Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 380568, 19 pages
doi:10.1155/2009/380568

Research Article

Existence of Periodic and Almost Periodic Solutions of Abstract Retarded Functional Difference Equations in Phase Spaces

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Received 20 November 2008; Revised 23 March 2009; Accepted 10 June 2009

Recommended by Donal O'Regan

The existence of periodic, almost periodic, and asymptotically almost periodic of periodic and almost periodic of abstract retarded functional difference equations in phase spaces is obtained by using stability properties of a bounded solution.

1. Introduction

In this paper, we study the existence of periodic, almost periodic, and asymptotic almost periodic solutions of the following functional difference equations with infinite delay:

$$x(n+1) = \mathbf{F}(n, x_n), \quad n \geq n_0 \geq 0, \tag{1.1}$$

assuming that this system possesses a bounded solution with some property of stability. In (1.1) $\mathbf{F} : \mathbb{N}(n_0) \times \mathcal{B} \to \mathbb{C}^r$, and $\mathcal{B}$ denotes an abstract phase space which we will define later.

The abstract space was introduced by Hale and Kato [1] to study qualitative theory of functional differential equations with unbounded delay. There exists a lot of literature devoted to this subject; we refer the reader to Corduneanu and Lakshmikantham [2], Hino et al. [3]. The theory of abstract retarded functional difference equations in phase space has attracted the attention of several authors in recent years. We only mention here Murakami [4, 5], Elaydi et al. [6], Cuevas and Pinto [7, 8], Cuevas and Vidal [9], and Cuevas and Del Campo [10].

As usual, we denote by $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{Z}^-$ the set of all integers, the set of all nonnegative integers, and the set of all nonpositive integers, respectively. Let $\mathbb{C}^r$ be the r-dimensional complex Euclidean space with norm $|\cdot|$. $\mathbb{N}(n_0)$ the set $\mathbb{N}(n_0) = \{n \in \mathbb{N} : n \geq n_0\}$.

If $x : \mathbb{Z} \to \mathbb{C}^r$ *is a function, we define for* $n \in \mathbb{N}(n_0)$, the function $x_n : \mathbb{Z}^- \to \mathbb{C}^r$ by $x_n(s) = x(n+s)$, $s \in \mathbb{Z}^-$. Furthermore $x_\bullet$ is the function given for $x_\bullet : \mathbb{N}(n_0) \to \mathcal{B}$, with $x_\bullet(n) = x_n$.

The abstract phase space $\mathcal{B}$, which is a subfamily of all functions from $\mathbb{Z}^-$ into $\mathbb{C}^r$ denoted by $\mathcal{F}(\mathbb{Z}^-, \mathbb{C}^r)$, is a normed space (with norm denoted by $\|\cdot\|_{\mathcal{B}}$) and satisfies the following axioms.

(A) *There is a positive constant* $J > 0$ *and nonnegative functions* $N(\cdot)$ *and* $M(\cdot)$ *on* $\mathbb{Z}^+$ *with the property that* $x : \mathbb{Z} \to \mathbb{C}^r$ *is a function, such that* $x_0 \in \mathcal{B}$, *then for all* $n \in \mathbb{Z}^+$, *the following conditions hold:*

(i) $x_n \in \mathcal{B}$,

(ii) $J|x(n)| \leq \|x_n\|_{\mathcal{B}}$,

(iii) $\|x_n\|_{\mathcal{B}} \leq N(n)\sup_{0\leq s\leq n}|x(s)| + M(n)\|x_0\|_{\mathcal{B}}$.

(B) *The space*$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$*is a Banach space.*

We need the following property on $\mathcal{B}$.

(C) *The inclusion map* $i : (B(\mathbb{Z}^-, \mathbb{C}^r), \|\cdot\|_\infty) \to (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *is continuous, that is, there is a constant* $K \geq 0$, *such that* $\|\varphi\|_{\mathcal{B}} \leq K\|\varphi\|_\infty$, *for all* $\varphi \in B(\mathbb{Z}^-, \mathbb{C}^r)$, *where* $B(\mathbb{Z}^-, \mathbb{C}^r)$ *represents the bounded functions from* $\mathbb{Z}^-$ *into* $\mathbb{C}^r$.

Axiom (C) says that any element of the Banach space of the bounded functions equipped with the supremum norm $(B(\mathbb{Z}^-, \mathbb{C}^r), \|\cdot\|_\infty)$ is on $\mathcal{B}$.

Remark 1.1. Using analogous ideas to the ones of [3], it is not difficult to prove that Axiom (C) is equivalente to the following.

(C′) If a uniformly bounded sequence $\{\varphi_n\}_n$ in $\mathcal{B}$ converges to a function φ compactly on $\mathbb{Z}^-$ (i.e., converges on any compact discrete interval in $\mathbb{Z}^-$) in the compact-open topology, then φ belong to $\mathcal{B}$ and $\|\varphi_n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

Remark 1.2. We will denote by $x(n, \tau, \varphi)$ ($\tau \geq n_0$, and $\varphi \in \mathcal{B}$) or simply by $x(n)$, the solution of (1.1) passing through (τ, φ), that is, $x(\tau, \tau, \varphi) = \varphi$, and the functional equation (1.1) is satisfied.

During this paper we will assume that the sequences $M(n)$ and $N(n)$ are bounded. The paper is organized as follows. In Section 2 we see some important implications of the fading memory spaces. Section 3 is devoted to recall definitions and some important basic results about almost periodic sequences, asymptotically almost periodic sequences, and uniformly asymptotically almost periodic functions. In Section 4 we analyze separately the cases where F is periodic and when it is almost periodic. Thus, in Section 4.1 assuming that the system (1.1) is periodic and the existence of a bounded solution (particular solution) which is uniformly stable and the phase space satisfies only the axioms (A)–(C), we prove the existence of an almost periodic solution and an asymptotically almost periodic solution. If additionally the particular solution is uniformly asymptotically stable, we prove the existence of a periodic solution. Similarly, in Section 4.2 considering that system (1.1) is almost periodic and the existence of a bounded solution and whenever the phase space satisfies the axioms (A)–(C), but here it is also necessary that $\mathcal{B}$ verifies the fading memory property. If the particular solution is asymptotically almost periodic, then system (1.1) has an almost periodic solution. While, if the particular solution is uniformly asymptotically stable, we prove the existence of an asymptotically almost periodic solution.

In [11, 12] the problem of existence of almost periodic solutions for functional difference equations is considered in the first case for the discrete Volterra equation and in the second reference for the functional difference equations with finite delay; in both cases the authors assume the existence of a bounded solution with a property of stability that gives information about the existence of an almost periodic solution. In an analogous way in [13] the problem of the existence of almost periodic solutions for functional difference equations with infinite delay is considered. These results can be applied to several kinds of discrete equations. However, our approach differs from Hamaya's because, firstly, in our work we consider both cases, namely, when F is periodic and when it is almost periodic in the first variable. And secondly, we analyze very carefully the implications of the existence of a bounded solution of (1.1) with each property: uniformly stable, uniformly asymptotically stable, and globally uniformly stable.

Furthermore, we cite the articles [14–16] which are devoted to study almost periodic solutions of difference equations, but a little is known about almost periodic solutions, and in particular, for periodic solutions of nonlinear functional difference equations in phase space via uniform stability, uniformly asymptotically stability, and globally uniformly stability properties of a bounded solution.

2. Fading Memory Spaces and Implications

Following the terminology given in [3], we introduce the family of operators on $\mathcal{B}$, $S(\cdot)$, as

$$[S(n)\varphi](\theta) = \begin{cases} \varphi(0), & \text{if } -n \leq \theta \leq 0, \\ \varphi(n+\theta), & \text{if } \theta < -n, \end{cases} \tag{2.1}$$

with $\varphi \in \mathcal{B}$. They constitute a family of linear operators on $\mathcal{B}$ having the semigroup property $S(n+m) = S(n)S(m)$ for $n, m \geq 0$. Immediately, the following result holds from Axiom (A):

$$\|S(n)\| \leq \frac{N(n)}{J} + M(n), \quad \text{for each } n \geq 0. \tag{2.2}$$

Now, given any function $x : \mathbb{Z} \to \mathbb{C}^r$ such that $x_0 \in \mathcal{B}$, we have the following decomposition:

$$x(n) = y(n) + z(n), \quad n \in \mathbb{Z}, \tag{2.3}$$

where

$$y(n) = \begin{cases} x(n), & \text{if } n \geq 0, \\ x(0), & \text{if } n \geq 0, \end{cases}$$
$$z(n) = \begin{cases} 0, & \text{if } n \geq 0, \\ x(n) - x(0), & \text{if } n < 0. \end{cases} \tag{2.4}$$

Then, we have the following decomposition of $x_n = y_n + z_n$, $y_n, z_n \in \mathcal{B}$ for $n \geq 0$, where

$$z_n = S(n)\left(x_0 - x(0)\chi\right), \tag{2.5}$$

and $\chi(\theta) = 1$ for all $\theta \leq 0$. Note that

$$z_n(0) = 0, \quad \text{for each } n \geq 0. \tag{2.6}$$

Let

$$\mathcal{B}_0 := \{\varphi \in \mathcal{B} : \varphi(0) = 0\} \tag{2.7}$$

be a subset of $\mathcal{B}$, and let $S_0(n) = S(n)|_{\mathcal{B}_0}$ be the restriction of S to $\mathcal{B}_0$. Clearly, the family $S_0(n)$, $n \in \mathbb{N}(n_0)$, is also a strongly continuous semigroup of bounded linear operators on $\mathcal{B}_0$. It is given explicitly by

$$\left[S_0(n)\varphi\right](\theta) = \begin{cases} 0, & -n \leq \theta \leq 0, \\ \varphi(n+\theta), & \theta < -n, \end{cases} \tag{2.8}$$

for $\varphi \in \mathcal{B}_0$.

