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Research Article

Maximal Regularity of the Discrete Harmonic Oscillator Equation

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We give a representation of the solution for the best approximation of the harmonic oscillator equation formulated in a general Banach space setting, and a characterization of l_p-maximal regularity—or well posedness—solely in terms of R-boundedness properties of the resolvent operator involved in the equation.

1. Introduction

In numerical integration of a differential equation, a standard approach is to replace it by a suitable difference equation whose solution can be obtained in a stable manner and without troubles from round off errors. However, often the qualitative properties of the solutions of the difference equation are quite different from the solutions of the corresponding differential equations.

For a given differential equation, a difference equation approximation is called *best* if the solution of the difference equation exactly coincides with solutions of the corresponding differential equation evaluated at a discrete sequence of points. Best approximations are not unique (cf. [1, Section 3.6]).

In the recent paper [2] (see also [1]), various discretizations of the harmonic oscillator equation $\ddot{y} + y = 0$ are compared. A best approximation is given by

$$\frac{\Delta^2 x_n}{(2\sin(\epsilon/2))^2} + x_{n+1} = 0, \tag{1.1}$$

where Δ denotes the forward difference operator of the first order, that is, for each $x : \mathbb{Z}_+ \to X$, and $n \in \mathbb{Z}_+$, $\Delta x_n = x_{n+1} - x_n$. On the other hand, in the article [3], a characterization of l_p-maximal regularity for a discrete second-order equation in Banach spaces was studied, but without taking into account the best approximation character of the equation. From an applied perspective, the techniques used in [3] are interesting when applied to concrete difference equations, but additional difficulties appear, because among other things, we need to get explicit formulas for the solution of the equation to be studied.

We study in this paper the discrete second-order equation

$$\Delta^2 x_n + A x_{n+1} = f_n, \tag{1.2}$$

on complex Banach spaces, where $A \in \mathcal{B}(X)$. Of course, in the finite-dimensional setting, (1.2) includes systems of linear difference equations, but the most interesting application concerns with partial difference equations. In fact, the homogeneous equation associated to (1.2) corresponds to the best discretization of the wave equation (cf. [1, Section 3.14]).

We prove that well posedness, that is, maximal regularity of (1.2) in l_p vector-valued spaces, is characterized on Banach spaces having the unconditional martingale difference property ((UMD) see, e.g., [4]) by the R-boundedness of the set

$$\left\{ \frac{(z-1)^2}{z}\left(\frac{(z-1)^2}{z} + A\right)^{-1} : |z| = 1,\ z \neq 1 \right\}. \tag{1.3}$$

The general framework for the proof of our statement uses a new approach based on operator-valued Fourier multipliers. In the continuous time setting, the relation between operator-valued Fourier multiplier and $R-$ boundedness of their symbols is well documented (see, e.g., [5–10]), but we emphasize that the discrete counterpart is too incipient and limited essentially a very few articles (see, e.g., [11, 12]). We believe that the development of this topic could have a strong applied potential. This would lead to very interesting problems related to difference equations arising in numerical analysis, for instance. From this perspective the results obtained in this work are, to the best of our knowledge, new.

We recall that in the continuous case, it is well known that the study of maximal regularity is very useful for treating semilinear and quasilinear problems. (see, e.g., Amann [13], Denk et al. [8], Clément et al. [14], the survey by Arendt [7] and the bibliography therein). However it should be noted that for nonlinear discrete time evolution equations some additional difficulties appear. In fact, we observe that this approach cannot be done by a direct translation of the proofs from the continuous time setting to the discrete time setting. Indeed, the former only allows to construct a solution on a (possibly very short) time interval, the global solution being then obtained by extension results. This technique will obviously fail in the discrete time setting, where no such thing as an arbitrary short time interval exists. In the recent work [15], the authors have found a way around the "short time interval" problem to treat semilinear problems for certain evolution equations of second order. One more case merits mentioning here is Volterra difference equations which describe processes whose current state is determined by their entire prehistory (see, e.g., [16, 17], and the references given there). These processes are encountered, for example, in mathematical models in population dynamics as well as in models of propagation of perturbation in matter with memory. In this direction one of the authors in [18] considered maximal regularity for Volterra difference equations with infinite delay.

