Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 256165, 10 pages
doi:10.1155/2009/256165

Research Article

A Fixed Point Approach to the Stability of a Quadratic Functional Equation in C^*-Algebras

Mohammad B. Moghimi,[1] Abbas Najati,[1] and Choonkil Park[2]

[1] *Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, 56199-11367 Ardabil, Iran*
[2] *Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea*

Correspondence should be addressed to Abbas Najati, a.nejati@yahoo.com

Received 18 May 2009; Accepted 31 July 2009

Recommended by Tocka Diagana

We use a fixed point method to investigate the stability problem of the quadratic functional equation $f(x+y)+f(x-y)=2f(\sqrt{xx^*+yy^*})$ in C^*-algebras.

1. Introduction and Preliminaries

In 1940, the following question concerning the stability of group homomorphisms was proposed by Ulam [1]: *Under what conditions does there exist a group homomorphism near an approximately group homomorphism?* In 1941, Hyers [2] considered the case of approximately additive functions $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon \tag{1.1}$$

for all $x, y \in E$. Aoki [3] and Th. M. Rassias [4] provided a generalization of the Hyers' theorem for additive mappings and for linear mappings, respectively, by allowing the Cauchy difference to be unbounded (see also [5]).

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon\left(\|x\|^p+\|y\|^p\right) \tag{1.2}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.3}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$ then inequality (1.2) *holds for $x, y \ne 0$ and* (1.4) *for $x \ne 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.*

The result of the Th. M. Rassias theorem has been generalized by Găvruţa [6] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [7–20]). We also refer the readers to the books [21–25]. A *quadratic functional equation* is a functional equation of the following form:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.5}$$

In particular, every solution of the quadratic equation (1.5) is said to be a *quadratic mapping.* It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x, x)$ for all x (see [16, 21, 26, 27]. The biadditive mapping B is given by

$$B(x, y) = \frac{1}{4}[f(x + y) - f(x - y)]. \tag{1.6}$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.5) was studied by Skof [28] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if we replace E_1 by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.5). Grabiec [11] has generalized these results mentioned above. Jun and Lee [14] proved the generalized Hyers-Ulam stability of a Pexiderized quadratic functional equation.

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2 (see [29]). *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.7}$$

for all nonnegative integers n or there exists a non-negative integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in E : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper A will be a C^*-algebra. We denote by $\sqrt{a}$ the unique positive element $b \in A$ such that $b^2 = a$ for each positive element $a \in A$. Also, we denote by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ the set of real, complex, and rational numbers, respectively. In this paper, we use a fixed point method (see [7, 15, 17]) to investigate the stability problem of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f\left(\sqrt{xx^* + yy^*}\right) \tag{1.8}$$

in C^*-algebras. A systematic study of fixed point theorems in nonlinear analysis is due to Hyers et al. [30] and Isac and Rassias [13].

2. Solutions of (1.8)

Theorem 2.1. *Let X be a linear space. If a mapping $f : A \to X$ satisfies $f(0) = 0$ and the functional equation* (1.8), *then f is quadratic.*

Proof. Letting $u = x + y$ and $v = x - y$ in (1.8), respectively, we get

$$f(u) + f(v) = 2f\left(\sqrt{\frac{uu^* + vv^*}{2}}\right) \tag{2.1}$$

for all $u, v \in A$. It follows from (1.8) and (2.1) that

$$f(u) + f(v) = f\left(\frac{u+v}{\sqrt{2}}\right) + f\left(\frac{u-v}{\sqrt{2}}\right) \tag{2.2}$$

for all $u, v \in A$. Letting $v = 0$ in (2.2), we get

$$2f\left(\frac{u}{\sqrt{2}}\right) = f(u) \tag{2.3}$$

for all $u \in A$. Thus (2.2) implies that

$$f(u+v) + f(u-v) = 2f(u) + 2f(v) \tag{2.4}$$

for all $u, v \in A$. Hence f is quadratic. □

Remark 2.2. A quadratic mapping does not satisfy (1.8) in general. Let $f : A \to A$ be the mapping defined by $f(x) = x^2$ for all $x \in A$. It is clear that f is quadratic and that f does not satisfy (1.8).

