

Choose $a > 0$ so that $a < (1/2)R_1$. For $a \le \|\varphi\|_C \le 2a$, we have $\|\varphi\|_C \le R_1$ and $a < (\sigma/\rho)b$. Thus,

$$f(\varphi) > (k_1\|\varphi\|_C)^{p-1} \ge (k_1 a)^{p-1} > \phi_p\left(\frac{a}{\sigma}\right) \quad \text{for } a \le \|\varphi\|_C \le 2a. \tag{2.22}$$

Thirdly, since $f_\infty > {k_2}^{p-1}$, there is $R_2 > 0$ sufficiently large such that

$$f(\varphi) > (k_2\|\varphi\|_C)^{p-1} \quad \text{for } \|\varphi\|_C \ge R_2. \tag{2.23}$$

Without loss of generality, suppose that $R_2 > 2b$. Choose $c \ge R_2 + d$. Then,

$$f(\varphi) > (k_2\|\varphi\|_C)^{p-1} \ge (k_2 c)^{p-1} > \phi_p\left(\frac{c}{\sigma}\right) \quad \text{for } c \le \|\varphi\|_C \le 2c. \tag{2.24}$$

We then have $0 < a < (\sigma/\rho)b < (\sigma/2\rho)(c-d)$, and now the conditions in Theorem 2.2 are all satisfied. By Theorem 2.2, BVP (1.1) has at least two positive solutions. The proof is complete. □

Theorem 2.5. *Let the following conditions be satisfied:*
(F) $f_0 < {k_3}^{p-1}$;
(G) *there exists a* $p_2 > 0$ *such that for* $0 \le \|\varphi\|_C \le 2p_2$, *one has* $f(\varphi) > (p_2/\sigma)^{p-1}$.
Then BVP (1.1) has at least two positive solutions.

The following corollaries are obvious.

Corollary 2.6. *Let the following conditions be satisfied:*
(D′) $f_0 = \infty$, $f_\infty = \infty$;
(E) *there exists a* $p_1 > 0$ *such that for* $0 \le \|\varphi\|_C \le 2p_1 + d$, *one has* $f(\varphi) < (p_1/\rho)^{p-1}$.
Then BVP (1.1) has at least two positive solutions.

Corollary 2.7. *Let the following conditions be satisfied:*
(F′) $f_0 = 0$;
(G) *there exists a* $p_2 > 0$ *such that for* $0 \le \|\varphi\|_C \le 2p_2$, *one has* $f(\varphi) > (p_2/\sigma)^{p-1}$.
Then BVP (1.1) has at least two positive solutions.

3. Example

Example 3.1. Consider BVP

$$\begin{gathered} \triangle\phi_p(\triangle x(t)) + r\left[x^{1/9}(t-1) + x^{1/3}(t-1)\right] = 0, \quad t \in [0,4], \\ x(t) = \psi(t), \quad t = -1,\ x(0) = 0,\ x(5) = x(6) = 1, \end{gathered} \tag{3.1}$$

where $\tau = 1$, $k = -1$, $N = 4$, $h = 3$, $\alpha = \beta = 0$, $r > 0$ is a constant satisfying $\sum_{n=h+\tau}^{N} r > 0$, $\psi(t) \ge 0$, $d = \|\psi\|_C = \max_{k=-1}|\psi(k)| > 0$, $p = 7/6$, $q = 7$, and $f(\varphi) = \varphi^{1/9}(-1) + \varphi^{1/3}(-1)$.

Suppose that $\varphi \in C^+$, then $\|\varphi\|_C = \varphi(-1)$.

As $\|\varphi\|_C \to 0$ or $\|\varphi\|_C \to +\infty$, we get

$$\begin{aligned} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} &= \frac{\varphi^{1/9}(-1) + \varphi^{1/3}(-1)}{\|\varphi\|_C^{p-1}} \\ &= \|\varphi\|_C^{(10-9p)/9} + \|\varphi\|_C^{(4-3p)/3} \longrightarrow +\infty. \end{aligned} \tag{3.2}$$

We deduce that

$$\rho = (\beta + h)\phi_q\left(\sum_{n=0}^{N} r(n)\right) = 3\left[\sum_{n=0}^{4} r\right]^6 = 46875r, \tag{3.3}$$

thus, for all $m > 0$ and $0 \le \|\varphi\|_C \le m + d$, one has

$$0 \le f(\varphi) \le (m+d)^{1/9} + (m+d)^{1/3} = (m+d)^{1/9}\left(m^{1-p} + \frac{(m+d)^{2/9}}{m^{p-1}}\right)m^{p-1}. \tag{3.4}$$

Define $H(m) = (m+d)^{1/9}(m^{1-p} + (m+d)^{2/9}/m^{p-1})$.

Suppose that r and d satisfy

$$(2d)^{1/9}\left(d^{-1/6} + 2^{2/9}d^{1/18}\right) < \left(\frac{1}{2\rho}\right)^{p-1}; \tag{3.5}$$

then $H(d) = (2d)^{1/9}(d^{-1/6} + 2^{2/9}d^{1/18}) < (1/2\rho)^{p-1}$ holds. So, we can find a $p_1 = d/2$ such that $f(\varphi) \le H(2p_1)(2p_1)^{p-1} < (p_1/\rho)^{p-1}$ for $0 \le \|\varphi\|_C \le 2p_1 + d$. By Corollary 2.6, we know that BVP (3.1) has at least two positive solutions.

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