Definition 2.1. A phase space $\mathcal{B}$ that satisfies axioms (A)-(B) and (C) or (C′) and such that the semigroup $S_0(n)$ is strongly stable is called a fading memory space.

Remark 2.2. Remember that a strongly continuous semigroup is strongly stable if for all $\varphi \in \mathcal{B}_0$, $S_0(n)\varphi \to 0$ as $n \to +\infty$.

Thus, we have the following result.

Lemma 2.3. *Let $x : \mathbb{Z} \to \mathbb{C}^r$, with $x_0 \in \mathcal{B}$, where $\mathcal{B}$ is a fading memory space. If $x(n) \to 0$ as $n \to +\infty$, then $x_n \to 0$ as $n \to +\infty$.*

Proof. Firstly, we note that as before, $x_n = y_n + S_0(n)[x_0 - x(0)\chi]$, where $\chi(\theta) = 1$, for $\theta \leq 0$ and

$$y(\theta) = \begin{cases} x(\theta), & \theta \geq 0, \\ x(0), & \theta < 0. \end{cases} \tag{2.9}$$

Then, by definition $S_0(n)[x_0 - x(0)\chi] \to 0$ as $n \to +\infty$ because $x_0 - x(0)\chi \in \mathcal{B}_0$. On the other hand, by hypothesis, $x(n) \to 0$ as $n \to +\infty$, so it follows from Axiom (C′) that $y_n \to 0$. Therefore, we conclude that $x_n \to 0$ as $n \to +\infty$. □

3. Notations and Preliminary Results

In this section, we review the definitions of (uniformly) almost periodic, asymptotically almost periodic sequence, which have been discussed by several authors and present some related properties.

For our purpose, we introduce the following definitions and results about almost periodic discrete processes which are given in [3, 17, 18] for the continuous case. For the discrete case we mention [11, 12].

Definition 3.1. A sequence $x : \mathbb{Z} \to \mathbb{C}^r$ is called an almost periodic sequence if the ϵ-translation set of x,

$$E\{\epsilon, x\} := \{\tau \in \mathbb{Z}/|x(n+\tau) - x(n)| < \epsilon,\ \forall n \in \mathbb{Z}\}, \tag{3.1}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists an integer $l = l(\epsilon) > 0$ such that each discrete interval of length l contains $\tau = \tau(\epsilon) \in E\{\epsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \epsilon, \quad \forall\ n \in \mathbb{Z}. \tag{3.2}$$

τ is called the ϵ-translation number of $x(n)$. We will denote by $\mathcal{AP}(\mathbb{Z};\mathbb{C}^r)$ the set of all such sequences. We will write that x is a.p. if $x \in \mathcal{AP}(\mathbb{Z};\mathbb{C}^r)$.

Definition 3.2. A sequence $x : \mathbb{Z} \to \mathbb{C}^r$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n), \tag{3.3}$$

where $p(n)$ is an almost periodic sequence, and $q(n) \to 0$ as $n \to +\infty$. We will denote by $\mathcal{AAP}(\mathbb{Z};\mathbb{C}^r)$ the set of all such sequences. We will write that x is a.a.p. if $x \in \mathcal{AAP}(\mathbb{Z};\mathbb{C}^r)$.

In general, we will consider $(X, \|\cdot\|_X)$ a Banach space.

Definition 3.3. A function or sequence $x : \mathbb{Z} \to X$ is said to be almost periodic (abbreviated a.p.) in $n \in \mathbb{Z}$ if for every $\epsilon > 0$ there is $N_\epsilon = N(\epsilon) > 0$ such that among N_ϵ consecutive integers there is one; call it p, such that

$$\|x(n+p) - x(n)\|_X < \epsilon, \quad \forall\ n \in \mathbb{Z}. \tag{3.4}$$

Denote by $\mathcal{AP}(\mathbb{Z});X)$ all such sequences, and x is said to be an almost periodic (a.p.) in X.

Definition 3.4. A sequence $\{x(n)\}_{n\in\mathbb{N}(n_0)}$, (or $\{x(n)\}_{n\in\mathbb{Z}}$), $x(n) \in X$, equivalently, a function $x : \mathbb{N}(n_0) \to X$ (or, $x : \mathbb{Z} \to X$) is called asymptotically almost periodic if $x = x_1 + x_2$, where $x_1 \in \mathcal{AP}(\mathbb{Z}; X)$ and $x_2 : \mathbb{N}(n_0) \to X$ (or, $x_2 : \mathbb{Z} \to X$) satisfying $\|x_2(n)\|_X \to 0$ as $n \to +\infty$

(or, $|n| \to +\infty$). Denote by $\mathcal{AAP}(\mathbb{N}(n_0);X)$ (or $\mathcal{AAP}(\mathbb{Z};X)$ all such sequences, and x is said to be an asymptotically almost periodic on $\mathbb{N}(n_0)$ (or on $\mathbb{Z}$) (a.a.p.) in X.

Remark 3.5. Almost periodic sequences can be also defined for any sequence $\{x(n)\}_{n\in J}$ $(J \subset \mathbb{Z})$ or $x : J \to X$ by requiring that $N_\epsilon = N(\epsilon) > 0$ consecutive integers are in J.

Definition 3.6. Let $\mathbf{f} : \mathbb{Z} \times \mathcal{B} \to \mathbb{C}^r$. $\mathbf{f}(n,\phi)$ is said to be almost periodic in n uniformly for $\phi \in \mathcal{B}$, if for any $\epsilon > 0$ and every compact $\Sigma \subset \mathcal{B}$, there exists a positive integer $l = l(\epsilon,\Sigma)$ such that any interval of length l (i.e., among l consecutive integers) contains an integer (or equivalently, there is one); call it τ, for which

$$\left|\mathbf{f}(n+\tau,\phi) - \mathbf{f}(n,\phi)\right| < \epsilon, \quad \forall n \in \mathbb{Z},\ \phi \in \Sigma. \tag{3.5}$$

τ is called the ϵ-translation number of $\mathbf{f}(n,\phi)$. We will denote by $\mathcal{UAP}(\mathbb{Z}\times\mathcal{B};\mathbb{C}^r)$ the set of all such sequences. In brief we will write that $\mathbf{f}$ is u.a.p. if $\mathbf{f} \in \mathcal{UAP}(\mathbb{Z}\times\mathcal{B};\mathbb{C}^r)$.

Definition 3.7. The hull of $\mathbf{f}$, denoted by $H(\mathbf{f})$, is defined by

$$H(\mathbf{f}) = \left\{g(n,\phi) : \lim_{k\to+\infty} \mathbf{f}(n+\tau_k,\phi) = g(n,\phi) \text{ uniformly on } \mathbb{Z}\times\Sigma\right\}, \tag{3.6}$$

for some sequence $\{\tau_k\}$, where Σ is any compact set in $\mathcal{B}$.

For our purpose, we introduce the following definitions and results about almost periodic discrete processes which are given in [3, 17, 18] for the continuous case. For the discrete case we mention [11, 12]. With the objective to make this manuscript self contained we decided to include the majority of the proofs.

Lemma 3.8. (a) *If $\{x(n)\}$ is an a.p. sequence, then there exists an almost periodic function $\mathbf{f}(t)$ such that $\mathbf{f}(n) = x(n)$ for $n \in \mathbb{Z}$.*

(b) *If $\mathbf{f}(t)$ is an a.p. function, then $\{\mathbf{f}(n)\}$ is an a.p. sequence.*

Lemma 3.9. (a) *If $\{x(n)\}$ is an a.p. sequence, then $\{x(n)\}$ is bounded.*

(b) *$\{x(n)\}$ is an a.p. sequence if and only if for any sequence $\{k_i'\} \subset \mathbb{Z}$ there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that $x(n+k_i)$ converges uniformly on $\mathbb{Z}$ as $i \to +\infty$. Furthermore, the limits sequence is also an almost periodic sequence.*

(c) *$\{x(n)\}$ $n \in \mathbb{Z}$ is an a.p. sequence if and only if for any sequence of integers $\{k_i'\}$, $\{l_i'\}$ there exist subsequences $k = \{k_i\} \subset \{k_i'\}$, $l = \{l_i\} \subset \{l_i'\}$ such that*

$$T_k T_l x(n) = T_{k+l} x(n), \quad \text{for } n \in \mathbb{Z}, \tag{3.7}$$

where $T_k x(n) = \lim_{i\to+\infty} x(n+k_i)$ for $n \in \mathbb{Z}$.

