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Research Article

On a k-Order System of Lyness-Type Difference Equations

G. Papaschinopoulos, C. J. Schinas, and G. Stefanidou

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We consider the following system of Lyness-type difference equations: $x_1(n+1) = (a_k x_k(n) + b_k)/x_{k-1}(n-1)$, $x_2(n+1) = (a_1 x_1(n) + b_1)/x_k(n-1)$, $x_i(n+1) = (a_{i-1}x_{i-1}(n) + b_{i-1})/x_{i-2}(n-1)$, $i = 3,4,\ldots,k$, where a_i, b_i, $i = 1,2,\ldots,k$, are positive constants, $k \geq 3$ is an integer, and the initial values are positive real numbers. We study the existence of invariants, the boundedness, the persistence, and the periodicity of the positive solutions of this system.

1. Introduction

Difference equations and systems of difference equations have many applications in biology, economy, and other sciences. So there exist many papers concerning systems of difference equations (see [1–10] and the references cited therein).

In [11], Kocić and Ladas investigated the existence of invariants, the boundedness, the persistence, the periodicity, and the oscillation of the positive solutions of the Lyness difference equation

$$x_{n+1} = \frac{x_n + A}{x_{n-1}}, \quad n = 0,1,\ldots, \tag{1.1}$$

where A is a positive constant and the initial conditions x_{-1}, y_{-1}, x_0, y_0 are positive real numbers.

In [6–8], the authors studied the behavior of the positive solutions of the system of two Lyness difference equations

$$x_{n+1} = \frac{by_n + c}{x_{n-1}}, \quad y_{n+1} = \frac{dx_n + e}{y_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where b, c, d, e are positive constants and the initial conditions x_{-1}, y_{-1}, x_0, y_0 are positive numbers.

Now in this paper, we consider the system of difference equations:

$$\begin{aligned} x_1(n+1) &= \frac{a_k x_k(n) + b_k}{x_{k-1}(n-1)}, \\ x_2(n+1) &= \frac{a_1 x_1(n) + b_1}{x_k(n-1)}, \\ x_i(n+1) &= \frac{a_{i-1} x_{i-1}(n) + b_{i-1}}{x_{i-2}(n-1)}, \quad i = 3, 4, \ldots, k, \end{aligned} \tag{1.3}$$

where a_i, b_i, $i = 1, 2, \ldots, k$, are positive constant numbers, $k \geq 3$ is an integer, and the initial values $x_i(-1)$, $x_i(0)$, $i = 1, 2, \ldots, k$, are positive real numbers. For simplicity, system (1.3) can be written as follows:

$$x_i(n+1) = \frac{a_{i-1} x_{i-1}(n) + b_{i-1}}{x_{i-2}(n-1)}, \quad i = 1, 2, \ldots, k, \tag{1.4}$$

where

$$a_0 = a_k, \qquad b_0 = b_k, \qquad x_j(n) = x_{k+j}(n), \quad j = -1, 0,\ n = -1, 0, \ldots. \tag{1.5}$$

We study the existence of invariants, the boundedness, the persistence, and the periodicity of the positive solutions of the system (1.3).

2. Boundedness and persistence

In this section, we study the boundedness and the persistence of the positive solutions of (1.3). For this goal, we show the following proposition in which we find conditions so that system (1.3) has an invariant.

Proposition 2.1. *Let $k \geq 3$ and*

$$\begin{aligned} \lambda_{k+i} &= \lambda_i, \quad i \in \{-2, -1, 0, 1, 2, 3, 4\}, \\ a_{k+i} &= a_i, \quad i \in \{-3, -2, -1, 0, 1\}, \\ b_{k+i} &= b_i, \quad i \in \{-2, -1, 0\}. \end{aligned} \tag{2.1}$$