Corollary 2.3. *Let X be a linear space. If a mapping $f : A \to X$ satisfies the functional equation* (1.8)*, then there exists a symmetric biadditive mapping $B : A \times A \to X$ such that $f(x) = B(x, x)$ for all $x \in A$.*

3. Generalized Hyers-Ulam Stability of (1.8) in C^*-Algebras

In this section, we use a fixed point method (see [7, 15, 17]) to investigate the stability problem of the functional equation (1.8) in C^*-algebras.

For convenience, we use the following abbreviation for a given mapping $f : A \to X$:

$$Df(x,y) := f(x+y) + f(x-y) - 2f\left(\sqrt{xx^* + yy^*}\right) \tag{3.1}$$

for all $x, y \in A$, where X is a linear space.

Theorem 3.1. *Let X be a linear space and let $f : A \to X$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.2}$$

for all $x, y \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\sqrt{2}x, \sqrt{2}y\right) \le 2L\varphi(x,y) \tag{3.3}$$

for all $x, y \in A$, then there exists a unique quadratic mapping $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2-2L}\phi(x) \tag{3.4}$$

for all $x \in A$, where

$$\phi(x) := \varphi(x, 0) + \varphi\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right). \tag{3.5}$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic, that is, $Q(tx) = t^2 Q(x)$ for all $x \in A$ and all $t \in \mathbb{R}$.

Proof. Replacing x and y by $(x+y)/2$ and $(x-y)/2$ in (3.2), respectively, we get

$$\left\| f(x)+f(y)-2f\left(\sqrt{\frac{xx^*+yy^*}{2}}\right)\right\| \le \varphi\left(\frac{x+y}{2},\frac{x-y}{2}\right) \tag{3.6}$$

for all $x,y\in A$. Replacing x and y by $x/\sqrt{2}$ and $y/\sqrt{2}$ in (3.2), respectively, we get

$$\left\| f\left(\frac{x+y}{\sqrt{2}}\right)+f\left(\frac{x-y}{\sqrt{2}}\right)-2f\left(\sqrt{\frac{xx^*+yy^*}{2}}\right)\right\| \le \varphi\left(\frac{x}{\sqrt{2}},\frac{y}{\sqrt{2}}\right) \tag{3.7}$$

for all $x,y\in A$. It follows from (3.6) and (3.7) that

$$\left\| f\left(\frac{x+y}{\sqrt{2}}\right)+f\left(\frac{x-y}{\sqrt{2}}\right)-f(x)-f(y)\right\| \le \varphi\left(\frac{x+y}{2},\frac{x-y}{2}\right)+\varphi\left(\frac{x}{\sqrt{2}},\frac{y}{\sqrt{2}}\right) \tag{3.8}$$

for all $x,y\in A$. Letting $y=x$ in (3.8), we get

$$\left\| f\left(\sqrt{2}x\right)-2f(x)\right\| \le \varphi(x,0)+\varphi\left(\frac{x}{\sqrt{2}},\frac{x}{\sqrt{2}}\right) \tag{3.9}$$

for all $x\in A$. By (3.3) we have $\phi(\sqrt{2}x)\le 2L\phi(x)$ for all $x\in A$. Let E be the set of all mappings $g:A\to X$ with $g(0)=0$. We can define a generalized metric on E as follows:

$$d(g,h) := \inf\{C\in[0,\infty] : \|g(x)-h(x)\| \le C\phi(x)\ \forall x\in A\}. \tag{3.10}$$

(E,d) is a generalized complete metric space [7].

