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Research Article

Convergence of a Mimetic Finite Difference Method for Static Diffusion Equation

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The numerical solution of partial differential equations with finite differences mimetic methods that satisfy properties of the continuum differential operators and mimic discrete versions of appropriate integral identities is more likely to produce better approximations. Recently, one of the authors developed a systematic approach to obtain mimetic finite difference discretizations for divergence and gradient operators, which achieves the same order of accuracy on the boundary and inner grid points. This paper uses the second-order version of those operators to develop a new mimetic finite difference method for the steady-state diffusion equation. A complete theoretical and numerical analysis of this new method is presented, including an original and nonstandard proof of the quadratic convergence rate of this new method. The numerical results agree in all cases with our theoretical analysis, providing strong evidence that the new method is a better choice than the standard finite difference method.

1. Introduction

Nowadays much effort has been devoted to create a discrete analog of vector and tensor calculus that could be used to accurately approximate continuum models for a wide range of physical and engineering problems which preserves, in a discrete sense, symmetries and conservation laws that are true in the continuum [1, 2]. This endeavor has led to the formulation of a set of mimetic finite difference discretization schemes to find high-order numerical solution of partial differential equations [3, 4]. These discretizations have been considered challenging even in the simplest case of one dimension on a uniform grid.

Particularly, in a recent article [5] a systematic way of constructing high-order mimetic discretizations for gradient, divergence operators with the same order of approximation at the boundary and inner region was presented. A key point of mimetic discretizations is to build discrete versions of these operators satisfying a discrete analog of the divergence or Green-Gauss theorem which implies that the discrete operators will satisfy a global conservation law. This condition also ensures that the discretizations of the boundary conditions and of the differential equation are compatible. In addition to having the advantage that its formulation is not more complex than standard finite differences, it has been known for some time that numerical methods based on mimetic discretization produce better results than standard finite differences.

In this article, we will provide a rigorous proof of quadratic convergence for a particular and unique mimetic finite difference method for the steady-state diffusion equation based on the second-order discrete gradient and divergence operators obtained and studied in [5–7]. Although the literature on second-order mimetic methods for the steady-state diffusion is fairly well established, our new method is not standard. Consequently, the theoretical analysis of convergence presented in this article is different from those reported previously, therefore representing a new contribution.

Comparative studies of the new method and other mimetic schemes have been reported previously [7, 8]. Those studies give evidence that the second-order gradient approximations of our scheme on the boundaries produce, in the worst cases, more accurate solution. However, all second-order mimetic schemes achieve quadratic convergence rate, which is not the situation for standard finite difference methods based on ghost points and extended grids. Comparison of the new scheme against sophisticated or very elaborated numerical techniques such as mixed finite elements is a current research topic but it is out the scope of this paper.

Without lose of generality and for methodological reasons, our analysis will be developed for the one-dimensional steady-state diffusion equation. In this case, the divergence theorem takes the form

$$\int_0^1 \frac{dv}{dx} f\,dx + \int_0^1 v\frac{df}{dx}dx = v(1)f(1) - v(0)f(0) \tag{1.1}$$

on which dv/dx plays the role of the divergence of the vector field $v(x)$, while df/dx plays the role of the gradient of the scalar field $f(x)$. Our numerical studies give evidence that theoretical results hold in several dimensions and the new scheme produces better approximations than standard finite difference methods.

The rest of this article is organized as follows. In Section 2, we introduce the continuum model for the steady-state diffusion equation with its respective boundary conditions relevant to this work. After that, in Section 3, the second-order discretized mimetic scheme for the gradient and divergence operators is presented. In Section 4, the new mimetic finite difference method for the steady-state diffusion equation is developed and described. Then, in Section 5 we present the proof of the quadratic convergence rate of the new method. Next, the solution and analysis of illustrative numerical test problems are given in Section 6. Finally, the conclusions and recommendations are summarized in Section 7.