(d) *$\{x(n)\}$, $n \in \mathbb{Z}^+$ (or, $n \in \mathbb{Z}$) is an a.a.p. sequence if and only if for any sequence $\{k_i'\} \subset \mathbb{Z}^+$ (or, $\mathbb{Z}$) such that $k_i' > 0$ and $k_i' \to +\infty$ as $i \to +\infty$ (or, $|k_i'| \to +\infty$ as $i \to +\infty$), there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that $x(n+k_i)$ converges uniformly on $\mathbb{Z}^+$ (or $\mathbb{Z}$) as $i \to +\infty$.*

Lemma 3.10. *Let* $\mathbf{x}(n)$ *be an a.a.p. periodic sequence. Then its decomposition,*

$$\mathbf{f}(n) = p(n) + q(n), \tag{3.8}$$

where $p(n)$ *is an a.p. sequence while* $q(n) \to 0$ *as* $n \to +\infty$*, is unique.*

Lemma 3.11. *Let* $\mathbf{f} : \mathbb{Z} \times \mathcal{B} \to \mathbb{C}^r$ *be almost periodic in* n *uniformly for* $\phi \in \mathcal{B}$ *and continuous in* ϕ*. Then* $\mathbf{f}(n, \phi)$ *is bounded and uniformly continuous on* $\mathbb{Z} \times \Sigma$ *for any compact set* Σ *in* $\mathcal{B}$*.*

Lemma 3.12. *Let* $\mathbf{f}(n, \phi)$ *be the same as in the previous lemma. Then, for any sequence* $\{h'_k\}$*, there exist a subsequence* $\{h_k\}$ *of* $\{h'_k\}$ *and a function* $g(n, \phi)$ *continuous in* ϕ *such that* $\mathbf{f}(n + h_k, \phi) \to g(n, \phi)$ *uniformly on* $\mathbb{Z} \times \Sigma$ *as* $k \to +\infty$*, where* Σ *is any compact set in* $\mathcal{B}$*. Moreover,* $g(n, \phi)$ *is also almost periodic in* n *uniformly for* $\phi \in \mathcal{B}$*.*

Lemma 3.13. *Let* $\mathbf{f}(n, \phi)$ *be the same as in the previous lemma. Then, there exists a sequence* $\{\alpha_k\}$*,* $\alpha_k \to +\infty$ *as* $k \to +\infty$ *such that* $\mathbf{f}(n + \alpha_k, \phi) \to f(n, \phi)$ *uniformly on* $\mathbb{Z} \times \Sigma$ *as* $k \to +\infty$*, where* Σ *is any compact set in* $\mathcal{B}$*.*

Lemma 3.14. *Let* $\mathbf{f} : \mathbb{Z} \times \mathcal{B} \to \mathbb{C}^r$ *be almost periodic in* n *uniformly for* $\phi \in \mathcal{B}$ *and continuous in* $\phi \in \mathcal{B}$*, and let* $p(n)$ *be an almost periodic sequence in* $\mathcal{B}$ *such that* $p(n) \in \Sigma$ *for all* $n \in \mathbb{Z}$*, where* Σ *is a compact set in* $\mathcal{B}$*. Then* $\mathbf{f}(n, p(n))$ *is almost periodic in* n*.*

Lemma 3.15. *Let* $\mathbf{f} : \mathbb{Z} \times \mathcal{B} \to \mathbb{C}^r$ *be almost periodic in* n *uniformly for* $\phi \in \mathcal{B}$ *and continuous in* $\phi \in \mathcal{B}$*, and let* $p(n)$ *be an almost periodic sequence in* $\mathbb{C}^r$ *such that* $p_n \in \Sigma$ *for all* $n \in \mathbb{Z}$*, where* Σ *is a compact set in* $\mathcal{B}$ *and* $p_n(s) = p(n + s)$ *for* $s \in \mathbb{Z}^-$*. Then* $\mathbf{f}(n, p_n)$ *is almost periodic in* n*.*

Remark 3.16. If $x : \mathbb{N}(n_0) \to X$ is a.a.p., then the decomposition $x = x_1 + x_2$, in the definition of an a.a.p. function, is unique (see [18]).

4. Existence of Almost Periodic Solutions

From now on we will assume that the system (1.1) has a unique solution for a given initial condition on $\mathcal{B}$ and without loss of generality $n_0 = 0$, thus $N_{n_0} = N_0 = \mathbb{Z}^+$.

We will make the following assumptions on (1.1).

(H1) $F : \mathbb{Z}^+ \times \mathcal{B} \to \mathbb{C}^r$ is continuous in the second variable for any fixed $n \in \mathbb{Z}^+$.

(H2) System (1.1) has a bounded solution $y = \{y(n)\}_{n \geq 0}$, passing through $(0, \varphi)$, $\varphi \in \mathcal{B}$, that is, $\sup_{n \geq 0} |y(n)| < \infty$.

For this bounded solution $\{y(n)\}_{n \geq 0}$, there is an $\alpha > 0$ such that $|y(n)| \leq \alpha$ for all n. So, we will have to assume that $\|y_n\|_{\mathcal{B}} \leq \alpha$ for all n, and $y_n \in \Sigma_\alpha = \{\phi \in \mathcal{B} / \|\phi\|_{\mathcal{B}} \leq \alpha\}$. Next, we will point out the definitions of stability for functional difference equations adapting it from the continuouscase according to Hino et al. in [3].

Definition 4.1. A bounded solution $x = \{x(n)\}_{n\geq 0}$ of (1.1) is said to be:

(i) *stable*, if for any $\epsilon > 0$ and any integer $\tau \geq 0$, there is $\delta := \delta(\epsilon, \tau) > 0$ such that $\|x_\tau - y_\tau\|_{\mathcal{B}} < \delta$ implies that $\|x_n - y_n\|_{\mathcal{B}} < \epsilon$ for all $n \geq \tau$, where $\{y(n)\}_{n\geq\tau}$ is any solution of (1.1);

(ii) *uniformly stable*, abbreviated as "$x \in \mathcal{US}$", if for any $\epsilon > 0$ and any integer $\tau \geq 0$, there is $\delta := \delta(\epsilon) > 0$ (δ does not depend on τ) such that $\|x_\tau - y_\tau\|_{\mathcal{B}} < \delta$ implies that $\|x_n - y_n\|_{\mathcal{B}} < \epsilon$ for all $n \geq \tau$, where $\{y(n)\}_{n\geq\tau}$ is any solution of (1.1);

(iii) *uniformly asymptotically stable*, abbreviated as "$x \in \mathcal{UAS}$", if it is uniformly stable and there is $\delta_0 > 0$ such that for any $\epsilon > 0$, there is a positive integer $N = N(\epsilon) > 0$ such that if $\tau \geq 0$ and $\|x_\tau - y_\tau\|_{\mathcal{B}} < \delta_0$, then $\|x_n - y_n\|_{\mathcal{B}} < \epsilon$ for all $n \geq \tau + N$, where $\{y(n)\}_{n\geq\tau}$ is any solution of (1.1);

(iv) *globally uniformly asymptotically stable*, abbreviated as "$x \in \mathcal{GUAS}$", if it is uniformly stable and $\|x_n - y_n\|_{\mathcal{B}} \to 0$ as $n \to +\infty$, whenever $\{y(n)\}_{n\geq\tau}$ is any solution of (1.1).

Remark 4.2. It is easy to see that an equivalent definition for $x = \{x(n)\}_{n\geq 0}$, being $\mathcal{UAS}$, is the following:

(iii)* $x = \{x(n)\}_{n\geq 0}$ is $\mathcal{UAS}$, if it is uniformly stable, and there exists $\delta_0 > 0$ such that if $\tau \geq 0$ and $\|x_\tau - y_\tau\|_{\mathcal{B}} < \delta_0$, then $\|x_n - y_n\|_{\mathcal{B}} \to 0$ as $n \to +\infty$, where $\{y_n\}_{n\geq\tau}$ is any solution of (1.1).

4.1. The Periodic Case

Here, we will assume what follows.

(H3) The function $F(n, \cdot)$ in (1.1) is periodic in $n \in \mathbb{Z}^+$, that is, there exists a positive integer T such that $F(n + T, \cdot) = F(n, \cdot)$ for all $n \in \mathbb{Z}^+$.

Moreover, we will assume what follows.

$(\widetilde{A})$ The sequences $M(n)$ and $N(n)$ in Axiom (A)(iii) are bounded by M and N, respectively and $M < 1$.