The paper is organized as follows. The second section provides the definitions and preliminary results to be used in the theorems stated and proved in this work. In particular to facilitate a comprehensive understanding to the reader we have supplied several basic R-boundedness properties. In the third section, we will give a geometrical link for the best discretization of the harmonic oscillator equation. In the fourth section, we treat the existence and uniqueness problem for (1.2). In the fifth section, we obtain a characterization about maximal regularity for (1.2).

2. Preliminaries

Let X and Y be the Banach spaces, let $\mathcal{B}(X,Y)$ be the space of bounded linear operators from X into Y. Let $\mathbb{Z}_+$ denote the set of nonnegative integer numbers, Δ the forward difference operator of the first order, that is, for each $x : \mathbb{Z}_+ \to X$, and $n \in \mathbb{Z}_+$, $\Delta x_n = x_{n+1} - x_n$. We introduce the means

$$\|(x_1,\dots,x_n)\|_R := \frac{1}{2^n}\sum_{\epsilon_j\in\{-1,1\}^n}\left\|\sum_{j=1}^{n}\epsilon_j x_j\right\|, \tag{2.1}$$

for $x_1,\dots,x_n \in X$.

Definition 2.1. Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there exists a constant $c \geq 0$ such that

$$\|(T_1x_1,\dots,T_nx_n)\|_R \leq c\|(x_1,\dots,x_n)\|_R, \tag{2.2}$$

for all $T_1,\dots,T_n \in \mathcal{T},\ x_1,\dots,x_n \in X,\ n \in \mathbb{N}$. The least c such that (2.2) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

An equivalent definition using the Rademacher functions can be found in [8]. We note that R-boundedness clearly implies uniformly boundedness. In fact, we have that $\sup_{T\in\mathcal{T}}\|T\| \leq R(\mathcal{T})$. If $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [5, Proposition 1.17]. Some useful criteria for R-boundedness are provided in [5, 8, 19]. We remark that the concept of R-boundedness plays a fundamental role in recent works by Clément and Da Prato [20], Clément et al. [21], Weis [9, 10], Arendt and Bu [5, 6], as well as Keyantuo and Lizama [22–25].

Remark 2.2. (a) Let $\mathcal{S},\mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded sets, then $\mathcal{S}+\mathcal{T} := \{S+T : S\in\mathcal{S},\ T\in\mathcal{T}\}$ is R- bounded.

(b) Let $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded sets, then $\mathcal{S}\cdot\mathcal{T} := \{S\cdot T : S\in\mathcal{S},\ T\in\mathcal{T}\} \subset \mathcal{B}(X,Z)$ is R- bounded and

$$R(\mathcal{S}\cdot\mathcal{T}) \leq R(\mathcal{S})\cdot R(\mathcal{T}). \tag{2.3}$$

(c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded whenever $\Omega \subset \mathbb{C}$ is bounded.

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV\left(\frac{1}{t}\right) * f. \tag{2.4}$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well-known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [4] and Burkholder [26]. The following result on operator-valued Fourier multipliers on $\mathbb{T}$, due to Blunck [11], is the key for our purposes. Note that for $f \in l_p(\mathbb{Z}; X)$ the Fourier transform on $\mathbb{T}$ is defined as

$$\mathcal{F}f(z) = \widehat{f}(z) = \sum_{j\in\mathbb{Z}} z^{-j} f(j), \quad z \in \mathbb{T}. \tag{2.5}$$

Theorem 2.3. *Let $p \in (1,\infty)$ and X be a UMD space. Let $\mathcal{T} := (-\pi, 0) \cup (0, \pi)$ and $M : \mathcal{T} \to \mathcal{B}(X)$ be a differentiable function such that the set*

$$\{M(t),\ (e^{it} - 1)(e^{it} + 1)M'(t) : t \in \mathcal{T}\} \tag{2.6}$$

is R-bounded. Then $T_M \in \mathcal{B}(l_p(\mathbb{Z}_+; X))$ for the following Fourier multiplier T_M:

$$\widehat{T_M f}(e^{it}) := M(t)\widehat{f}(e^{it}), \quad t \in \mathcal{T},\ \widehat{f} \in L_\infty(\mathbb{T}; X) \text{ of compact support}. \tag{2.7}$$

Recall that $T \in \mathcal{B}(X)$ is called analytic if the set

$$\{n(T - I)T^n : n \in \mathbb{N}\} \tag{2.8}$$

is bounded. For recent and related results on analytic operators we refer to [27].