2. The continuum model

Our model problem will be presented in terms of the steady diffusion equation. Being one of the most important and widely used equations of the mathematical physics, the

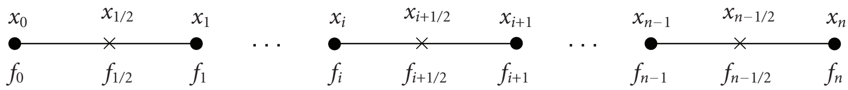

Figure 3.1. Uniform staggered (nonuniform point distributed) grid ($f_i \equiv f(x_i)$).

range of physical and engineering problems modeled by this equation includes heat transfer, flow through porous medium, and the pricing of some financial instruments [9–11]. Accordingly, the wide range of applications of the diffusion equation justify the effort and time devoted in finding ways of obtaining high-quality numerical solution of it on different contexts.

In the one-dimensional case, the diffusion equation takes the form

$$\frac{d}{dx}\left(K(x)\frac{df(x)}{dx}\right) = F(x), \tag{2.1}$$

where $f(x)$ is an unknown function, $F(x)$ is a given source term, and $K(x)$ is a positive function. To have a properly posed boundary value problem by (2.1), we will be imposing boundary conditions of the Robin (mixed) type

$$\alpha_0 f(0) - f'(x)\big|_{x=0} = \gamma_0, \tag{2.2a}$$

$$\alpha_1 f(1) + f'(x)\big|_{x=1} = \gamma_1, \tag{2.2b}$$

where α_0, α_1, γ_0, and γ_1 are known constants. In this article, we are analyzing the case where $K(x)$ is the identity. In this situation, it has been reported that the problem posed by (2.1) and (2.2) has a unique solution when α's coefficients are not null [12, 13].

3. The mimetic discretization

The description of the mimetic discretization will be presented by using the one-dimensional *uniform staggered* or *nonuniform point distributed* grid represented in Figure 3.1 in the interval $[x_0 = 0, x_n = 1]$.

In this context, the region of interest (interval $[0,1]$) is partitioned into n equally spaced cells $[x_i, x_{i+1}]$ with $0 \le i \le n-1$. The end points of each cell are called nodes, being the boundary of the grid the nodes x_0 and x_n, respectively. The center of each cell is indexed with half integer, namely $x_{i+1/2} = (1/2)(x_i + x_{i+1})$.

In this grid, the vector field v of (1.1) can be considered as a vector whose components are the vector field v evaluated at the grid's nodes including the boundary, while the scalar function $f(x)$ can be considered as a vector having components corresponding to the evaluation of the function f at the center of each cell and at the boundary nodes

$$v \equiv \left(v(x_0), v(x_1), v(x_2), \dots, v(x_{n-1}), v(x_n)\right)^T, \tag{3.1a}$$

$$f \equiv \left(f(x_0), f(x_{1/2}), f(x_{3/2}), \dots, f(x_{n-1/2}), f(x_n)\right)^T. \tag{3.1b}$$

On uniform grids, following the notation of Figure 3.1, the approach of [5, 7] can be applied to obtain second-order mimetic discretizations $\mathbf{D}$ and $\mathbf{G}$ for the divergence and gradient continuum operators, respectively, yielding that the gradient at the boundary points x_0 and x_n has the form

$$(\mathbf{G}f)_0 = \frac{-(8/3)f_0 + 3f_{1/2} - (1/3)f_{3/2}}{h}, \tag{3.2a}$$

$$(\mathbf{G}f)_n = \frac{(8/3)f_n - 3f_{n-1/2} + (1/3)f_{n-3/2}}{h}, \tag{3.2b}$$

while at the inner points (cell or edges), represented as crosses in Figure 3.1, the gradient and divergence approximations coincide with standard central difference schemes

$$(\mathbf{G}f)_i = \frac{f_{i+1/2} - f_{i-1/2}}{h}, \quad i = 1,\dots,n-1, \tag{3.3a}$$

$$(\mathbf{D}f)_{i+1/2} = \frac{f_{i+1} - f_i}{h}, \quad i = 0,\dots,n-1. \tag{3.3b}$$