Lemma 4.3. *Suppose that condition $(\widetilde{A})$ holds. If $\{y(n)\}$ is a bounded solution of* (1.1) *such that $y_0 \in \mathcal{B}$, then y_n is also bounded in $\mathbb{Z}^+$.*

Proof. Let us say that $|y(n)| \leq R$ for all $n \in \mathbb{Z}$. Then by Axiom (A)(iii) and hypothesis $(\widetilde{A})$ we have

$$\|y_n\|_{\mathcal{B}} \leq N \sup_{0\leq s\leq n} |y(s)| + M\|y_0\|_{\mathcal{B}} \leq NR + M\|y_0\|_{\mathcal{B}}, \quad \forall n \in \mathbb{Z}^+. \tag{4.1}$$

□

Lemma 4.4. *Suppose that condition $(\widetilde{A})$ holds. Let $\{y^k(n)\}_{k\geq 1}$ be a sequence in $\mathbb{C}^r$ such that $y_0^k \in \mathcal{B}$ for all $k \geq 1$. Assume that $y^k(s) \to \eta(s)$ as $k \to +\infty$ for every $s \in \mathbb{Z}$ and $\eta_0 \in \mathcal{B}$, then $y_n^k \to \eta_n$ in $\mathcal{B}$ as $k \to +\infty$ for each $n \in \mathbb{Z}^+$. In particular, if $y^k(s) \to \eta(s)$ as $k \to +\infty$ uniformly in $s \in \mathbb{Z}$, then $y_n^k \to \eta_n$ in $\mathcal{B}$ as $k \to +\infty$ uniformly in $n \in \mathbb{Z}^+$.*

Proof. By Axiom (A)(iii) and hypotheses we have that

$$\|y_n^k - \eta_n\|_{\mathcal{B}} \le N \sup_{0\le s\le n} \left| y^k(s) - \eta(s) \right| + M\|y_0^k - \eta_0\|_{\mathcal{B}}, \quad \text{for any } n \ge 0. \tag{4.2}$$

In the particular case $n = 0$ we obtain

$$\|y_0^k - \eta_0\|_{\mathcal{B}} \le \frac{N}{1-M} \left| y^k(0) - \eta(0) \right|, \tag{4.3}$$

and so $\|y_0^k - \eta_0\|_{\mathcal{B}} \longrightarrow 0$ as $k \longrightarrow +\infty$. On the other hand, since n is fixed, it follows that

$$\sup_{0\le s\le n} \left| y^k(s) - \eta(s) \right| \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty, \tag{4.4}$$

for each $n \in \mathbb{Z}^+$. Therefore, we have concluded the proof. □

Theorem 4.5. *Suppose that condition* $(\widetilde{A})$ *and (H1)–(H3) hold. If the bounded solution* $\{y(n)\}_{n\ge 0}$ *of* (1.1) *is* $\mathcal{US}$*, then* $\{y(n)\}$ *is an a.a.p. sequence in* $\mathbb{C}^r$*, equivalently,* (1.1) *has an a.a.p. solution.*

Proof. By Lemma 4.3 there exists $\alpha \in \mathbb{R}^+$ such that $\|y_n\|_{\mathcal{B}} \le \alpha$ for all $n \in \mathbb{Z}^+$, and a bounded (or compact) set $\Sigma_\alpha \subset \mathcal{B}$ such that $y_n \in \Sigma_\alpha$ for all $n \ge 0$. Let $\{n_k\}_{k\ge 1}$ be any integer sequence such that $n_k > 0$ and $n_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$. For each n_k, there exists a nonnegative integer m_k such that $m_k T \le n_k \le (m_k+1)T$. Set $n_k = m_k T + \tau_k$. Then $0 \le \tau_k < T$ for all $k \ge 1$. Since $\{\tau_k\}_{k\ge 1}$ is a bounded set, we can assume that, taking a subsequence if necessary, $\tau_k = j_*$ for all $k \ge 1$, where $0 \le j_* < T$. Now, set $y^k(n) = y(n+n_k)$. Thus,

$$y^k(n+1) = y(n+n_k+1) = F(n+n_k, y_{n+n_k}) = F\left(n+n_k, y_n^k\right) = F\left(n+j_*, y_n^k\right), \tag{4.5}$$

which implies that $\{y^k(n)\}$ is a solution of the system,

$$x(n+1) = F(n+j_*, x_n), \tag{4.6}$$

through $(0, y_{n_k})$. It is clear that if $\{y(n)\}_{n\ge 0}$ is $\mathcal{US}$, then $\{y^k(n)\}_{n\ge 0}$ is also $\mathcal{US}$ with the same pair $(\epsilon, \delta(\epsilon))$ as the one for $\{y(n)\}_{n\ge 0}$.

Since $\{y(n+n_k)\}$ is bounded for all n and n_k, we can use the diagonal method to get a subsequence $\{n_{k_j}\}$ of $\{n_k = m_k T + j_*\}$ such that $y(n+n_{k_j})$ converges for each $n \in \mathbb{Z}$ as $j \longrightarrow +\infty$. Thus, we can assume that the sequence $y(n+n_k)$ converges for each $n \in \mathbb{Z}$ as $k \longrightarrow +\infty$. Since $y_0^k = y_{n_k} \in \mathcal{B}$, by Lemma 4.4 it follows that y_n^k is also convergent for each $n \in \mathbb{Z}$. In particular, for any $\epsilon > 0$ there exists a positive integer $N_1(\epsilon)$ such that if $k, m \ge N_1(J\epsilon)$ (J is the constant given in Axiom A(ii)), then

$$\|y_0^k - y_0^m\|_{\mathcal{B}} < \delta(\epsilon), \tag{4.7}$$

where $\delta(\epsilon)$ is the number given by the uniform stability of $\{y(n)\}_{n\geq 0}$. Since $y^k(n) \in \mathcal{US}$, it follows from Definition 4.1 and (4.7) that

$$\|y_n^k - y_n^m\|_{\mathcal{B}} < J\epsilon, \quad \forall n \geq 0, \tag{4.8}$$

and by Axiom A(ii) it follows that

$$\left|y^k(n) - y^m(n)\right| < \epsilon, \quad \forall n \geq 0, \; k, m \geq N_1(\epsilon). \tag{4.9}$$

This implies that for any positive integer sequence n_k, $n_k \to +\infty$ as $k \to +\infty$, there is a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ for which $\{y(n + n_{k_j})\}$ converges uniformly on $\mathbb{Z}^+$ as $j \to +\infty$. Thus, the conclusion of the theorem follows from Lemma 3.9(d). □

Before proving our following result we remark that if y is a.a.p. then there are unique sequences $p, q : \mathbb{Z} \to \mathbb{C}^r$ such that $y(n) = p(n) + q(n)$, with p a.p. and $q(n) \to 0$ as $n \to 0$ as $n \to +\infty$. By Lemma 3.9(a) it follows that p is bounded and thus $p \in B(\mathbb{Z}^-, \mathbb{C}^r)$. Hence, by Axiom (C) we must have that $p_n \in \mathcal{B}$ for all $n \geq 0$. In particular, $q_n = y_n - p_n \in \mathcal{B}$ for all $n \geq 0$.

Theorem 4.6. *Suppose that* $(\widetilde{A})$ *and (H1)–(H3) hold and the bounded solution* $\{y(n)\}_{n\geq 0}$ *of* (1.1) *is* $\mathcal{US}$*, then system* (1.1) *has an a.p. solution, which is also* $\mathcal{US}$*.*

Proof. It follows from Theorem 4.5 that y is an a.a.p. Set $y(n) = p(n) + q(n)$ $(n \geq 0)$, where $\{p(n)\}_{n\geq 0}$ is a.p. sequence and $q(n) \to 0$ as $n \to +\infty$. For the positive integer sequence $\{n_k T\}$, by Lemma 3.9(b)–(d) and arguments of the previous theorem, we can choice a subsequence $\{n_{k_j} T\}$ of $\{n_k T\}$ such that $y(n + n_{k_j} T)$ converges uniformly in $n \in \mathbb{Z}$ and $p(n + n_{k_j} T) \to \eta(n)$ uniformly on $\mathbb{Z}$ as $j \to +\infty$ and $\{\eta(n)\}$ is also a.p. Then, $y(n + n_{k_j} T) \to \eta(n)$ uniformly in $n \in \mathbb{Z}$, and thus by Lemma 4.4 $y_{n+n_{k_j}T} \to \eta_n$ uniformly in $n \in \mathbb{Z}^+$ on $\mathcal{B}$ as $j \to +\infty$ and $\eta_n \in \mathcal{B}$. Since

$$\eta(n+1) \longleftarrow y\left(n + n_{k_j} T + 1\right) = F\left(n + n_{k_j} T, y_{n+n_{k_j}T}\right) = F\left(n, y_{n+n_{k_j}T}\right) \longrightarrow F(n, \eta_n) \tag{4.10}$$

as $j \to +\infty$, we have $\eta(n+1) = F(n, \eta_n)$ for $n \geq 0$, that is, the system (1.1) has an almost periodic solution, and so we have proved the first statement of the theorem.