Let $\Lambda : E\to E$ be the mapping defined by

$$(\Lambda g)(x) = \frac{1}{2}g\left(\sqrt{2}x\right) \quad \forall g\in E \text{ and all } x\in A. \tag{3.11}$$

Let $g,h\in E$ and let $C\in[0,\infty]$ be an arbitrary constant with $d(g,h)\le C$. From the definition of d, we have

$$\|g(x)-h(x)\| \le C\phi(x) \tag{3.12}$$

for all $x\in A$. Hence

$$\|(\Lambda g)(x)-(\Lambda h)(x)\| = \frac{1}{2}\left\|g\left(\sqrt{2}x\right)-h\left(\sqrt{2}x\right)\right\| \le \frac{1}{2}C\phi\left(\sqrt{2}x\right) \le CL\phi(x) \tag{3.13}$$

for all $x\in A$. So

$$d(\Lambda g,\Lambda h) \le Ld(g,h) \tag{3.14}$$

for any $g,h \in E$. It follows from (3.9) that $d(\Lambda f, f) \leq 1/2$. According to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point Q of Λ, that is,

$$Q : A \to X, \qquad Q(x) = \lim_{k\to\infty}\left(\Lambda^k f\right)(x) = \lim_{k\to\infty}\frac{1}{2^k} f\left(2^{k/2}x\right), \tag{3.15}$$

and $Q(\sqrt{2}x) = 2Q(x)$ for all $x \in A$. Also,

$$d(Q,f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{2-2L}, \tag{3.16}$$

and Q is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f,g) < \infty\}$. Thus the inequality (3.4) holds true for all $x \in A$. It follows from the definition of Q, (3.2), and (3.3) that

$$\|DQ(x,y)\| = \lim_{k\to\infty}\frac{1}{2^k}\|Df\left(2^{k/2}x, 2^{k/2}y\right)\| \leq \lim_{k\to\infty}\frac{1}{2^k}\varphi\left(2^{k/2}x, 2^{k/2}y\right) = 0 \tag{3.17}$$

for all $x,y \in A$. By Theorem 2.1, the function $Q : A \to X$ is quadratic.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then by the same reasoning as in the proof of [4] Q is $\mathbb{R}$-quadratic. □

Corollary 3.2. *Let $0 < r < 2$ and θ, δ be non-negative real numbers and let $f : A \to X$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y)\| \leq \delta + \theta\left(\|x\|^r + \|y\|^r\right) \tag{3.18}$$

for all $x,y \in A$. Then there exists a unique quadratic mapping $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\delta}{2-2^{r/2}} + \frac{2+2^{r/2}}{2^{r/2}\left(2-2^{r/2}\right)}\theta\|x\|^r \tag{3.19}$$

for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

The following theorem is an alternative result of Theorem 3.1 and we will omit the proof.

Theorem 3.3. *Let $f : A \to X$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \to [0,\infty)$ satisfying (3.2) for all $x,y \in A$. If there exists a constant $0 < L < 1$ such that*

$$2\varphi(x,y) \leq L\varphi\left(\sqrt{2}x, \sqrt{2}y\right) \tag{3.20}$$

for all $x,y \in A$, then there exists a unique quadratic mapping $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{2-2L}\phi(x) \tag{3.21}$$

for all $x \in A$, where $\phi(x)$ is defined as in Theorem 3.1. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

Corollary 3.4. *Let $r > 2$ and θ be non-negative real numbers and let $f : A \to X$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.22}$$

for all $x, y \in A$. Then there exists a unique quadratic mapping $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \le \frac{2 + 2^{r/2}}{2^{r/2}(2^{r/2} - 2)}\theta\|x\|^r \tag{3.23}$$

for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

For the case $r = 2$ we use the Gajda's example [31] to give the following counterexample (see also [9]).

Example 3.5. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} |x|^2, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \ge 1. \end{cases} \tag{3.24}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{4^n}\phi(2^n x). \tag{3.25}$$

It is clear that f is continuous and bounded by $4/3$ on $\mathbb{C}$. We prove that

$$\left| f(x+y) + f(x-y) - 2f\left(\sqrt{|x|^2 + |y|^2}\right)\right| \le \frac{64}{3}\left(|x|^2 + |y|^2\right) \tag{3.26}$$

for all $x, y \in \mathbb{C}$. To see this, if $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \ge 1/4$, then

$$\left| f(x+y) + f(x-y) - 2f\left(\sqrt{|x|^2 + |y|^2}\right)\right| \le \frac{16}{3} \le \frac{64}{3}\left(|x|^2 + |y|^2\right). \tag{3.27}$$