Assume that the system of $2k$ equations, with k unknowns $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the form

$$\begin{aligned} \lambda_{i+2} b_{i-1} + \lambda_{i+3} a_i a_{i-1} &= \lambda_{i-2} b_{i-3} + \lambda_{i-3} a_{i-4} a_{i-3}, \quad i \in \{1, 2, \ldots, k\}, \\ \lambda_{i+4} a_{i+1} b_i &= \lambda_{i-1} a_{i-2} b_{i-1}, \quad i \in \{1, 2, \ldots, k\}, \end{aligned} \tag{2.2}$$

has a nontrivial solution $\lambda_1,\lambda_2,\dots\lambda_k$. Then system (1.3) has an invariant of the form

$$\begin{aligned}I_n &= \sum_{i=1}^{k}\lambda_{i+2}x_i(n)+\sum_{i=1}^{k}\lambda_{i+2}x_i(n-1)\\&\quad+\sum_{i=1}^{k}\left(\lambda_i b_{i-1}+\lambda_{i-1}a_{i-1}a_{i-2}\right)\frac{1}{x_i(n)}+\sum_{i=1}^{k}\left(\lambda_i b_{i-1}+\lambda_{i-1}a_{i-1}a_{i-2}\right)\frac{1}{x_i(n-1)}\\&\quad+\sum_{i=1}^{k}\lambda_{i-1}a_{i-2}b_{i-1}\frac{1}{x_i(n)x_{i-1}(n-1)}+\sum_{i=1}^{k}\lambda_i a_{i-1}\frac{x_{i-1}(n-1)}{x_i(n)}\\&\quad+\sum_{i=1}^{k}\lambda_{i+3}a_i\frac{x_i(n)}{x_{i-1}(n-1)}.\end{aligned}\tag{2.3}$$

Proof. From (1.5), (1.4), (2.1), (2.2), and (2.3), we have

$$\begin{aligned}I_{n+1} &= \sum_{i=1}^{k}\lambda_{i+2}a_{i-1}\frac{x_{i-1}(n)}{x_{i-2}(n-1)}+\sum_{i=1}^{k}\lambda_{i+2}b_{i-1}\frac{1}{x_{i-2}(n-1)}+\sum_{i=1}^{k}\lambda_{i+2}x_i(n)\\&\quad+\sum_{i=1}^{k}\left(\lambda_i b_{i-1}+\lambda_{i-1}a_{i-1}a_{i-2}+\lambda_{i-1}a_{i-2}b_{i-1}\frac{1}{x_{i-1}(n)}+\lambda_i a_{i-1}x_{i-1}(n)\right)\\&\quad\times\frac{x_{i-2}(n-1)}{a_{i-1}x_{i-1}(n)+b_{i-1}}\\&\quad+\sum_{i=1}^{k}\left(\lambda_i b_{i-1}+\lambda_{i-1}a_{i-1}a_{i-2}\right)\frac{1}{x_i(n)}+\sum_{i=1}^{k}\lambda_{i+3}a_i a_{i-1}\frac{1}{x_{i-2}(n-1)}\\&\quad+\sum_{i=1}^{k}\lambda_{i+3}a_i b_{i-1}\frac{1}{x_{i-1}(n)x_{i-2}(n-1)}\\&=\sum_{i=1}^{k}\lambda_{i+2}x_i(n)+\sum_{i=1}^{k}\lambda_i x_{i-2}(n-1)\\&\quad+\sum_{i=1}^{k}\left(\lambda_i b_{i-1}+\lambda_{i-1}a_{i-1}a_{i-2}\right)\frac{1}{x_i(n)}+\sum_{i=1}^{k}\left(\lambda_{i+2}b_{i-1}+\lambda_{i+3}a_i a_{i-1}\right)\frac{1}{x_{i-2}(n-1)}\\&\quad+\sum_{i=1}^{k}\lambda_{i+3}a_i b_{i-1}\frac{1}{x_{i-1}(n)x_{i-2}(n-1)}+\sum_{i=1}^{k}\lambda_{i-1}a_{i-2}\frac{x_{i-2}(n-1)}{x_{i-1}(n)}\\&\quad+\sum_{i=1}^{k}\lambda_{i+2}a_{i-1}\frac{x_{i-1}(n)}{x_{i-2}(n-1)}=I_n.\end{aligned}\tag{2.4}$$