It should be noted that the discretized divergence operator (3.3b) is only defined at the inner nodes or cell centers, while the gradient is defined at nodal points. As is worked out by [5], the construction of the discretized mimetic gradient $\mathbf{G}$ follows from the discretized mimetic divergence $\mathbf{D}$ as a consequence of imposing that both operators must satisfy a discrete version of Green-Gauss theorem. This can be seen by defining an extension of $\mathbf{D}$, denoted by $\widehat{\mathbf{D}}$, which satisfies $(\widehat{\mathbf{D}}f)_0 = 0$, $(\widehat{\mathbf{D}}f)_n = 0$, and $(\widehat{\mathbf{D}}f)_{i+1/2} = (\mathbf{D}f)_{i+1/2}$. The discrete expression for the fundamental equation (1.1) can be written in the form

$$\langle \widehat{\mathbf{D}}v, f\rangle_{\mathbf{Q}} + \langle v, \mathbf{G}f\rangle_{\mathbf{P}} = \langle \widetilde{\mathbf{B}}v, f\rangle_{\mathbf{I}}, \tag{3.4}$$

where $\langle a,b\rangle_{\mathbf{M}} = b^T\mathbf{M}a$ defines a generalized weighted inner product, $\mathbf{Q}$ and $\mathbf{P}$ are weighting diagonal matrices, $\mathbf{I}$ is the identity matrix, and $\widetilde{\mathbf{B}}$ is a matrix called boundary operator. It is shown in [7] that matrix $\mathbf{Q}$ is the identity, and the diagonal coefficients of $\mathbf{P}$ satisfy $\mathbf{P}(1,1) = \mathbf{P}(n+2,n+2) = 3/8$, $\mathbf{P}(2,2) = \mathbf{P}(n+1,n+1) = 9/8$ with $\mathbf{P}(i,i) = 1$ for $2 < i < n+1$. Similarly, it can be proved that boundary operator $\widetilde{\mathbf{B}}$ is an $n+2\times n+1$ matrix define by $\widetilde{\mathbf{B}} = \mathbf{Q}\widehat{\mathbf{D}} + (\mathbf{G})^t\mathbf{P}$. This operator is just an algebraic expression determined by (3.4) which allows the inner product $\langle \widetilde{\mathbf{B}}v, f\rangle_{\mathbf{I}}$ to be second-order consistence with $v(1)f(1) - v(0)f(0)$ under mesh refinement. Moreover, a simple calculation indicates that if v or f vectors, in (3.1a) or (3.1b), are constant then the following relation holds:

$$\langle \widetilde{\mathbf{B}}v, f\rangle_{\mathbf{I}} = \begin{Bmatrix} (f(1) - f(1)) \cdot v \text{ if } v \equiv \text{ constant} \\ (v(1) - v(1)) \cdot f \text{ if } f \equiv \text{ constant} \end{Bmatrix}. \tag{3.5}$$

This means that (3.4) becomes a discrete version of the fundamental theorem of Calculus.

Since gradient, divergence, and boundary operator discretizatons presented in this section satisfy (3.4) and (3.5), then they are called mimetic discretizations. Moreover, any numerical scheme based exclusively on them is conservative.

4. Mimetic method for the steady diffusion equation

Our continuum model problem of Section 2, given by (2.1) and (2.2), can be approximated by using the results of Section 3, in the following form:

$$(\mathbf{MI} \equiv \widehat{\mathbf{A}} + \widetilde{\mathbf{B}}\mathbf{G} + \widehat{\mathbf{D}}\mathbf{K}\mathbf{G})f = b. \tag{4.1}$$

In this expression $\widehat{\mathbf{A}}$ is the $(n+2)\times(n+2)$ matrix having as nonzero entries those elements in its diagonal which corresponds to the boundary nodes. The values in those entries are the associated α value given by (2.2). In the one-dimensional case its only non null entries are $\widehat{\mathbf{A}}(1,1) = \alpha_0$ and $\widehat{\mathbf{A}}(n+2,n+2) = \alpha_1$. The operator $\mathbf{K}$ is a diagonal tensor whose known values are positive and evaluated at grid block edges. Sometimes the product $(\mathbf{KG})f$ is called flux. The operators $\mathbf{G}$, $\mathbf{D}$, and $\widetilde{\mathbf{B}}$ are the mimetic discretizations for gradient, divergence, and normal boundary conditions presented in previous section. The right-hand side, vector b, has the form

$$b = \left(\gamma_0, F_{1/2}, F_{3/2}, \ldots, F_{n-1/2}, \gamma_1\right)^T \tag{4.2}$$

and f represents the mimetic approximation. Since all differential operators in (2.1) and (2.2) are approximated by mimetic discretizations in (4.1) then it represents a mimetic method for the steady state diffusion equations. The original idea of introducing (4.1) as a mimetic method for Poisson equation was given in [7]. However, a rigorous proof of its convergence has not been provided yet. In this article we are filling this gap.