In order to prove the second affirmation, notice that $y(n+n_{k_j} T) \in \mathcal{US}$ since $y \in \mathcal{US}$. For any $n_0 \in \mathbb{Z}^+$, let $\{x(n)\}_{n\geq 0}$ be a solution of (1.1) such that $x_0 \in \mathcal{B}$ and $\|\eta_{n_0} - x_{n_0}\|_{\mathcal{B}} := \mu < \delta(\epsilon)$. Again, by Lemma 4.4 $y_n^{k_j} \to \eta_n$ as $j \to +\infty$ for each $n \geq 0$, so there is a positive integer $J_1 > 0$ such that if $j \geq J_1$, then

$$\|y_{n_0}^{k_j} - \eta_{n_0}\|_{\mathcal{B}} < \delta(\epsilon) - \mu. \tag{4.11}$$

Thus, for $j \geq J_1$, we have

$$\|y_{n_0+n_{k_j}T} - x_{n_0}\|_{\mathcal{B}} \leq \|y_{n_0+n_{k_j}T} - \eta_{n_0}\|_{\mathcal{B}} + \|\eta_{n_0} - x_{n_0}\|_{\mathcal{B}} < \delta(\epsilon). \tag{4.12}$$

Then,

$$\|y_{n+n_{k_j}T} - x_n\|_{\mathcal{B}} < \epsilon \quad \forall n \geq n_0. \tag{4.13}$$

Therefore, there is $J_2 > 0$ such that if $j \geq J_2$, then

$$\|\eta_{n_0} - y_{n_0+n_{k_j}T}\|_{\mathcal{B}} < \delta(\nu), \tag{4.14}$$

and hence, $\|\eta_n - y_{n+n_{k_j}T}\|_{\mathcal{B}} < \nu$ for all $n \geq n_0$, where $(\nu, \delta(\nu))$ is a pair for the uniform stability of $y(n + n_{k_j}T)$. This shows that if $j \geq \max\{J_1, J_2\}$, then

$$\|\eta_n - x_n\|_{\mathcal{B}} \leq \|\eta_n - y_{n+n_{k_j}T}\|_{\mathcal{B}} + \|y_{n+n_{k_j}T} - x_n\|_{\mathcal{B}} < \epsilon + \nu, \tag{4.15}$$

for all $n \geq n_0$, which implies that $\|\eta_n - x_n\|_{\mathcal{B}} \leq \epsilon$ for all $n \geq n_0$ if $\|\eta_{n_0} - x_{n_0}\|_{\mathcal{B}} < \delta(\epsilon)$ because ν is arbitrary. This proves that $\eta(n)$ is $\mathcal{US}$. □

In the case when we have an asymptotically stable solution of (1.1) we obtain the following result.

Theorem 4.7. *Suppose that* $(\widetilde{A})$ *and (H1)–(H3) hold and the bounded solution* $\{y(n)\}_{n\geq 0}$ *of* (1.1) *is* $\mathcal{UAS}$*, then the system* (1.1) *has a periodic solution of period* mT *for some positive integer* m*, which is also* $\mathcal{UAS}$*.*

Proof. Set $y^k(n) = y(n + kT)$, $k = 1, 2, \ldots$. By the proof of Theorem 4.5, there is a subsequence $\{y^{k_j}(n)\}$ which converges to a solution $\{\eta(n)\}$ of (4.6) for each $n \in \mathbb{Z}$ and hence by Lemma 4.4, $y_0^{k_j} \to \eta_0$ as $j \to +\infty$. Thus, there is a positive integer p such that $\|y_0^{k_p} - y_0^{k_p+1}\|_{\mathcal{B}} < \delta_0$ $(0 \leq k_p < k_p + 1)$, where δ_0 is obtained from the uniformly asymptotic stability of $\{y(n)\}_{n\geq 0}$. Let $m = k_{p+1} - k_p$, and notice that $y^m(n) = y(n + mT)$ is a solution of (1.1). Since $y^m_{k_pT}(j) = y^m(k_pT + j) = y(k_{p+1}T + j) = y_{k_{p+1}T}(j)$ for $j \in \mathbb{Z}^-$, that is, $y^m_{k_pT} = y_{k_{p+1}+T}$, we have

$$\|y^m_{k_pT} - y_{k_pT}\|_{\mathcal{B}} = \|y_{k_{p+1}T} - y_{k_pT}\|_{\mathcal{B}} = \|y_0^{k_{p+1}} - y_0^{k_p}\|_{\mathcal{B}} < \delta_0, \tag{4.16}$$

and hence,

$$\|y^m_n - y_n\|_{\mathcal{B}} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{4.17}$$

because $\{y(n)\}_{n\geq 0}$ is $\mathcal{UAS}$ (see also Remark 4.2). On the other hand, $\{y(n)\}_{n\geq 0}$ is a.a.p. by Theorem 4.5, then

$$y(n) = p(n) + q(n), \quad n \geq 0, \tag{4.18}$$

where $\{p(n)\}_{n\in\mathbb{Z}}$ is a.p. and $q(n) \to 0$ as $n \to +\infty$. It follows from (4.17) and (4.18) that

$$|p(n) - p(n + mT)| \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \tag{4.19}$$

which implies that $p(n) = p(n + mT)$ for all $n \in \mathbb{Z}$ because $\{p(n)\}$ is a.p.

For the integer sequence $\{kmT\}$, $k = 1, 2, \ldots$, we have $y(n+kmT) = p(n)+q(n+kmT)$. Then $y(n + kmT) \to p(n)$ uniformly for all $n \in \mathbb{Z}$ as $k \to +\infty$, and again by Lemma 4.4, $y_{n+kmT} \to p_n$ uniformly in $n \in \mathbb{Z}^+$ as $k \to +\infty$. Since $y(n + kmT + 1) = F(n, y_{n+kmT})$, we have $p(n + 1) = F(n, p_n)$ for $n \geq 0$, which implies that (1.1) has a periodic solution $\{p(n)\}_{n\geq 0}$ of period mT.

Now, we will proceed to prove that $p \in \mathcal{UAS}$ by the use of definition (ii)* in Remark 4.2. Notice that since $y \in \mathcal{UAS}$ then $y^{k_j}(n)$ is a $\mathcal{UAS}$ solution of (1.1) with the same δ_0 as the one for $\{y(n)\}$. Let $\{x(n)\}$ be any solution of (1.1) such that $\|p_{n_0} - x_{n_0}\|_{\mathcal{B}} < \delta_0$. Set $\|p_{n_0} - x_{n_0}\|_{\mathcal{B}} := \mu < \delta_0$. Again, for sufficient large j, we have the similar relations (4.12) and (4.14) with $\|y_{n_0+n_{k_j}T} - x_{n_0}\|_{\mathcal{B}} < \delta_0$ and $\|y_{n_0+n_{k_j}T} - \eta_{n_0}\|_{\mathcal{B}} < \delta_0$. Thus,

$$\|\eta_n - x_n\|_{\mathcal{B}} \leq \|\eta_n - y_{n+n_{k_j}T}\|_{\mathcal{B}} + \|y_{n+n_{k_j}T} - x_n\|_{\mathcal{B}} \longrightarrow 0, \tag{4.20}$$

as $n \to +\infty$ if $\|y_{n_0} - x_{n_0}\|_{\mathcal{B}} < \delta_0$, because y^{k_j}, x, and $\eta(n)$ satisfy (1.1). This completes the proof. □

Finally, if the particular solution is $\mathcal{GUAS}$, we will prove that system (1.1) has a periodic solution.

Theorem 4.8. *Suppose that* $(\widetilde{A})$ *and (H1)–(H3) hold and that the bounded solution* $\{y(n)\}_{n\geq 0}$ *of* (1.1) *is* $\mathcal{GUAS}$*, then the system* (1.1) *has a periodic solution of period* T*.*

Proof. By Theorem 4.5, y is a.a.p. Then $y(n) = p(n) + q(n)$ $(n \geq 0)$, where $\{p(n)\}$ $(n \in \mathbb{Z})$ is an a.p. sequence and $q(n) \to 0$ as $n \to +\infty$. Notice that $y(n + T)$ is also a solution of (1.1) satisfying $y_T \in \Sigma_\alpha$. Since $\{y(n)\}$ is $\mathcal{GUAS}$, we have that $\|y_n - y_{n+T}\|_{\mathcal{B}} \to 0$ as $n \to +\infty$, which implies that $p(n) = p(n+T)$ for all $n \in \mathbb{Z}$. Using same technique as in the proof of Theorem 4.7, we can show that $\{p(n)\}$ is a T-periodic solution of (1.1). □

4.2. The Almost Periodic Case

Here, we will assume that

(H4) the function $F(n, \cdot)$ in (1.1) is almost periodic in $n \in \mathbb{Z}^+$ uniformly in the second variable.

By $C(F)$ we denote the uniform closure of F, that is, $C(F) = \{G/\exists\, \alpha_k$ such that $\alpha_k \to +\infty$ and $F(n + \alpha_k, \cdot) \to G(n, \cdot)$ uniformly on $\mathbb{Z}^+ \times \Sigma$ as $k \to +\infty$ where Σ is any compact set in $\mathcal{B}\}$. Note that $C(F) \subset \mathcal{AP}(\mathbb{Z}^+ \times \mathcal{B}, \mathbb{C}^r)$ by Lemma 3.12 and $F \in C(F)$ by Lemma 3.13.