This completes the proof of the proposition. □

Corollary 2.2. *Let $k=3$. Then system (1.3) for $k=3$ has the following invariant:*

$$\begin{aligned}
I_n ={}& b_1x_1(n)+b_2x_2(n)+b_3x_3(n)+b_1x_1(n-1)+b_2x_2(n-1)\\
&+b_3x_3(n-1)+(b_2b_3+b_1a_2a_3)\frac{1}{x_1(n)}+(b_3b_1+b_2a_3a_1)\frac{1}{x_2(n)}\\
&+(b_1b_2+b_3a_1a_2)\frac{1}{x_3(n)}+(b_2b_3+b_1a_2a_3)\frac{1}{x_1(n-1)}\\
&+(b_3b_1+b_2a_3a_1)\frac{1}{x_2(n-1)}+(b_1b_2+b_3a_1a_2)\frac{1}{x_3(n-1)}\\
&+b_1a_2b_3\frac{1}{x_1(n)x_3(n-1)}+b_2a_3b_1\frac{1}{x_2(n)x_1(n-1)}\\
&+b_3a_1b_2\frac{1}{x_3(n)x_2(n-1)}+b_2a_3\frac{x_3(n-1)}{x_1(n)}+b_3a_1\frac{x_1(n-1)}{x_2(n)}\\
&+b_1a_2\frac{x_2(n-1)}{x_3(n)}+b_2a_1\frac{x_1(n)}{x_3(n-1)}+b_3a_2\frac{x_2(n)}{x_1(n-1)}+b_1a_3\frac{x_3(n)}{x_2(n-1)}.
\end{aligned}\tag{2.5}$$

Proof. From (2.1) and (2.2), we get $\lambda_2b_2=\lambda_1b_3$, $\lambda_3b_3=\lambda_2b_1$, $\lambda_1b_1=\lambda_3b_2$. We set $\lambda_1=b_2$, $\lambda_2=b_3$, $\lambda_3=b_1$. Then from (2.3), the proof follows immediately. □

Corollary 2.3. *Let $k=4$. Suppose that*

$$b_1=b_2=b_3=b_4=b.\tag{2.6}$$

Then system (1.3) for $k=4$ has an invariant of the form

$$\begin{aligned}
I_n ={}& a_1x_1(n)+a_2x_2(n)+a_3x_3(n)+a_4x_4(n)+a_1x_1(n-1)\\
&+a_2x_2(n-1)+a_3x_3(n-1)+a_4x_4(n-1)+(a_3b+a_4a_2a_3)\frac{1}{x_1(n)}\\
&+(a_4b+a_4a_3a_1)\frac{1}{x_2(n)}+(a_1b+a_4a_1a_2)\frac{1}{x_3(n)}\\
&+(a_2b+a_3a_1a_2)\frac{1}{x_4(n)}+(a_3b+a_4a_2a_3)\frac{1}{x_1(n-1)}+(a_4b+a_4a_3a_1)\frac{1}{x_2(n-1)}\\
&+(a_1b+a_4a_1a_2)\frac{1}{x_3(n-1)}+(a_2b+a_3a_1a_2)\frac{1}{x_4(n-1)}\\
&+a_3a_2b\frac{1}{x_1(n)x_4(n-1)}+a_3a_4b\frac{1}{x_2(n)x_1(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&+a_1a_4b\frac{1}{x_3(n)x_2(n-1)}+a_1a_2b\frac{1}{x_4(n)x_3(n-1)}\\
&+a_1a_4\frac{x_1(n-1)}{x_2(n)}+a_1a_2\frac{x_2(n-1)}{x_3(n)}+a_3a_2\frac{x_3(n-1)}{x_4(n)}+a_3a_4\frac{x_4(n-1)}{x_1(n)}\\
&+a_1a_4\frac{x_4(n)}{x_3(n-1)}+a_3a_4\frac{x_3(n)}{x_2(n-1)}+a_3a_2\frac{x_2(n)}{x_1(n-1)}+a_1a_2\frac{x_1(n)}{x_4(n-1)}.
\end{aligned}
\tag{2.7}
$$

Proof. From (2.1), (2.2), and (2.6), we obtain

$$
\begin{gathered}
\lambda_2b+\lambda_3a_4a_3=\lambda_2b+\lambda_1a_4a_1,\\
\lambda_1b+\lambda_2a_2a_3=\lambda_1b+\lambda_4a_4a_3,\\
\lambda_3b+\lambda_4a_4a_1=\lambda_3b+\lambda_2a_2a_1,\\
\lambda_4b+\lambda_1a_2a_1=\lambda_4b+\lambda_3a_2a_3,\\
\lambda_1a_4b=\lambda_2a_3b,\\
\lambda_2a_1b=\lambda_3a_4b,\\
\lambda_3a_2b=\lambda_4a_1b,\\
\lambda_4a_3b=\lambda_1a_2b.
\end{gathered}
\tag{2.8}
$$