The mimetic method for the steady diffusion equations (4.1) is too general for the purpose of our theoretical analysis. Therefore two simplifications are in order. They are the assumption that $\mathbf{K}$ is the identity operator and the restriction to one-dimensional problems. The first assumption is widely used in numerical analysis, because the tensor coefficient $\mathbf{K}$ is usually differentiable and it is well known that lower-order terms do not play any role in convergence analysis [1]. Restriction to one-dimensional problems for purposes of studying new numerical schemes on uniform, logical rectangular, Cartesian grids is a standard restriction. This is justified by the fact that all techniques and arguments of the one-dimensional proof can be translated without change to the higher dimensional cases by analogy [1, 2, 5].

Under these two simplifications we proceed to develop the explicit equations for the new mimetic method (4.1), which represents an $n+2\times n+2$ linear system. Its first equation represents the discretization of the boundary condition given by (2.2a), it is of the form

$$\left(\frac{8}{3h} + \alpha_0\right) f_0 - \frac{3}{h} f_{1/2} + \frac{1}{3h} f_{3/2} = \gamma_0. \tag{4.3}$$

The second equation comes from the discretization of the one-dimensional stationary diffusion equation at the cell center $x_{1/2}$,

$$\left(\frac{8}{3h^2} - \frac{1}{3h}\right) f_0 - \left(\frac{4}{h^2} - \frac{1}{2h}\right) f_{1/2} + \left(\frac{4}{3h^2} - \frac{1}{6h}\right) f_{3/2} = F_{1/2}. \tag{4.4}$$

Notice that the coefficients in this equation contain terms of the form (const/h) which are not common in standard finite differences discretizations. The third equation, at the cell center $x_{3/2}$, has the form

$$\frac{1}{3h}f_0+\left(\frac{1}{h^2}-\frac{1}{2h}\right)f_{1/2}-\left(\frac{2}{h^2}-\frac{1}{6h}\right)f_{3/2}+\frac{1}{h^2}f_{5/2}=F_{3/2}. \tag{4.5}$$

This is an extremely unusual equation, which is a major distinction of the mimetic finite difference discretization scheme with other discretization approaches. The next set of difference equations, at cell centers $x_{i+1/2}$ for $i=2,\dots,n-3$, takes the form of standard second-order central finite difference discretization for the second derivative. To close the system of difference equations arisen from the mimetic discretization of the one dimensional steady diffusion equation, three more equations are obtained from (4.1) at the points $x_{n-3/2}$, $x_{n-1/2}$, and $x_n=1$. However, they are symmetric to (4.3), (4.4), (4.5) so they will not be written down.

It is important to note that standard Taylor expansions of these equations around their associated points lead to truncation errors of order $O(h)$ for (4.4), (4.5), and their symmetric associates. On the other hand (4.3), its symmetric, and the remaining equations have truncation errors of order $O(h^2)$.

5. Analysis of convergence

Standard finite difference analysis does not provide optimum convergence rate for the mimetic method described in this article. Consequently, its convergence analysis will be divided in two parts. In the first part a finite difference scheme associated to the mimetic method is developed. Truncations analysis and optimum convergence results for this associated scheme can be obtained by traditional techniques. The second section gives the nonstandard convergence proof for the mimetic method based on the convergence of the associated scheme.