Lemma 4.9. *Suppose that Axiom (C) is true, and that* $\{x(n)\}_{n\in\mathbb{Z}}$ *is an a.p. sequence with* $x_0 \in \mathcal{B}$*, then* x_n *is a.p.*

Proof. We know that, given $\epsilon > 0$, there exists an integer $l = l(\epsilon) > 0$ such that each discrete interval of length l contains a $\tau = \tau(\epsilon) \in E\{\epsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \frac{\epsilon}{K}, \quad \forall\, n \in \mathbb{Z}. \tag{4.21}$$

By Axiom (C) we have

$$\begin{aligned} \|x_{n+\tau} - x_n\|_{\mathcal{B}} &\le K\|x_{n+\tau} - x_n\|_{\infty} \\ &= K\sup_{\theta\le 0}|x_{n+\tau}(\theta) - x_n(\theta)| \\ &= K\sup_{\theta\le 0}|x(n+\tau+\theta) - x(n+\theta)| \\ &< \epsilon. \end{aligned} \tag{4.22}$$

□

Lemma 4.10. *Suppose that $\mathcal{B}$ is a fading memory space and $\{x(n)\}_{n\in\mathbb{Z}}$ is a.a.p. with $x_0 \in \mathcal{B}$, then x_n is a.a.p.*

Proof. Since $x(n)$ is a.a.p. there are unique sequences $y(n)$ and $v(n)$ such that y is a.p. and $v(n) \to 0$ as $n \to +\infty$. Then by Lemma 4.9 it follows that y_n is a.p., and by Lemma 2.3 it follows that $v_n \to 0$ as $n \to +\infty$. Therefore, $x_n = y_n + v_n$ is a.a.p. □

Theorem 4.11. *Suppose that conditions $(\widetilde{A})$, (H1)-(H2), and (H4) hold and that $\mathcal{B}$ is a fading memory space. If the bounded solution $\{y(n)\}_{n\ge 0}$ of* (1.1) *is an a.a.p. sequence, then the system* (1.1) *has an a.p. solution.*

Proof. Since the solution $\{y(n)\}_{n\ge0}$ is a.a.p., it follows from Lemma 3.10 that $y(n)$ has a unique decomposition $y(n) = p(n) + q(n)$, where $\{p(n)\}_{n\in\mathbb{Z}}$ is a.p. and $q(n) \to 0$ as $n \to +\infty$. Notice that $\{y(n)\}$ is bounded. By Lemma 4.3 there is a compact set Σ_α in $\mathcal{B}$ such that $y_n, p_n \in \Sigma_\alpha$ for all $n \ge 0$. By Lemma 3.13, there is an integer sequence $\{n_k\}$, $n_k > 0$, such that $n_k \to +\infty$ as $k \to +\infty$ and $F(n+n_k, \phi) \to F(n, \phi)$ uniformly on $\mathbb{Z} \times \Sigma_\alpha$ as $k \to +\infty$. Taking a subsequence if necessary, we can also assume that $p(n+n_k) \to \widetilde{p}(n)$ uniformly on $\mathbb{Z}$, and by Lemma 3.9(b) we have that $\{\widetilde{p}(n)\}$ is also an a.p. sequence. For any $s \in \mathbb{Z}^-$, there is a positive integer k_0 such that if $k > k_0$, then $s + n_k \ge 0$. In this case, we see that $y(n+n_k) \to \widetilde{p}(n)$ uniformly for all n as $k \to +\infty$, and hence by Lemma 4.4 $y_{n+n_k} \to \widetilde{p}_n$ in $\mathcal{B}$ in $n \in \mathbb{Z}^+$ as $k \to +\infty$. Since

$$\begin{aligned} y(n+n_k+1) &= F(n+n_k, y_{n+n_k}) \\ &= \left[F(n+n_k, y_{n+n_k}) - F(n+n_k, \widetilde{p}_n)\right] \\ &\quad + \left[F(n+n_k, \widetilde{p}_n) - F(n, \widetilde{p}_n)\right] + F(n, \widetilde{p}_n), \end{aligned} \tag{4.23}$$

and from the previous considerations the first term of the right-hand side of (4.23) tends to zero as $k \to +\infty$ and since $F(n+n_k, \widetilde{p}_n) - F(n, \widetilde{p}_n) \to 0$ as $k \to +\infty$, we have that $\widetilde{p}(n+1) = F(n, \widetilde{p}_n)$ for all $n \in \mathbb{Z}^+$, which implies that (1.1) has an a.p. solution $\{\widetilde{p}(n)\}_{n\ge0}$ passing through $(0, \widetilde{p}_0)$, where $\widetilde{p}_0(j) = \widetilde{p}(j)$ for $j \in \mathbb{Z}^-$. □

We are now in a position to prove the following result.

Theorem 4.12. *Suppose that the assumptions* $(\widetilde{A})$, *(H1), (H2), and (H4) hold, and that* $\mathcal{B}$ *is a fading memory space. If the bounded solution* $\{y(n)\}_{n\geq 0}$ *of* (1.1) *is* $\mathcal{UAS}$, *then* $\{y(n)\}_{n\geq 0}$ *is a.a.p. Consequently,* (1.1) *has an a.p. solution which is* $\mathcal{UAS}$.

Proof. Let the bounded solution y of (1.1) be $\mathcal{UAS}$ with the triple $(\delta(\epsilon),\delta_0,N(\epsilon))$. Let $\{n_k\}_{k\geq 1}$ be any positive integer such that $n_k \to +\infty$ as $k \to +\infty$. Set $y^k(n) = y(n+n_k)$. As previously $y^k(n)$ is a solution of

$$x(n+1) = F(n+n_k, x_n), \tag{4.24}$$

and $\{y^k(n)\}$ is $\mathcal{UAS}$ with the same triple $(\delta(\epsilon),\delta_0,N(\epsilon))$. By Lemma A.2, for the set Σ_α and any $0<\epsilon<1$ there exists $\delta_1(\epsilon)>0$ such that $|h(n)|<\delta_1(\epsilon)$ and $\|x^k_{n_0}-x_{n_0}\|_{\mathcal{B}}<\delta_1(\epsilon)$ for some $n_0\geq 0$ implies that $\|x^k_n - x_n\|_{\mathcal{B}}<\epsilon/2$ for all $n\geq n_0$, where $\{x(n)\}_{n\geq n_0}$ is a bounded solution of

$$x(n+1) = F(n+n_k, x_n) + h(n), \tag{4.25}$$

passing through (n_0, x_{n_0}) and $x_n\in\Sigma_\alpha$ for $n\geq n_0$. Since $y^k(j)$ is uniformly bounded for all $k\geq 1$ and $j\in\mathbb{Z}$, taking a subsequence if necessary, we can assume that $\{y^k(j)\}$ is convergent for each $j\in\mathbb{Z}$ and $F(n+n_k,\phi)\to G(n,\phi)$ uniformly on $\mathbb{Z}^+\times\Sigma_\alpha$, for some a.p. function G. In this case, by Lemma 4.4 there is a positive integer $k_1(\epsilon)$ such that if $m,k\geq k_1(\epsilon)$, then

$$\|y^k_0 - y^m_0\|_{\mathcal{B}} < \delta_1(\epsilon). \tag{4.26}$$

On the other hand, $y^m_n\in\Sigma_\alpha$ for $n\in\mathbb{Z}^+$ is a solution of (4.25) with $h(n)=h_{k,m}(n)$, that is,

$$x(n+1) = F(n+n_k, x_n) + h_{k,m}(n), \tag{4.27}$$

where $h_{k,m}(n)$ is defined by the relation

$$h_{k,m}(n) = F\left(n+n_m, y^m_n\right) - F\left(n+n_k, y^m_n\right), \quad n\in\mathbb{Z}^+. \tag{4.28}$$

To apply Lemma A.2 to (4.24) and its associated equation (4.27), we will point out some properties of the sequence $\{h_{k,m}(n)\}_{n\geq 0}$. Since $F(n+n_k,\phi)\to G(n,\phi)$ uniformly on $\mathbb{Z}^+\times\Sigma_\alpha$, for the above $\delta_1(\epsilon)>0$, there is a positive integer $k_2(\epsilon)>k_1(\epsilon)$ such that if $k,m\geq k_2(\epsilon)$, then

$$\|F(n+n_m,\phi) - F(n+n_k,\phi)\| < \delta_1(\epsilon), \quad \forall n\in\mathbb{Z}^+,\ \phi\in\Sigma_\alpha, \tag{4.29}$$

which implies that $|h_{k,m}(n)| = |F(n+n_m,y^m_n) - F(n+n_k,y^m_n)| < \delta_1(\epsilon)$ for all $n\in\mathbb{Z}$. Applying Lemma A.2 to (4.24) and its associated equation (4.27) with the above arguments and condition (4.26), we conclude that for any positive integer sequence $\{n_k\}_{k\geq 1}$, $n_k\to+\infty$ as $k\to+\infty$, and $\epsilon>0$, there is a positive integer $k_2(\epsilon)>0$ such that

$$\|y^k_n - y^m_n\|_{\mathcal{B}} < \frac{\epsilon}{J}, \quad n\geq 0 \text{ if } k,m>k_2(\epsilon), \tag{4.30}$$

and hence by Axiom A(ii) $|y^k(n) - y^m(n)| < \epsilon$ for all $n \geq 0$ if $k, m > k_2(\epsilon)$. This implies that the bounded solution $\{y(n)\}_{n\geq 0}$ of (1.1) is a.a.p. by Lemma 3.9(d). Furthermore, (1.1) has an a.p. solution, which is $\mathcal{UAS}$ by Theorem 4.11. This ends the proof. □

Appendix

The proof of the following lemmas used ideas developed by Hino et al. in [3] for the functional differential equations with infinite delay and by Song [12] for functional difference equations with finite delay.