We set in (2.8) $\lambda_1=a_3$, $\lambda_2=a_4$, $\lambda_3=a_1$, $\lambda_4=a_2$. Then from (2.3), the proof follows immediately. ☐

Corollary 2.4. *Consider system (1.3), where $k=5$. Suppose that*

$$
\begin{aligned}
a_4a_5&=b_2,\\
a_3a_2&=b_5,\\
a_5a_1&=b_3,\\
a_4a_3&=b_1,\\
a_1a_2&=b_4.
\end{aligned}
\tag{2.9}
$$

Then system (1.3), with $k=5$, has an invariant of the form

$$
\begin{aligned}
I_n={}&\lambda_3x_1(n)+\lambda_4x_2(n)+\lambda_5x_3(n)+\lambda_1x_4(n)+\lambda_2x_5(n)\\
&+\lambda_3x_1(n-1)+\lambda_4x_2(n-1)+\lambda_5x_3(n-1)\\
&+\lambda_1x_4(n-1)+\lambda_2x_5(n-1)\\
&+(\lambda_1a_2a_3+\lambda_5a_4a_5)\frac{1}{x_1(n)}+(\lambda_2a_3a_4+\lambda_1a_5a_1)\frac{1}{x_2(n)}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\lambda_3 a_4 a_5 + \lambda_2 a_1 a_2) \frac{1}{x_3(n)} + (\lambda_4 a_1 a_5 + \lambda_3 a_2 a_3) \frac{1}{x_4(n)} \\
&+ (\lambda_5 a_1 a_2 + \lambda_4 a_3 a_4) \frac{1}{x_5(n)} + (\lambda_1 a_2 a_3 + \lambda_5 a_4 a_5) \frac{1}{x_1(n-1)} \\
&+ (\lambda_2 a_3 a_4 + \lambda_1 a_5 a_1) \frac{1}{x_2(n-1)} + (\lambda_3 a_4 a_5 + \lambda_2 a_1 a_2) \frac{1}{x_3(n-1)} \\
&+ (\lambda_4 a_1 a_5 + \lambda_3 a_2 a_3) \frac{1}{x_4(n-1)} + (\lambda_5 a_1 a_2 + \lambda_4 a_3 a_4) \frac{1}{x_5(n-1)} \\
&+ \lambda_5 a_4 a_3 a_2 \frac{1}{x_1(n) x_5(n-1)} + \lambda_1 a_5 a_4 a_3 \frac{1}{x_2(n) x_1(n-1)} \\
&+ \lambda_2 a_1 a_4 a_5 \frac{1}{x_3(n) x_2(n-1)} + \lambda_3 a_2 a_5 a_1 \frac{1}{x_4(n) x_3(n-1)} \\
&+ \lambda_4 a_3 a_1 a_2 \frac{1}{x_5(n) x_4(n-1)} \\
&+ \lambda_1 a_5 \frac{x_5(n-1)}{x_1(n)} + \lambda_2 a_1 \frac{x_1(n-1)}{x_2(n)} + \lambda_3 a_2 \frac{x_2(n-1)}{x_3(n)} \\
&+ \lambda_4 a_3 \frac{x_3(n-1)}{x_4(n)} + \lambda_5 a_4 \frac{x_4(n-1)}{x_5(n)} \\
&+ \lambda_4 a_1 \frac{x_1(n)}{x_5(n-1)} + \lambda_5 a_2 \frac{x_2(n)}{x_1(n-1)} + \lambda_1 a_3 \frac{x_3(n)}{x_2(n-1)} \\
&+ \lambda_2 a_4 \frac{x_4(n)}{x_3(n-1)} + \lambda_3 a_5 \frac{x_5(n)}{x_4(n-1)},
\end{aligned}
\tag{2.10}
$$

where λ_i, $i = 1,2,3,4,5$, are real numbers.