3. Spectral Properties and Open Problems

In this section we first give a geometrical link between the best discretization (1.2) and the equations of the form

$$\Delta^2 x_n + A x_{n+k} = f_n, \quad x_0 = x_1 = 0, \quad k \in \{0, 1, 2\}. \tag{3.1}$$

The motivation comes from the recent article of Cieśliński and Ratkiewicz [2], where several discretizations of second-order linear ordinary differential equations with constant coefficients are compared and discussed. More precisely, concerning the harmonic oscillator equation $\ddot{x} + x = 0$ the following three discrete equations are considered:

$$\begin{aligned}\Delta^2 x_n + \epsilon^2 x_n &= 0;\\ \Delta^2 x_n + \epsilon^2 x_{n+1} &= 0;\\ \Delta^2 x_n + \epsilon^2 x_{n+2} &= 0.\end{aligned} \tag{3.2}$$

In particular, it is proved in [2] that the best (called "exact" in that paper) discretization of the harmonic oscillator is given by

$$\Delta^2 x_n + \left(2\sin\left(\frac{\epsilon}{2}\right)\right)^2 x_{n+1} = 0, \tag{3.3}$$

which reminds the "symmetric" version of Euler's discretization scheme, but ϵ that appears in the discretization of the second derivative is replaced by $2\sin(\epsilon/2)$.

Remark 3.1. Observe that (3.1) can be rewritten as

$$x_{n+2} = 2x_{n+1} - x_n - Ax_{n+k} + f_n. \tag{3.4}$$

If $k \in \mathbb{Z}$ in (3.1), then we have a well-defined recurrence relation of order 2 in case $k = 0$ or 1 (and of order $(2-k)$) in case $k < 0$.

In case $k = 2$, we have $(I + A)x_{n+2} = 2x_{n+1} - x_n + f_n$, that is, a recurrence relation of order 2, which can be not well defined unless $-1 \in \rho(A)$. Finally, in case $k > 2$, $x_{n+k} = A^{-1}(2x_{n+1} - x_n - x_{n+2} + f_n)$ is of order k (note that here we need $0 \in \rho(A)$).

Taking (formally) Fourier transform to (3.1), we obtain

$$(z-1)^2\widehat{x}(z) + Az^k\widehat{x}(z) = \widehat{f}(z). \tag{3.5}$$

Hence the operator $(z-1)^2 + z^k A$ is invertible if and only if $-(z-1)^2/z^k$ belongs to the resolvent set $\rho(A)$ of A. Define the function

$$\Gamma_\alpha(t) = -\frac{(e^{it}-1)^2}{e^{i\alpha t}}, \quad \alpha \in \mathbb{R},\ t \in (0, 2\pi). \tag{3.6}$$

Then, for each α fixed, $\Gamma_\alpha(t)$ describes a curve in the complex plane such that $\Gamma_\alpha(0) = \Gamma_\alpha(2\pi) = 0$.

Proposition 3.2. *The curve Γ_α attains the minimum length at $\alpha = 1$.*

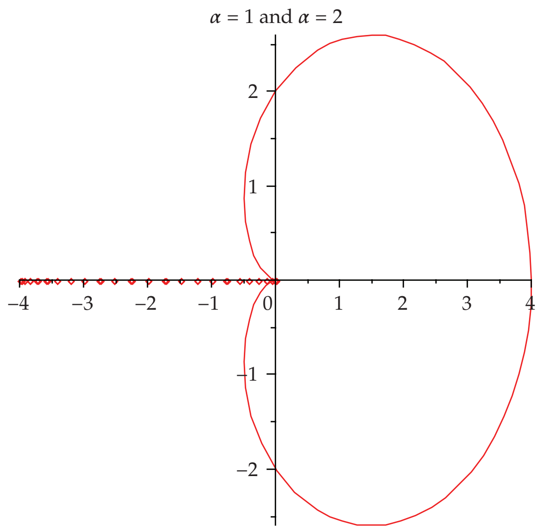

Figure 1

Proof. A calculation gives $\Gamma'_\alpha(t) = -2ie^{-i(\alpha/2)t}((\alpha-1)(1-\cos t) + i\sin t)$. Hence the length of Γ_α is given by

$$l(\alpha) = \int_0^{2\pi} \left|\Gamma'_\alpha(t)\right| dt = 2\int_0^{2\pi} \sqrt{(\alpha-1)^2(1-\cos t)^2 + \sin^2 t}\, dt. \tag{3.7}$$