Lemma A.1. *Suppose that* $(\widetilde{A})$, *(H1), (H2), and (H4) hold and that* $\mathcal{B}$ *is a fading memory space. Let* y *be the bounded solution of* (1.1). *Let* $\{n_k\}_{k\geq 1}$ *be a positive integer sequence such that* $n_k \to +\infty$, $y_{n_k} \to \phi$, *and* $F(n + n_k, \phi) \to G(n, \phi)$ *uniformly on* $\mathbb{Z} \times \Sigma$ *as* $k \to +\infty$, *where* Σ *is any compact subset in* $\mathcal{B}$ *and* $G \in C(F)$. *If the bounded solution* $\{y(n)\}_{n\geq 0}$ *is* $\mathcal{US}$, *then the solution* $\{\eta(n)\}_{n\geq 0}$ *of*

$$x(n+1) = G(n, x_n), \tag{A.1}$$

through $(0, \phi)$, *is* $\mathcal{US}$. *In addition, if* $\{y(n)\}_{n\geq 0}$ *is* $\mathcal{UAS}$, *then* $\{\eta(n)\}_{n\geq 0}$ *is also* $\mathcal{UAS}$.

Proof. Set $y^k(n) = y(n + n_k)$. It is easy to see that $y^k(n)$ is a solution of

$$x(n+1) = F(n + n_k, x_n), \quad n \geq 0, \tag{A.2}$$

passing though $(0, y_{n_k})$ and $y_n^k \in \Sigma_\alpha$ for all k. Since $\{y(n)\}_{n\geq 0}$ is $\mathcal{US}$, then $\{y^k(n)\}$ is also $\mathcal{US}$ with the same pair $(\epsilon, \delta(\epsilon))$ as the one for $\{y(n)\}_{n\geq 0}$. Taking a subsequence if necessary, we can assume that $\{y^k(n)\}_{k\geq 1}$ converges to a vector $\eta(n)$ for each $n \geq 0$ as $k \to +\infty$. From (4.23) with $\widetilde{p}_n = \eta_n$, we can see that $\{\eta(n)\}_{n\geq 0}$ is the unique solution of (A.1), satisfying $\eta_0 = \phi$ because $y_{n_k} \to \phi$.

To show that the solution $\{\eta(n)\}_{n\geq 0}$ of (A.1) is $\mathcal{US}$, we need to prove that for any $\epsilon > 0$ and any integer $n_0 \geq 0$, there exists $\delta^*(\epsilon) > 0$ such that $\|\eta_{n_0} - y_{n_0}\|_{\mathcal{B}} < \delta^*(\epsilon)$ implies that $\|\eta_n - y_n\|_{\mathcal{B}} < \epsilon$ for all $n \geq n_0$, where $\{y(n)\}_{n\geq n_0}$ is a solution of (A.1) with $y_{n_0} = \chi \in \mathcal{B}$.

We know from Lemma 4.4 that $y_n^k \to \eta_n$ as $k \to +\infty$ for each n; thus, for any given $n_0 \in \mathbb{Z}^+$, if k is sufficiently large; say $k \geq k_0 > 0$, we have

$$\|y_{n_0}^k - \eta_{n_0}\|_{\mathcal{B}} < \frac{1}{2}\delta\left(\frac{\epsilon}{2}\right), \tag{A.3}$$

where $\delta(\epsilon)$ comes from the uniform stability of $\{y(n)\}_{n\geq 0}$. Let $\chi \in \mathcal{B}$ be such that

$$\|\chi - \eta_{n_0}\|_{\mathcal{B}} < \frac{1}{2}\delta\left(\frac{\epsilon}{2}\right), \tag{A.4}$$

and let $\{x(n)\}_{n\geq n_0}$ be the solution of (1.1) such that $x_{n_0+n_k} = \phi$. Then $\{x^k(n) = x(n+n_k)\}$ is a solution of (A.2) with $x^k_{n_0} = \phi$. Since $\{y^k(n)\}$ is $\mathcal{US}$ and $\|x^k_{n_0} - y^k_{n_0}\|_{\mathcal{B}} < \delta(\epsilon/2)$ for $k \geq k_0$, we have

$$\|y^k_n - x^k_n\|_{\mathcal{B}} < \frac{\epsilon}{2} \quad \forall n \geq n_0,\ k \geq k_0. \tag{A.5}$$

It follows from (A.5) that

$$\|x^k_n\|_{\mathcal{B}} \leq \|y^k_n\|_{\mathcal{B}} + \frac{\epsilon}{2} < \alpha + \frac{\epsilon}{2} \quad n \geq n_0,\ k \geq k_0. \tag{A.6}$$

Then there exists a number $\alpha^* > 0$ such that $x^k_n \in S_{\alpha^*}$ for all $n \geq 0$ and $k \geq k_0$, which implies that there is a subsequence of $\{x^k(n)\}_{k\geq 0}$ for each $n \geq n_0 - \tau$, denoted by $\{x^k(n)\}$ again, such that $x^k(n) \to y(n)$ for each $n \geq n_0 - \tau$, and hence by Lemma 4.4 $x^k_n \to y_n$ for all $n \geq n_0$ as $k \to +\infty$. Clearly, $y_{n_0} = \chi$, and the set S_{α^*} is compact set $\mathcal{B}$. Since $F(n,\phi)$ is almost periodic in n uniformly for $\phi \in \mathcal{B}$, we can assume that, taking a subsequence if necessary, $F(n+n_k,\phi) \to G(n,\phi)$ uniformly on $\mathbb{Z} \times S_{\alpha^*}$ as $k \to +\infty$. Taking $k \to +\infty$ in $x^k(n+1) = F(n+n_{n_k}, x^n_k)$, we have $y(n+1) = G(n, y_n)$, namely, $\{y(n)\}$ is the unique solution of (A.1), passing through (n_0,χ) with $y_{n_0} = \chi \in \mathcal{B}$. On the other hand, for any integer $N > 0$, there exists $k_N \geq k_0$ such that if $k \geq k_N$, then

$$\|x^k_n - y_n\|_{\mathcal{B}} < \frac{\epsilon}{4}, \quad \|y^k_n - \eta_n\|_{\mathcal{B}} < \frac{\epsilon}{4} \quad \text{for } n_0 \leq n \leq n_0 + N. \tag{A.7}$$

From (A.5) and (A.7), we obtain

$$\|\eta_n - y_n\|_{\mathcal{B}} < \epsilon \quad \text{for } n_0 \leq n \leq n_0 + N. \tag{A.8}$$

Since N is arbitrary, we have $\|\eta_n - y_n\|_{\mathcal{B}} < \epsilon$ for all $n \geq n_0$ if $\|\chi - \eta_{n_0}\|_{\mathcal{B}} < \delta(\epsilon/2)/2$ and $\phi \in \mathcal{B}$, which implies that the solution $\{\eta(n)\}_{n\geq 0}$ of (A.1) is $\mathcal{US}$.

Now, we consider the case where $\{y(n)\}_{n\geq 0}$ is $\mathcal{UAS}$. Then the solution $\{y^k(n)\}$ of (A.2) is also $\mathcal{UAS}$ with the same pair $(\delta_0, \epsilon, N(\epsilon))$ as the one for $\{y(n)\}_{n\geq 0}$. Let $(\delta^*(\epsilon), \epsilon)$ be the pair for uniform stability of $\{\eta(n)\}$.

For any given $n_0 \in \mathbb{Z}^+$, if k is sufficiently large; say $k \geq k_0 > 0$, we have

$$\|y^k_{n_0} - \eta_{n_0}\|_{\mathcal{B}} < \frac{1}{2}\delta_0, \tag{A.9}$$

where δ_0 is the one for uniformly asymptotic stability of $\{y(n)\}_{n\geq 0}$. Let $\phi \in \mathcal{B}$ such that $\|\phi - \eta_{n_0}\|_{\mathcal{B}} < (\delta_0/2)$, and let $\{x(n)\}_{n\geq n_0}$, for each fixed $k \geq k_0$, be the solution of (1.1) such that $x_{n_0+n_k} = \chi$. Then x^k is a solution of (A.2) with $x^k_{n_0} = \chi$. Since $\{y^k(n)\}$ is $\mathcal{UAS}$ and $\|x^k_{n_0} - y^k_{n_0}\|_{\mathcal{B}} < (\delta_0/2)$ for each fixed $k \geq k_0$, we have

$$\|y^k_n - x^k_n\|_{\mathcal{B}} < \frac{\epsilon}{2} \quad \forall n \geq n_0 + N\left(\frac{\epsilon}{2}\right),\ k \geq k_0. \tag{A.10}$$

By the same argument as above, there is a subsequence of n_k, which we will continue calling n_k, such that $\{x^k(n)\}$ converges to the solution $\{y(n)\}$ of (A.1) through (n_0, χ) and $F(n + n_k, \phi) \to G(n, \phi)$ uniformly on $\mathbb{Z} \times S_{\alpha^*}$ as $k \to +\infty$, where S_{α^*} is a compact set in $\mathcal{B}$ with $|x^k(n)| \leq \alpha^*$ for all $k \geq k_0$ and $n \in \mathbb{Z}$. Then $\{y(n)\}$ is the unique solution of (A.1), passing through (n_0, χ) with $y_{n_0} = \chi \in \mathcal{B}$. On the other hand, by Lemma 4.4 for any integer $N > 0$ there exists $k_N \geq k_0$ such that if $k \geq k_N$, then

$$\|x_n^k - y_n\|_{\mathcal{B}} < \frac{\epsilon}{4}, \quad \|y_n^k - \eta_n\|_{\mathcal{B}} < \frac{\epsilon}{4} \quad \text{for } n_0 + N\left(\frac{\epsilon}{2}\right) \leq n \leq n_0 + N\left(\frac{\epsilon}{2}\right) + N, \tag{A.11}$$

and hence $\|y_n - \eta_n\|_{\mathcal{B}} < \epsilon$ for $n_0 + N(\epsilon/2) \leq n \leq n_0 + N(\epsilon/2) + N$. Since N is arbitrary, we have

$$\|y_n - \eta_n\|_{\mathcal{B}} < \epsilon, \quad \forall n \geq n_0 + N\left(\frac{\epsilon}{2}\right), \tag{A.12}$$

if $\|\phi - \eta_{n_0}\|_{\mathcal{B}} < (\delta_0/2)$ and $\phi \in \mathcal{B}$; thus, $\eta_n \in \mathcal{UAS}$ and the proof is complete. □

Now, we need to prove the following important lemma.