5.1. Associated finite difference scheme equations. As it was stated in the mimetic method description, all its equations are standard with the exception of (4.4) and (4.5), which contains nonstandard (const/h) terms. It can be easily shown that if those terms are omitted then an associated finite difference scheme results. This associated scheme can be represented in function of the fundamental operators $\mathbf{G}$, $\mathbf{D}$, $\hat{\mathbf{A}}$ described previously, and a boundary operator $\mathbf{B}$ with non null entries, $\mathbf{B}(1,1)=-1$ and $\mathbf{B}(n+2,n+1)=1$. With these operators the associated scheme can be represented by the following expression:

$$(\mathbf{M}\equiv\hat{\mathbf{A}}+\mathbf{B}\mathbf{G}+\hat{\mathbf{D}}\mathbf{G})f_{\mathrm{a}}=b, \tag{5.1}$$

where f_{a} refers to the associated finite difference approximation. This scheme is conservative and new, but it is not mimetic. Its analysis has been recently developed in [14]. It is proved, by an application of the modulus maximum principle, that it has an optimum second-order convergence rate

$$|f_{\mathrm{ex}}-f_{\mathrm{a}}|\le O(h^2), \tag{5.2}$$

where f_{ex} denotes the exact solution to the continuum problem and $|\cdot|$ refers to the maximum norm. This order of convergence is not evident, because two of the $n+2$ equations in (5.1) have linear truncation errors.

A relation between the mimetic method and the associated scheme is obtained through operator $\mathbf{E}$ defined as follows:

$$\mathbf{E} = (\widetilde{\mathbf{B}} - \mathbf{B})\mathbf{G}. \tag{5.3}$$

It can be established by inspection that operators $\mathbf{MI}$ and $\mathbf{M}$, in (4.1) and (5.1), satisfy the relation

$$\mathbf{MI} = \mathbf{M} + \mathbf{E}. \tag{5.4}$$

This means that the mimetic method is a perturbation of the associated scheme by the operator $\mathbf{E}$.

In was proved in [7] that operators $\mathbf{M}^{-1}$ and $\mathbf{E}$ satisfy the following estimate:

$$|\mathbf{M}^{-1}\mathbf{E}| \leq O(h). \tag{5.5}$$

5.2. Convergence of mimetic method. In order to establish the convergence of the mimetic method, its proof will be divided in two parts. In the first one it will be shown that the scheme converges. After that, in the second part, it will be proved that its optimal convergence rate is quadratic.

Let us begin by combining (4.1), (5.1), (5.4) to obtain

$$\mathbf{M}(f_{\text{a}} - f_{\text{ex}}) = (\mathbf{M} + \mathbf{E})(f - f_{\text{ex}}) + \mathbf{E}f_{\text{ex}}. \tag{5.6}$$

After multiplication by $\mathbf{M}^{-1}$, taking norm and applying the triangle inequality, we obtain the expression

$$|f - f_{\text{ex}}| \leq |f_{\text{a}} - f_{\text{ex}}| + |\mathbf{M}^{-1}\mathbf{E}|\,|f - f_{\text{ex}}| + |\mathbf{M}^{-1}\mathbf{E}|\,|f_{\text{ex}}|. \tag{5.7}$$

On the other hand, using (5.2), (5.5), and (5.7) the following estimated expression always holds:

$$|f - f_{\text{ex}}| \leq O(h^2) + O(h)|f - f_{\text{ex}}| + O(h)|f_{\text{ex}}|. \tag{5.8}$$

This inequality is solved for $|f - f_{\text{ex}}|$ and the convergence estimate results

$$|f - f_{\text{ex}}| \leq O(h). \tag{5.9}$$

This expression shows a linear convergence rate for the mimetic method although it is not the best possible estimate.

To continue with the second part of the proof, let us consider the following identity:

$$(\mathbf{E}f_{\text{ex}}) = (\widetilde{\mathbf{B}} - \mathbf{B})\mathbf{G}f_{\text{ex}} \tag{5.10}$$

Table 6.1. Numerical errors for 1D problem maximum norm.