From which the conclusion follows. □

Remark 3.3. As a consequence, the value $k = 1$ in (3.1) is *singular* in the sense that the curve described by (3.6) attains the minimum length if and only if $\alpha = 1$ (see Figure 1). This singular character is reinforced by observing that

$$\Gamma_1(\epsilon) = \left(2\sin\left(\frac{\epsilon}{2}\right)\right)^2, \tag{3.8}$$

and that this value exactly corresponds to the step size in the best discretization of the harmonic oscillator obtained in [2]. We conjecture that there is a general link between the geometrical properties of curves related to classes of difference equations and the property of best approximation. This is possibly a very difficult task, which we do not touch in this paper.

In what follows we denote $T := A + I$; $\mathbb{D}(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ and $\mathbb{T} = \partial\mathbb{D}(0, 1)$. The following result relates the values of $\Gamma_1(t)$ with the spectrum of the operator A. It will be essential in the proof of our characterization of well posedness for (1.2) in l_p-vector-valued spaces given in Section 5 (cf. Theorem 5.2).

Proposition 3.4. *Suppose that T is analytic. Then $\sigma(I - T) \subseteq \mathbb{D}(1, 1) \cup \{0\}$. In particular,*

$$-\Gamma_1((0, 2\pi)) \subset \rho(I - T). \tag{3.9}$$

Proof. Let $M > 0$ such that $M/n \geq \|T^n(T-I)\|$ for all $n \in \mathbb{N}$. Define $p(z) = z^{n+1} - z^n$. By the spectral mapping theorem, we have

$$\begin{aligned}\|T^n(T-I)\| &\geq \sup_{\lambda \in \sigma(p(T))} |\lambda| \\ &= \sup_{\lambda \in p(\sigma(T))} |\lambda| \\ &= \sup_{z \in \sigma(T)} \left| z^n (z-1) \right| \\ &= \sup_{w \in \sigma(I-T)} \left| w(1-w)^n \right| \geq |w||1-w|^n,\end{aligned} \tag{3.10}$$

for all $w \in \sigma(I-T)$, $n \in \mathbb{N}$. Hence

$$\sigma(I-T) \subseteq \mathbb{D}(1,1) \cup \{0\}. \tag{3.11}$$

Finally, we observe that $-\Gamma_1(t) = -[2\sin(t/2)]^2 \in (-4, 0)$. □

4. Existence and Uniqueness

In this section, we treat the existence and uniqueness problem for the equation

$$\begin{gathered}\Delta^2 x_n - (I-T)x_{n+1} = f_n, \quad n \in \mathbb{Z}_+, \\ x_0 = x_1 = 0.\end{gathered} \tag{4.1}$$

Remark 4.1. If $z = (z_n)$ is solution of the equation

$$\begin{gathered}\Delta^2 z_n - (I-T)z_{n+1} = 0, \quad n \in \mathbb{Z}_+, \\ z_0 = z_1 = 0,\end{gathered} \tag{4.2}$$

then $z \equiv 0$. It follows from induction. In fact, suppose that $z_n = 0$ for all $n < m$, choosing $n = m-2$ in (4.2) we get $z_m = 0$.

Recall that the convolution of two sequences x_n and y_n is defined by

$$(x * y)(n) = \sum_{j=0}^{n} x(n-j)y(j) = \sum_{j=0}^{n} x(n)y(n-j). \tag{4.3}$$

Also we note that the convolution theorem for the discrete Fourier transform holds, that is, $\widehat{x * y}(z) = \hat{x}(z)\hat{y}(z)$. Further properties can be found in [28, Section 5.1]. Our main result in this section, on existence and uniqueness of solution for (4.1), read as follows.