Lemma A.2. *Suppose that the assumptions* $(\widetilde{A})$*, (H1), (H2), and (H4) hold, that* $\mathcal{B}$ *is a fading memory space, that the bounded solution* y *of* (1.1) *is* $\mathcal{UAS}$*, and that for each* $G \in \mathcal{C}(F)$*, the solution of* (A.1) *is unique for any given initial data. Let* $S \supset \Sigma_\alpha$ *be a given compact set in* $\mathcal{B}$*. Then for any* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon) > 0$ *such that if* $n_0 \geq 0$*,* $\|y_{n_0} - x_{n_0}\|_{\mathcal{B}} < \delta$*, and* $\{h(n)\}$ *is a sequence with* $|h(n)| \leq \delta$ *for* $n \geq n_0$*, one has* $\|y_n - x_n\|_{\mathcal{B}} < \epsilon$ *for all* $n \geq n_0$*, where* $\{x(n)\}$ *is any bounded solution of the system*

$$x(n+1) = F(n, x_n) + h(n), \quad n \geq n_0, \tag{A.13}$$

passing through (n_0, x_{n_0}) *and such that* $x_n \in S$ *for all* $n \geq n_0$*.*

Proof. Suppose that the bounded solution $\{y(n)\}_{n\geq 0}$ of (1.1) is $\mathcal{UAS}$ with the triple $(\delta(\bullet), \delta_0, N(\bullet))$. The proof will be by contradiction, we assume that Lemma A.2 is not true. Then for some compact set $S_* \supsetneqq \Sigma_\alpha$, there exist ϵ, $0 < \epsilon < \delta_0$, sequences $\{n_k\} \subset \mathbb{Z}^+$, $\{r_k\} \subset \mathbb{Z}^+$, mapping sequences $h_k : [n_k, +\infty) \to \mathbb{C}^r$, $\varphi^k : (-\infty, n_k] \to \mathbb{C}^r$, and

$$\begin{gathered} \|y_{n_k} - x_{n_k}^k\|_{\mathcal{B}} < \frac{1}{k}, \quad |h_k(n)| \leq \frac{1}{k} \quad \text{for } n \geq n_k, \\ \|y_n - x_n^k\|_{\mathcal{B}} \leq \epsilon \quad \text{for } n_k \leq n \leq n_k + r_k - 1, \ \|y_{n_k+r_k} - x_{n_k+r_k}^k\|_{\mathcal{B}} \leq \epsilon, \end{gathered} \tag{A.14}$$

for sufficiently large k, where $\{x^k(n)\}$ is a solution of

$$x(n+1) = F(n, x_n) + h_k(n), \quad n \geq n_k, \tag{A.15}$$

passing through (n_k, φ^k) such that $x_n^k \in S_*$ for all $n \geq n_k$ and $k \geq 1$. Since S_* is a bounded subset of $\mathcal{B}$, it follows that $\{x^k(n_k+r_k+n)\}_{k\geq 1}$ and $\{x^k(n_k+n)\}_{k\geq 1}$ are uniformly bounded for all n_k and $n \geq -\infty$. We first consider the case where $\{r_k\}_{k\geq 1}$ contains an unbounded subsequence. Set $N = N(\epsilon) > 1$. Taking a subsequence if necessary, we may assume from Lemmas 3.12

and 3.9(b) that there is $G \in \mathcal{C}(F)$ such that $F(n+n_k+r_k-N,\phi) \to G(n,\phi)$ uniformly on $\mathbb{Z}^+ \times S_*$, $x^k(n+n_k+r_k-N) \to z(n)$, and $y(n+n_k+r_k-N) \to w(n)$ for $n \in \mathbb{Z}^+$ as $k \to \infty$, where $z, w : \mathbb{Z}^+ \to \mathbb{C}^r$ are some bounded functions. Since

$$x^k(n+n_k+r_k-N+1) = F\Big(n+n_k+r_k-N, x^k_{n+n_k+r_k-N}\Big) + h_k(n+n_k+r_k-N), \qquad \text{(A.16)}$$

passing to the limit as $k \to +\infty$, by the similar arguments in the proof of Theorem 4.11, we conclude that $\{z(n)\}_{n\geq 0}$ is the solution of the following equation:

$$x(n+1) = G(n, x_n), \quad n \in \mathbb{Z}^+. \qquad \text{(A.17)}$$

Similarly, $\{w(n)\}_{n\geq 0}$ is also a solution of (A.17). By Lemma 4.4 $x^k_{nk+r_k-N} \to z_0$ and $y_{n_k+r_k-N} \to w_0$ in $\mathcal{B}$ as $k \to +\infty$; it follows from (A.14) that $\|w_0 - z_0\|_{\mathcal{B}} \leq \lim_{k\to+\infty}\|w_{n_k+r_k-N} - z_{n_k+r_k-N}\|_{\mathcal{B}} \leq \epsilon < \delta_0$. Notice that $\{w(n)\}_{n\geq 0}$ is a solution of (A.17), passing through $(0, w_0)$, and is $\mathcal{UAS}$ by Lemma A.1. We have $\|w_N - z_N\|_{\mathcal{B}} < \epsilon$. On the other hand, since

$$\begin{aligned} y_{n_k+r_k}(j) &= y(N+j+n_k+r_k-N) \longrightarrow w(N+j) = w_N(j), \\ x^k_{n_k+r_k}(j) &= x^k(N+j+n_k+r_k-j) \longrightarrow z(N+j) = z_N(j) \end{aligned} \qquad \text{(A.18)}$$

as $k \to +\infty$ for each $j \in (-\infty, 0]$, it follows from (A.14) that

$$\|w_N - z_N\|_{\mathcal{B}} = \lim_{k\to+\infty}\|y_{n_k+r_k} - x^k_{n_k+r_k}\|_{\mathcal{B}} \geq \epsilon. \qquad \text{(A.19)}$$

This is a contradiction. Thus, the sequence $\{r_k\}$ must be bounded. Taking a subsequence if necessary, we can assume that $0 < r_k \equiv r_0 < \infty$. Moreover, we may assume that $x^k(n_k+n) \to \tilde{z}(n)$ and $y(n_k+n) \to \tilde{w}(n)$ for each $n \in \mathbb{Z}$, and $F(n+n_k,\phi) \to \tilde{G}(n,\phi)$ uniformly on $\mathbb{Z} \times S_*$, for some functions $\tilde{z}(n)$, $\tilde{w}(n)$ on $\mathbb{Z}^+$, and $\tilde{G} \in \mathcal{C}(F)$. Since $y_{n_k} \to \tilde{w}_0$ and $x^k_{n_k}\tilde{z}(j) = \tilde{z}_0$ in $\mathcal{B}$ as $k \to +\infty$, we have $\|\tilde{w}_0 - \tilde{z}_0\|_{\mathcal{B}} = \lim_{k\to+\infty}\|y_{n_k} - x^k_{n_k}\|_{\mathcal{B}} = \lim_{k\to+\infty}\|y_{n_k} - \varphi^k\|_{\mathcal{B}} = 0$ by (A.14), and hence $\tilde{w}_0 \equiv \tilde{z}_0$, that is, $\tilde{w}(j) = \tilde{z}(j)$ for all $j \in (-\infty, 0]$. Moreover, $\tilde{z}(n)$ and $\tilde{w}(n)$ satisfy the same relation:

$$x(n+1) = \tilde{G}(n, x_n), \quad n \in \mathbb{Z}^+. \qquad \text{(A.20)}$$

The uniqueness of the solutions for the initial value problems implies that $\tilde{z}(n) \equiv \tilde{w}(n)$ for $n \in \mathbb{Z}^+$, and hence $\|\tilde{w}_{r_0} - \tilde{z}_{r_0}\|_{\mathcal{B}} = 0$. On the other hand, and again from Lemma 4.4, $y_{n_k+r_0} \to \tilde{w}_{r_0}$ and $x^k_{n_k+r_0} \to \tilde{z}_{r_0}$ in $\mathcal{B}$ as $k \to +\infty$, then from (A.14) we have

$$\|\tilde{w}_{r_0} - \tilde{z}_{r_0}\|_{\mathcal{B}} = \lim_{k\to+\infty}\|y_{n_k+r_k} - x^k_{n_k+r_k}\|_{\mathcal{B}} \geq \epsilon. \qquad \text{(A.21)}$$

This is a contradiction, that proves Lemma A.2. □

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