Proof. Using (2.1), (2.2), and (2.9), we get

$$
\begin{aligned}
\lambda_1 a_1 a_5 + \lambda_2 a_4 a_3 &= \lambda_2 a_3 a_4 + \lambda_1 a_5 a_1, \\
\lambda_2 a_2 a_1 + \lambda_3 a_4 a_5 &= \lambda_3 a_4 a_5 + \lambda_2 a_1 a_2, \\
\lambda_5 a_4 a_5 + \lambda_1 a_3 a_2 &= \lambda_1 a_2 a_3 + \lambda_5 a_4 a_5, \\
\lambda_3 a_3 a_2 + \lambda_4 a_1 a_5 &= \lambda_4 a_1 a_5 + \lambda_3 a_2 a_3, \\
\lambda_4 a_4 a_3 + \lambda_5 a_2 a_1 &= \lambda_5 a_1 a_2 + \lambda_4 a_3 a_4, \\
\lambda_1 a_3 a_4 a_5 &= \lambda_1 a_3 a_4 a_5, \\
\lambda_2 a_4 a_5 a_1 &= \lambda_2 a_4 a_5 a_1, \\
\lambda_3 a_5 a_1 a_2 &= \lambda_3 a_5 a_1 a_2, \\
\lambda_4 a_1 a_2 a_3 &= \lambda_4 a_1 a_2 a_3, \\
\lambda_5 a_2 a_3 a_4 &= \lambda_5 a_2 a_3 a_4,
\end{aligned}
\tag{2.11}
$$

which are satisfied for any real numbers λ_i, $i = 1,2,3,4,5$. Then from (2.3), the corollary is proved. □

3. Periodicity

We study the periodicity of the positive solutions of (1.3) by investigating three cases: $k = 3$, $k = 4$, and $k \in \{5,6,\dots\}$. For the first case, we show the following proposition.

Proposition 3.1. *Consider system (1.3) for $k = 3$. If*

$$\begin{aligned} a_1 &= a_2 = a_3 = a, \\ b_1 &= b_2 = b_3 = b, \\ a^2 &= b, \end{aligned} \tag{3.1}$$

then every positive solution of system (1.3) is periodic of period 15.

Proof. We have

$$\begin{aligned} x_1(n+5) &= \frac{ax_3(n+4)+a^2}{x_2(n+3)} = \frac{a\big((ax_2(n+3)+a^2)/x_1(n+2)\big)+a^2}{x_2(n+3)} \\ &= \frac{a^2x_2(n+3)+a^3+a^2x_1(n+2)}{x_1(n+2)x_2(n+3)} \\ &= \frac{a^2\big((ax_1(n+2)+a^2)/x_3(n+1)\big)+a^3+a^2x_1(n+2)}{x_1(n+2)\big((ax_1(n+2)+a^2)/x_3(n+1)\big)} \\ &= \frac{a^3x_1(n+2)+a^4+a^3x_3(n+1)+a^2x_1(n+2)x_3(n+1)}{x_1(n+2)\big[ax_1(n+2)+a^2\big]} \\ &= \frac{\big[ax_1(n+2)+a^2\big]\big[ax_3(n+1)+a^2\big]}{x_1(n+2)\big[ax_1(n+2)+a^2\big]} \\ &= \frac{ax_3(n+1)+a^2}{x_1(n+2)} = x_2(n). \end{aligned} \tag{3.2}$$

Working in a similar way, we can prove that

$$\begin{aligned} x_2(n+5) &= x_3(n), \\ x_3(n+5) &= x_1(n). \end{aligned} \tag{3.3}$$

Thus,

$$x_1(n+15) = x_2(n+10) = x_3(n+5) = x_1(n). \tag{3.4}$$

Similarly,

$$\begin{aligned} x_2(n+15) &= x_2(n), \\ x_3(n+15) &= x_3(n), \end{aligned} \tag{3.5}$$

and the proof of the proposition is complete. □

In the sequel, we prove the following proposition which concerns the case $k = 4$.

Proposition 3.2. *Consider system (1.3) for $k = 4$. If*

$$\begin{aligned} &a_1 = a_2 = a_3 = a_4 = a, \\ &b_1 = b_2 = b_3 = b_4 = b, \\ &a^2 = b, \end{aligned} \tag{3.6}$$

then every positive solution of system (1.3) is periodic of period 20.