Theorem 4.2. *Let $T \in \mathcal{B}(X)$, then there exists a unique solution of* (4.1) *which is given by $x_{m+1} = (B * f)_m$, where $B(n) \in \mathcal{B}(X)$satisfies the following equation:*

$$\begin{gathered}\Delta^2 B(n) - (I-T)B(n+1) = 0, \\ B(0) = 0, \qquad B(1) = I.\end{gathered} \tag{4.4}$$

If T is an analytic operator, one has that

$$B(n) = \frac{1}{2\pi i}\oint_C R\left(\frac{(z-1)^2}{z}, I-T\right) z^{n-1}\, dz, \tag{4.5}$$

where C is a circle, centered at the origin of the z-plane that enclosed all poles of

$$R\left(\frac{(z-1)^2}{z}, I-T\right) z^{n-1}. \tag{4.6}$$

Hence,

$$\widehat{B}(z) = R\left(\frac{(z-1)^2}{z}, I-T\right). \tag{4.7}$$

Proof. Let $V_n := [x_n, \Delta x_n]$, $F_n = [0, f_n]$, and $R_T \in \mathcal{B}(X \times X)$ defined by

$$R_T[x,y] = [x+y, x+2y-T(x+y)]. \tag{4.8}$$

Then it is not difficult to see that (4.1) is equivalent to

$$\begin{gathered} V_{n+1} - R_T V_n = F_n, \quad n \in \mathbb{Z}_+, \\ V_0 = [0,0], \end{gathered} \tag{4.9}$$

which has the solution

$$V_{m+1} = \sum_{n=0}^{m} R_T^n F_{m-n}. \tag{4.10}$$

Denote

$$R_T = \begin{bmatrix} I & I \\ I-T & 2I-T \end{bmatrix}. \tag{4.11}$$

Then a calculation shows us that there is an operator $B(n) \in \mathcal{B}(X)$ with $(I-T)B(n) = B(n)(I-T)$ such that

$$R_T^n = \begin{bmatrix} \Delta B(n) - B(n)(I-T) & B(n) \\ B(n)(I-T) & \Delta B(n) \end{bmatrix}. \tag{4.12}$$

$B(n)$ satisfy the following equation:

$$\begin{gathered} B(n+2) = (3I-T)B(n+1) - B(n), \\ B(0) = 0, \qquad B(1) = I, \end{gathered} \tag{4.13}$$

which is equivalent to

$$\Delta^2 B(n) - (I-T)B(n+1) = 0,$$
$$B(0) = 0, \qquad B(1) = I. \tag{4.14}$$

We can see that there are two sequences $a_k(2n)$, $b_k(2n+1)$ in $\mathbb{N}$ such that

$$B(2n) = \sum_{k=1}^{n} (-1)^{n-k} a_k(2n)(3I-T)^{2k-1}, \quad n \geq 1,$$
$$B(2n+1) = \sum_{k=0}^{n} (-1)^{n-k} b_k(2n+1)(3I-T)^{2k}, \quad n \geq 1. \tag{4.15}$$

Since $B(2n) = (3I-T)B(2n-1) - B(2(n-1))$, we have

$$a_k(2n) = b_{k-1}(2n-1) + a_k(2(n-1)), \quad k = 1, \dots, n-1,$$
$$a_n(2n) = b_{n-1}(2n-1) = 1, \qquad a_{n-1}(2n) = 2n-2,$$
$$a_1(2n) = n, \qquad b_0(2n-1) = 0, \qquad b_{n-1}(2n+1) = 2n-1. \tag{4.16}$$

On the other hand, using (4.12), we have

$$x_{m+1} = (B * f)_m,$$
$$\Delta x_{m+1} = (\Delta B * f)_m. \tag{4.17}$$

Hence, applying Fourier transform in (4.17), we obtain

$$\widehat{\Delta B}(z)\hat{f}(z) = (z-1)\hat{B}(z)\hat{f}(z). \tag{4.18}$$

Given $x \in X$ we define

$$f_n^0 = \begin{cases} x & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \tag{4.19}$$

A direct calculation shows that $\hat{f}^0(z) = x$, for $z \in \mathbb{T}$. Then by (4.18), we get

$$\widehat{\Delta B}(z)x = (z-1)\hat{B}(z)x, \quad x \in X, \ z \in \mathbb{T}. \tag{4.20}$$