Grid size	Error finite difference	Error mimetic method
16	1.3666	0.7654
64	0.3469	0.0507
256	0.0946	0.0032
Convergence rate	0.9689	1.9995

which takes the following form:

$$(\mathbf{E}f_{\text{ex}}) = \Bigg(0, \frac{f_{\text{ex}}'(x_0) - f_{\text{ex}}'(x_1) + O(h^2)}{8}, \frac{-f_{\text{ex}}'(x_0) + f_{\text{ex}}'(x_1) + O(h^2)}{8},$$
$$0,\dots,0, \frac{-f_{\text{ex}}'(x_{n-1}) + f_{\text{ex}}'(x_n) + O(h^2)}{8}, \frac{f_{\text{ex}}'(x_{n-1}) - f_{\text{ex}}'(x_n) + O(h^2)}{8}, 0\Bigg)^T, \tag{5.11}$$

second-order terms in this expression come from the truncation error for mimetic gradient approximation $\mathbf{G}$. Since the exact solution f_{ex} is assumed to be smooth and infinity differentiable, then differences $f_{\text{ex}}'(x_0) - f_{\text{ex}}'(x_1)$, $f_{\text{ex}}'(x_{n-1}) - f_{\text{ex}}'(x_n)$, and their opposites can be interpreted as central differences to obtain an optimum estimate. Therefore, the vector $(\mathbf{E}f_{\text{ex}})$ satisfies the simplified relation

$$(\mathbf{E}f_{\text{ex}}) = O(h^2)w, \tag{5.12}$$

where $w = (0,1,1,0,\dots,0,1,1,0)^T$.

By substitution of (5.10) and (5.12) into the identity relation (5.6), we have the following relation:

$$(f - f_{\text{ex}}) = (f_{\text{a}} - f_{\text{ex}}) - (\mathbf{M}^{-1}\mathbf{E})(f - f_{\text{ex}}) - O(h^2)\mathbf{M}^{-1}w. \tag{5.13}$$

After taking norm and applying the triangle inequality, we obtain the estimate

$$|f - f_{\text{ex}}| \leq |f_{\text{a}} - f_{\text{ex}}| + |\mathbf{M}^{-1}\mathbf{E}|\,|f - f_{\text{ex}}| + O(h^2)|w|. \tag{5.14}$$

On the other hand, we know from (5.2), (5.5), and (5.9) that the following inequality, derived from (5.14), will hold:

$$|f - f_{\text{ex}}| \leq O(h^2). \tag{5.15}$$

The last estimate shows that the mimetic method has a quadratic convergence rate, which is the best possible result.

6. Numerical study

This section will present the numerical study of two boundary value problems in one and two dimensions, computed via the mimetic method developed in previous sections and the standard finite difference method. The main parameters analyzed in these test problems are the rate of convergence and the number of exact digits in the approximated solutions. Additional numerical studies, which differ in details and objectives from the ones presented in this article, can be found in [7, 8]. Those studies give evidence that our new second-order mimetic scheme produces, in some cases, better solutions than well-known mimetic schemes with first-order one-side gradient approximation at the boundary. However, this affirmation is problem dependent. In general, all the mimetic schemes produce comparable solutions. On the other hand, comparison of our new scheme against standard finite differences schemes based on ghost points and extended staggered grids has not been fully analyzed previously. Numerical results show that our new mimetic scheme always produces better approximations than standard finite differences method. Therefore, that comparison will be developed in this numerical study.

The one-dimensional boundary value problem in this test is formulated in terms of the ordinary differential equation $f''(x) = \lambda(\lambda-1)((1-x)^{(\lambda-2)} - x^{(\lambda-2)})$ defined on the interval $(0,1)$, and its solution must satisfy Robin boundary conditions shown in (2.2). A well-posed problem is obtained with $\alpha_0 = \alpha_1 = 1$ and $\gamma_0 = -\gamma_1 = (\lambda+1)$, λ being an arbitrary nonnull integer number. This problem has a unique analytical solution given by $f(x) = (1-x)^\lambda - x^\lambda$, representing a boundary layer for large values of λ. Correspondingly, it is an excellent test problem to evaluate numerical schemes with different discretization alternatives for boundary conditions. Numerical results for this test problem are presented in Table 6.1, after implementing both numerical methods on the staggered grid described in Figure 3.1 and setting $\lambda = 25$ (similar results and conclusions are obtained for even larger values of λ). The shown numerical errors, computed in the maximum norm, indicate that on refined grids the mimetic method achieved at least two exact digits in its approximation, while standard finite difference methods obtained only one exact digit. Such results indicate a clear advantage of the mimetic scheme. In the same table at the bottom, the numerical convergence rates for each method are also presented. A quadratic convergence rate was obtained for the mimetic method, as one would expect from previous theoretical analysis on the convergence rate of the method. Standard finite differences schemes get a first-order numerical convergence rate, which is a direct effect of having a first-order discretization for the Laplacian at nodes $x_{1/2}$ and $x_{n+1/2}$. In the extended ghost point grid, those two nodes become internal nodes away from the ghost boundary. Consequently, modulus maximum principle implies that first-order truncation error in the Laplacian will be transferred completely to the convergence rate and it cannot be canceled or balanced with second order discretizations at boundary nodes.