Proof. We have

$$\begin{aligned} x_1(n+5) &= \frac{ax_4(n+4)+a^2}{x_3(n+3)} = \frac{a\big((ax_3(n+3)+a^2)/x_2(n+2)\big)+a^2}{x_3(n+3)} \\ &= \frac{a^2x_3(n+3)+a^3+a^2x_2(n+2)}{x_2(n+2)x_3(n+3)} \\ &= \frac{a^2\big((ax_2(n+2)+a^2)/x_1(n+1)\big)+a^3+a^2x_2(n+2)}{x_2(n+2)\big((ax_2(n+2)+a^2)/x_1(n+1)\big)} \\ &= \frac{a^3x_2(n+2)+a^4+a^3x_1(n+1)+a^2x_1(n+1)x_2(n+2)}{x_2(n+2)\big[ax_2(n+2)+a^2\big]} \\ &= \frac{\big[ax_2(n+2)+a^2\big]\big[ax_1(n+1)+a^2\big]}{x_2(n+2)\big[ax_2(n+2)+a^2\big]} \\ &= \frac{ax_1(n+1)+a^2}{x_2(n+2)} = x_4(n). \end{aligned} \tag{3.7}$$

Arguing as above, we can show that

$$\begin{aligned} x_2(n+5) &= x_1(n), \\ x_3(n+5) &= x_2(n), \\ x_4(n+5) &= x_3(n). \end{aligned} \tag{3.8}$$

So,

$$x_1(n+20) = x_4(n+15) = x_3(n+10) = x_2(n+5) = x_1(n). \tag{3.9}$$

Similarly,

$$\begin{aligned} x_2(n+20) &= x_2(n), \\ x_3(n+20) &= x_3(n), \\ x_4(n+20) &= x_4(n), \end{aligned} \tag{3.10}$$

and the proof of the proposition is complete. □

Finally, we study the case $k \in \{5,6,\dots\}$. To this end, we have at first to prove the following lemma.

Lemma 3.3. *Let $k \geq 5$. If*

$$a_1 = a_2 = \cdots = a_k = a, \qquad b_1 = b_2 = \cdots = b_k = b, \qquad a^2 = b \tag{3.11}$$

then

$$\begin{aligned} x_i(n+5) &= x_{k-5+i}(n), \quad i \in \{1,2,\dots,5\}, \\ x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,k\}. \end{aligned} \tag{3.12}$$

Proof. From (1.3), we have

$$\begin{aligned} x_1(n+5) &= \frac{ax_k(n+4)+a^2}{x_{k-1}(n+3)} = \frac{a\left(\left(ax_{k-1}(n+3)+a^2\right)/x_{k-2}(n+2)\right)+a^2}{x_{k-1}(n+3)} \\ &= \frac{a^2x_{k-1}(n+3)+a^3+a^2x_{k-2}(n+2)}{x_{k-2}(n+2)x_{k-1}(n+3)} \\ &= \frac{a^2\left(\left(ax_{k-2}(n+2)+a^2\right)/x_{k-3}(n+1)\right)+a^3+a^2x_{k-2}(n+2)}{x_{k-2}(n+2)\left(\left(ax_{k-2}(n+2)+a^2\right)/x_{k-3}(n+1)\right)} \\ &= \frac{a^3x_{k-2}(n+2)+a^4+a^3x_{k-3}(n+1)+a^2x_{k-2}(n+2)x_{k-3}(n+1)}{x_{k-2}(n+2)\left(ax_{k-2}(n+2)+a^2\right)} \\ &= \frac{\left(ax_{k-2}(n+2)+a^2\right)\left(ax_{k-3}(n+1)+a^2\right)}{x_{k-2}(n+2)\left(ax_{k-2}(n+2)+a^2\right)}. \end{aligned} \tag{3.13}$$