Hence

$$\widehat{\Delta B}(z) = (z-1)\hat{B}(z), \quad z \in \mathbb{T}. \tag{4.21}$$

On the other hand, since $V_{m+1} = [(B * f)_m, (\Delta B * f)_m]$ is solution of (4.9), we have

$$(B * f)_m = (B * f)_{m-1} + (\Delta B * f)_{m-1}, \tag{4.22}$$

and hence

$$\begin{aligned}(\Delta B * f)_m &= (I - T)\left[(B * f)_{m-1} + (\Delta B * f)_{m-1}\right] + (\Delta B * f)_{m-1} + f_m \\ &= (I - T)(B * f)_m + (\Delta B * f)_{m-1} + f_m.\end{aligned} \tag{4.23}$$

Therefore,

$$(\Delta B * f)_m - (\Delta B * f)_{m-1} = (I - T)(B * f)_m + f_m. \tag{4.24}$$

Applying Fourier transform in (4.24) and taking into account (4.21), we have

$$\left[\frac{(z-1)^2}{z} - (I - T)\right]\widehat{B}(z) = I. \tag{4.25}$$

If T is analytic, we get

$$\widehat{B}(z) = R\left(\frac{(z-1)^2}{z}, I - T\right), \tag{4.26}$$

and the proof is finished. □

5. Maximal Regularity

In this section, we obtain a spectral characterization about maximal regularity for (1.2). The following definition is motivated in the paper [11] (see also [3]).

Definition 5.1. Let $1 < p < +\infty$. One says that (4.1) has discrete maximal regularity if $Kf = (I - T)B * f$ defines a bounded operator $K \in \mathcal{B}(l_p(\mathbb{Z}_+; X))$.

As consequence of the definition, if (1.2) has discrete maximal regularity, then (1.2) has discrete l_p-maximal regularity in the following sense: for each $(f_n) \in l_p(\mathbb{Z}_+; X)$ we have $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+; X)$, where (x_n) is the solution of the equation $\Delta^2 x_n - (I - T)x_{n+1} = f_n$, for all $n \in \mathbb{Z}_+$, $x_0 = 0$, $x_1 = 0$. Moreover,

$$\Delta^2 x_n = \sum_{k=1}^{n}(I - T)B(k)f_{n-k} + f_n = ((I - T)B * f)_n + f_n. \tag{5.1}$$

A similar analysis as above can be carried out when we consider more general initial conditions, but the price to pay for this is that the proof would certainly require additional l_p-summability condition on $B(n)$. The following is the main result of this paper.

Theorem 5.2. *Let X be a UMD space and let $T \in \mathcal{B}(X)$ analytic. Then the following assertions are equivalent.*

(i) *Equation* (1.2) *has discrete maximal regularity.*

(ii) $\{((z-1)^2/z)R((z-1)^2/z, I-T)/|z|=1,\ z\neq 1\}$ *is R-bounded.*

Proof. (i)⇒(ii) Define $k_T : \mathbb{Z} \to \mathcal{B}(X)$ by

$$k_T(n) = \begin{cases} (I-T)B(n) & \text{for } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

and the corresponding operator $K_T : l_p(\mathbb{Z}_+; X) \to l_p(\mathbb{Z}_+; X)$ by

$$(K_T f)(n) = \sum_{j=0}^{n} k_T(j) f_{n-j} = (k_T * f)(n), \quad n \in \mathbb{Z}_+. \tag{5.3}$$

By hypothesis, K_T is well defined and bounded on $l_p(\mathbb{Z}_+; X)$. By Proposition 3.4, $(z-1)^2/z \in \rho(I-T)$ whenever $|z|=1$, $z\neq 1$. Then, by Theorem 4.2 we have

$$\begin{aligned} \widehat{k}_T(z) &= (I-T)\widehat{B}(z) \\ &= (I-T)R\left(\frac{(z-1)^2}{z}, I-T\right) \\ &= \frac{(z-1)^2}{z}R\left(\frac{(z-1)^2}{z}, I-T\right) - I, \quad z \in \mathbb{T},\ z\neq 1. \end{aligned} \tag{5.4}$$

We observe that there exists $L_M \in \mathcal{B}(l_p(\mathbb{Z}_+; X))$ such that

$$\mathcal{F}(L_M f)(z) := \frac{(z-1)^2}{z}R\left(\frac{(z-1)^2}{z}, I-T\right)\widehat{f}(z). \tag{5.5}$$

Explicitly, L_M is given by $(L_M f)(n) = (K_T f)(n) + f(n)$. We conclude, from [11, Proposition 1.4], that the set in (ii) is R-bounded.