Now suppose that $0 < |x|^2 + |y|^2 < 1/4$. Then there exists a positive integer k such that

$$\frac{1}{4^{k+1}} \le |x|^2 + |y|^2 < \frac{1}{4^k}. \tag{3.28}$$

Thus

$$2^{k-1}|x \pm y|,\ 2^{k}\sqrt{|x|^2+|y|^2} \in (-1,1). \tag{3.29}$$

Hence

$$2^{m}|x \pm y|,\ 2^{m}\sqrt{|x|^2+|y|^2} \in (-1,1) \tag{3.30}$$

for all $m = 0, 1, \ldots, k-1$. It follows from the definition of f and (3.28) that

$$\begin{aligned}
&\left|f(x+y)+f(x-y)-2f\left(\sqrt{|x|^2+|y|^2}\right)\right| \\
&\quad = \left|\sum_{n=k}^{\infty}\frac{1}{4^n}\left[\phi(2^n(x+y))+\phi(2^n(x-y))-2\phi\left(2^n\sqrt{|x|^2+|y|^2}\right)\right]\right| \\
&\quad \le 4\sum_{n=k}^{\infty}\frac{1}{4^n} = \frac{64}{3\times 4^{k+1}} \le \frac{64}{3}\left(|x|^2+|y|^2\right).
\end{aligned} \tag{3.31}$$

Thus f satisfies (3.26). Let $Q : \mathbb{C} \to \mathbb{C}$ be a quadratic function such that

$$|f(x)-Q(x)| \le \beta|x|^2 \tag{3.32}$$

for all $x \in \mathbb{C}$, where β is a positive constant. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all $x \in \mathbb{Q}$. So we have

$$|f(x)| \le (\beta+|c|)|x|^2 \tag{3.33}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If $x_0 \in (0, 2^{-m}) \cap \mathbb{Q}$, then $2^n x_0 \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. So

$$f(x_0) \ge \sum_{n=0}^{m-1}\frac{1}{4^n}\phi(2^n x_0) = m|x_0|^2 > (\beta+|c|)|x_0|^2 \tag{3.34}$$

which contradicts (3.33).

Acknowledgment

The third author was supported by Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00041).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[6] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[7] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," in *Iteration Theory*, vol. 346 of *Grazer Mathematische Berichte*, pp. 43–52, Karl-Franzens-Universitaet Graz, Graz, Austria, 2004.

[8] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.

[9] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[10] V. A. Faĭziev, Th. M. Rassias, and P. K. Sahoo, "The space of (ψ, γ)-additive mappings on semigroups," *Transactions of the American Mathematical Society*, vol. 354, no. 11, pp. 4455–4472, 2002.

[11] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.

[12] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[13] G. Isac and Th. M. Rassias, "Stability of Ψ-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[14] K.-W. Jun and Y.-H. Lee, "On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality," *Mathematical Inequalities & Applications*, vol. 4, no. 1, pp. 93–118, 2001.

[15] S.-M. Jung and T.-S. Kim, "A fixed point approach to the stability of the cubic functional equation," *Boletín de la Sociedad Matemática Mexicana*, vol. 12, no. 1, pp. 51–57, 2006.

[16] Pl. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.

[17] M. Mirzavaziri and M. S. Moslehian, "A fixed point approach to stability of a quadratic equation," *Bulletin of the Brazilian Mathematical Society*, vol. 37, no. 3, pp. 361–376, 2006.

[18] C.-G. Park, "On the stability of the linear mapping in Banach modules," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 711–720, 2002.

[19] Th. M. Rassias, "On a modified Hyers-Ulam sequence," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 1, pp. 106–113, 1991.

[20] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[21] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[22] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[23] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[24] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[25] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[26] D. Amir, *Characterizations of Inner Product Spaces*, vol. 20 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1986.

[27] P. Jordan and J. von Neumann, "On inner products in linear, metric spaces," *Annals of Mathematics*, vol. 36, no. 3, pp. 719–723, 1935.

[28] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[29] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

[30] D. H. Hyers, G. Isac, and Th. M. Rassias, *Topics in Nonlinear Analysis & Applications*, World Scientific, River Edge, NJ, USA, 1997.

[31] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.