Then since, from (1.3),

$$x_{k-4}(n) = \frac{ax_{k-3}(n+1)+a^2}{x_{k-2}(n+2)}, \tag{3.14}$$

it follows that

$$x_1(n+5) = x_{k-4}(n). \tag{3.15}$$

Similarly, we can prove that

$$x_i(n+5) = x_{k-5+i}(n), \quad i \in \{2,3,\dots,5\}. \tag{3.16}$$

Let $i \in \{6,7,\dots,k\}$. Then

$$\begin{aligned}
x_i(n+5) &= \frac{ax_{i-1}(n+4)+a^2}{x_{i-2}(n+3)} = \frac{a\big((ax_{i-2}(n+3)+a^2)/x_{i-3}(n+2)\big)+a^2}{x_{i-2}(n+3)} \\
&= \frac{a^2x_{i-2}(n+3)+a^3+a^2x_{i-3}(n+2)}{x_{i-2}(n+3)x_{i-3}(n+2)} \\
&= \frac{a^2\big((ax_{i-3}(n+2)+a^2)/x_{i-4}(n+1)\big)+a^3+a^2x_{i-3}(n+2)}{x_{i-3}(n+2)\big((ax_{i-3}(n+2)+a^2)/x_{i-4}(n+1)\big)} \\
&= \frac{a^3x_{i-3}(n+2)+a^4+a^3x_{i-4}(n+1)+a^2x_{i-3}(n+2)x_{i-4}(n+1)}{x_{i-3}(n+2)\big(ax_{i-3}(n+2)+a^2\big)} \\
&= \frac{\big(ax_{i-3}(n+2)+a^2\big)\big(ax_{i-4}(n+1)+a^2\big)}{x_{i-3}(n+2)\big(ax_{i-3}(n+2)+a^2\big)}.
\end{aligned} \tag{3.17}$$

Then since, from (1.3),

$$x_{i-5}(n) = \frac{ax_{i-4}(n+1)+a^2}{x_{i-3}(n+2)}, \tag{3.18}$$

it follows that

$$x_i(n+5) = x_{i-5}(n), \quad i \in \{6,7,\dots,k\}. \tag{3.19}$$

Now we can show the following proposition. □

Proposition 3.4. *Consider system (1.3), where $k \geq 5$. Assume that relations (3.11) hold. Then the following statements are true.*

(i) *Every positive solution of system (1.3) is periodic of period k if $k = 5r$, $r = 1,2,\dots$.*

(ii) *Every positive solution of system (1.3) is periodic of period $5k$ if $k \neq 5r$, $r = 1,2,\dots$.*

Proof. Consider an arbitrary solution $(x_1(n),\dots,x_k(n))$ of (1.3).

(i) Suppose that $k = 5r$, $r = 1,2,\dots$. Then from (3.12), we have

$$\begin{aligned}
x_i(n+5) &= x_{5r-5+i}(n), \quad i \in \{1,2,\dots,5\}, \\
x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,5r\}.
\end{aligned} \tag{3.20}$$

We claim that for $i = 1,2,\dots,5$,

$$x_i(n+5s) = x_{5r-5s+i}(n), \quad s = 1,2,\dots,r. \tag{3.21}$$

From (3.20), it is obvious that (3.21) is true for $s = 1$. Suppose that for $i = 1,2,\dots,5$, relation (3.21) is true for $s = 1,2,\dots,r-1$. Then since $6 \leq 5r-5s+i \leq 5r$, from (3.20) and (3.21), we get for $i = 1,2,\dots,5$,

$$x_i(n+5+5s) = x_{5r-5s+i}(n+5) = x_{5r-5(s+1)+i}(n), \tag{3.22}$$

and so (3.21) is true. Then from (3.21) for $s = r$, we have

$$x_i(n+5r) = x_i(n), \quad i = 1,2,\dots,5. \tag{3.23}$$

Therefore the sequences $x_i(n)$, $i = 1,2,\dots,5$ are periodic of period 5. Then from (3.20), all the sequences $x_i(n)$, $i = 1,2,\dots,k$, are periodic of period k.

(ii) Suppose that $k \neq 5r$, $r = 1,2,\dots$. Let $k = 5r+1$, $r = 1,2,\dots$. Then from (3.12), we have

$$\begin{aligned} x_i(n+5) &= x_{5r-4+i}(n), \quad i \in \{1,2,\dots,5\}, \\ x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,5r+1\}. \end{aligned} \tag{3.24}$$

Applying (3.24) and using the same argument to show (3.21), we can prove that for $i = 1,2,\dots,5$

$$x_i(n+5s) = x_{5r-5s+i+1}(n), \quad s = 1,2,\dots,r. \tag{3.25}$$

So from (3.24) and (3.25) for $i = 1$, $s = r$, we get

$$\begin{gathered} x_1(n+25r+5) = x_2(n+20r+5) = x_3(n+15r+5) = x_4(n+10r+5), \\ x_5(n+5r+5) = x_6(n+5) = x_1(n). \end{gathered} \tag{3.26}$$

Therefore $x_1(n)$ is a periodic sequence of period $5(5r+1) = 5k$. Hence by (3.24), all the sequences $x_i(n)$, $i = 1,2,\dots,k$, are periodic of period $5k$.