(ii)⇒(i) Define $M(t) = e^{-it}(e^{it}-1)^2 R(e^{-it}(e^{it}-1)^2, I-T) - I$ for $t \in \mathcal{T}$. Then $M(t)$ is R-bounded by hypothesis and Remark 2.2. Define

$$N(t) = (e^{it}-1)^2 R(e^{it}(e^{it}-1)^2, I-T) - e^{it}I, \tag{5.6}$$

then $M(t) = e^{-it}N(t)$ and $\{N(t)\}$ is R-bounded. A calculation shows that $M'(t) = -ie^{-it}N(t) + e^{-it}N'(t)$. Note that $M(t)$ is R-bounded if and only if $N(t)$ is R-bounded (cf. Remark 2.2). Moreover,

$$(e^{it}-1)N'(t) = 2ie^{it}\left[N(t)+e^{it}I\right] - \left(2-i+ie^{-it}\right)\left[N(t)+e^{it}I\right]^2 - ie^{it}\left(e^{it}-1\right)I. \tag{5.7}$$

It shows that the set $\{(e^{it}-1)M'(t)\}_{t\in\mathcal{T}}$ is R-bounded, thanks to Remark 2.6 again. It follows the R-boundedness of the set $\{(e^{it}+1)(e^{it}-1)M'(t)\}$. Then, by Theorem 2.7 we obtain that there exists $T_M \in \mathcal{B}(l_p(\mathbb{Z},X))$ such that

$$\mathcal{F}(T_M f)(z) = \frac{(z-1^2)}{z} R\left(\frac{(z-1)^2}{z}, I-T\right)\hat{f}(z) - \hat{f}(z), \quad z\in\mathbb{T},\ z\neq 1. \tag{5.8}$$

By Theorem 4.2, we have

$$\mathcal{F}(Kf)(z) = (I-T)R\left(\frac{(z-1)^2}{z}, I-T\right)\hat{f}(z) = \mathcal{F}(T_M f)(z). \tag{5.9}$$

Then, by uniqueness of the Fourier transform, we conclude that $K\in\mathcal{B}(l_p(\mathbb{Z}_+,X))$. □

Remark 5.3. Note that

$$\left\{\frac{(z-1)^2}{z} R\left(\frac{(z-1)^2}{z}, I-T\right) / |z| = 1,\ z\neq 1\right\} \tag{5.10}$$

is R-bounded if and only if

$$\left\{(z-1)^2 R\left(\frac{(z-1)^2}{z}, I-T\right) / |z| = 1,\ z\neq 1\right\} \tag{5.11}$$

is R-bounded.

Corollary 5.4. *Let H be a Hilbert space and let $T\in\mathcal{B}(H)$ be an analytic operator. Then the following assertions are equivalent.*

(i) *Equation* (1.2) *has discrete maximal regularity.*

(ii) $\sup_{|z|=1,\ z\neq 1}\|((z-1)^2/z)((z-1)^2/z-(I-T))^{-1}\| < \infty$.

Remark 5.5. Letting $H=\mathbb{C}$ and $T=\rho I$ with $0\le\rho<1$, we get that the hypothesis of the preceding corollary are satisfied. We conclude that the scalar equation

$$\Delta^2 x_n - (1-\rho)x_{n+1} = f_n, \quad n\in\mathbb{Z}_+,\ x_0 = x_1 = 0, \tag{5.12}$$

has the property that for all $(f_n)\in l_p(\mathbb{Z}_+)$ we get $(\Delta^2 x_n)\in l_p(\mathbb{Z}_+)$. In particular $x_n \to 0$, that is, the solution is stable. Note that using (4.7) we can infer that

$$B(n) = \frac{1}{a-b}(a^n - b^n), \tag{5.13}$$

where a and b are the real roots of $z^2+(\rho-3)z-1=0$. Moreover, the solution is given by

$$x_{m+1} = (B*f)_m = \sum_{j=0}^{m}\frac{1}{a-b}\left(a^{(m-j)} - b^{(m-j)}\right)f(j). \tag{5.14}$$

Remark 5.6. We emphasize that from a more theoretical perspective, our results also are true when we consider the more general equation (3.1) instead of (1.1), but additional hypothesis will be needed (cf. Remark 3.1). Until now literature about this subject is too incipient and should be developed.

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