Though our theoretical results were established for one-dimensional problem, they also hold on higher dimensional logical rectangular Cartesian grids. This paragraph presents the numerical study of a two-dimensional test problem, defined by the two-dimensional Poisson equation $\triangle u = (128/(\exp(16)-1))\exp(8(x+y))$ on the region $\Omega = (0,1)\times(0,1)$. Its solution is given by expression $u(x,y) = (1/(\exp(16)-1))(\exp(8(x+y))-1)$, satisfying the corresponding Robin boundary condition with coefficient $\alpha = (-16\exp(16))/(\exp(16)-1)$. The solution behavior is that of a boundary layer toward the $(1,1)$ corner along the main diagonal of Ω. Details of the staggered grid implementation used in this problem are fully developed in [8]. Table 6.2 provides a summary of

Table 6.2. Numerical errors for 2D problem maximum norm.

Grid size	Error finite difference	Error mimetic method
3×3	0.0645	0.0084
10×10	0.0255	0.0010
20×20	0.0137	0.0002
Convergence rate	0.8176	1.9702

numerical errors computed with the maximum norm. As in the one-dimensional case, the errors obtained by the mimetic method are smaller than those obtained by standard finite differences. Also, the numerical convergence rate for each method in the last row of the table is shown. It gives a second-order convergence rate for the mimetic scheme as was predicted by our theoretical analysis in the one dimensional problem. In addition, it can be noticed that for the same reasons given in the previous 1D problem, the first-order truncation error associated to standard finite differences on the two-dimensional staggered grid is passed to the convergence rate.

It is important to note that it is possible to improve the convergence rate of the standard finite difference method by using a different grid configuration such as block center grids. However, in all cases the same mimetic method always produce a solution that is comparable to or better than standard finite differences. This means that the mimetic method is very robust and systematic, which is a great advantage and improvement for a numerical method based on finite differences molecules.

At this point it is of interest to mention that there is an important property related to the mimetic scheme, which cannot be matched by standard finite difference approximation. It is essentially the rigorous treatment given in the mimetic discretization method to both the boundary conditions and the differential equation. This advantage can easily be observed if the nonhomogeneous term in the differential equation has a singularity at the boundary. In such case, the mimetic method produces a robust code whose numerical results are of high accuracy. On the contrary, standard finite difference codes developed on any grid based on ghost point will break down because they require the regularity of the nonhomogeneous term up to the boundary. This last condition is artificial and it is one of the main deficiencies of standard second-order finite difference schemes. Such deficiencies are eliminated in the mimetic scheme.

7. Remarks and conclusions

A complete analytical and numerical study for a new second-order mimetic finite difference scheme for the diffusion equation has been presented. Theoretical and numerical analysis of its quadratic convergence rate is a new contribution. This is not an obvious result in view of the first-order truncation errors and the nonstandard linear equations in its mathematical formulation.

The convergence proof gives a possible strategy to obtain similar results for higher order mimetic methods based on the mimetic operators given in [5].

The new method was applied to a selected set of test problems. The numerical results indicate its main advantages over the most common second-order standard finite difference methods based on ghost points and extended grids. In addition, the numerical study ratifies the theoretical quadratic converge rate of the new scheme.

The most important advantages of the numerical method developed in this article are: it is mimetic or conservative; its formulations at inner and boundary nodes are consistent; its numerical implementation is more robust than most common second-order finite differences schemes, it is not based on ghost point techniques, and it gives a rigorous discretization of both the boundary conditions and the differential equation.

In view, of the above considerations it is fair to say that our new second-order mimetic scheme is a better choice than standard finite differences schemes for solving the static diffusion equation.

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