Let $k = 5r+2$. Then from (3.12), we have

$$\begin{aligned} x_i(n+5) &= x_{5r-3+i}(n), \quad i \in \{1,2,\dots,5\}, \\ x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,5r+2\}. \end{aligned} \tag{3.27}$$

Then from (3.27) and using the same argument to prove (3.21), we can prove that for $i = 1,2,\dots,5$,

$$x_i(n+5s) = x_{5r-5s+i+2}(n), \quad s = 1,2,\dots,r. \tag{3.28}$$

Then from (3.27) and (3.28) for $i = 1$, $s = r$, we get

$$\begin{aligned} x_1(n+25r+10) &= x_3(n+20r+10) = x_5(n+15r+10) \\ &= x_7(n+10r+10) = x_2(n+10r+5) = x_4(n+5r+5) \\ &= x_6(n+5) = x_1(n), \end{aligned} \tag{3.29}$$

which implies that $x_1(n)$ is a periodic sequence of period $5k$. Then by relations (3.27), we can prove that the sequences $x_i(n)$, $i = 2,3,\dots,k$, are periodic of period $5k$.

Let $k = 5r+3$. Then from (3.12), we have

$$\begin{aligned} x_i(n+5) &= x_{5r-2+i}(n), \quad i \in \{1,2,\dots,5\}, \\ x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,5r+3\}. \end{aligned} \tag{3.30}$$

Then from (3.30) and using the same argument to show (3.21), we can prove that for $i = 1,2,\dots,5$,

$$x_i(n+5s) = x_{5r-5s+i+3}(n), \quad s = 1,2,\dots,r. \tag{3.31}$$

Then from (3.30) and (3.31) for $i = 1$, $s = r$, we get

$$\begin{aligned} x_1(n+25r+15) &= x_4(n+20r+15) = x_7(n+15r+15) \\ &= x_2(n+15r+10) = x_5(n+10r+10) \\ &= x_8(n+5r+10) = x_3(n+5r+5) = x_6(n+5) = x_1(n), \end{aligned} \tag{3.32}$$

which implies that $x_1(n)$ is a periodic sequence of period $5(5r+3) = 5k$. Then from relations (3.30), we can prove that the sequences $x_i(n)$, $i = 1,2,\dots,k$, are periodic of period $5k$.

Let $k = 5r+4$. Then from (3.12), we have

$$\begin{aligned} x_i(n+5) &= x_{5r-1+i}(n), \quad i \in \{1,2,\dots,5\}, \\ x_i(n+5) &= x_{i-5}(n), \quad i \in \{6,7,\dots,5r+4\}. \end{aligned} \tag{3.33}$$

Then from (3.33) and using the same argument to show (3.21), we can prove that for $i = 1,2,\dots,5$,

$$x_i(n+5s) = x_{5r-5s+i+4}(n), \quad s = 1,2,\dots,r. \tag{3.34}$$

Then from (3.33) and (3.34) for $i = 1$, $s = r$, we get

$$\begin{aligned} x_1(n+25r+20) &= x_5(n+20r+20) = x_9(n+15r+20) \\ &= x_4(n+15r+15) = x_8(n+10r+15) = x_3(n+10r+10) \\ &= x_7(n+5r+10) = x_2(n+5r+5) = x_6(n+5) = x_1(n), \end{aligned} \tag{3.35}$$

which implies that $x_1(n)$ is a periodic sequence of period $5(5r+4) = 5k$. Then from relations (3.33), we can prove that the sequences $x_i(n)$, $i = 1,2,\dots,k$, are periodic of period $5k$. This completes the proof of the proposition. □

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G. Papaschinopoulos: School of Engineering, Democritus University of Thrace,
67100 Xanthi, Greece
Email address: gpapas@ee.duth.gr

C. J. Schinas: School of Engineering, Democritus University of Thrace,
67100 Xanthi, Greece
Email address: cschinas@ee.duth.gr

G. Stefanidou: School of Engineering, Democritus University of Thrace,
67100 Xanthi, Greece
Email address: gstefmath@yahoo.gr