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Research Article

Stability Results for a Class of Difference Systems with Delay

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Considering the linear delay difference system $\mathbf{x}(n+1) = a\mathbf{x}(n) + B\mathbf{x}(n-k)$, where $a \in (0,1)$, B is a $p \times p$ real matrix, and k is a positive integer, the stability domain of the null solution is completely characterized in terms of the eigenvalues of the matrix B. It is also shown that the stability domain becomes smaller as the delay increases. These results may be successfully applied in the stability analysis of a large class of nonlinear difference systems, including discrete-time Hopfield neural networks.

1. Introduction

In this paper, we will characterize the stability region of the null solution for the following class of linear delay difference systems:

$$\mathbf{x}(n+1) = a\mathbf{x}(n) + B\mathbf{x}(n-k) \quad \forall n \geq k, \tag{1.1}$$

where $a \in (0,1)$, B is a $p \times p$ real matrix, and k is a positive integer.

Similar linear difference systems have been recently investigated by Levitskaya [1] (focusing on the special case $a = 1$) and by Kipnis and Komissarova [2] (studying the special case $a = -1$). Two-dimensional systems of this form have been thoroughly investigated by Matsunaga, in the case $a = 1$ [3, 4] and in the general case $a \in \mathbb{R}$ [5]. The common starting point of all these results is the well-known papers of Kuruklis [6] and Papanicolaou [7], which focus on the scalar difference equation

$$x(n+1) - ax(n) + bx(n-k) = 0, \tag{1.2}$$

where $a, b \in \mathbb{R}$ and $k \in \mathbb{N}^*$. A more recent discussion of the stability properties of this scalar difference equation, based on relatively simple arguments, is presented in [8]. As for the nondelayed case $k = 0$, we refer to the recent paper in [9] and the references therein.

System (1.1) can be regarded as the linearization (at the origin) of the following nonlinear delay difference system:

$$\mathbf{x}(n+1) = F(\mathbf{x}(n), \mathbf{x}(n-k)) \quad \forall n \geq k, \tag{1.3}$$

where the function $F : \Omega \times \Omega \to \Omega$ (with $\mathbf{0} \in \Omega \subset \mathbb{R}^p$) satisfies $F(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, $D_{\mathbf{x}_1} F(\mathbf{0}, \mathbf{0}) = a \cdot \mathrm{Id}$ (where Id denotes the p-dimensional identity matrix), and $D_{\mathbf{x}_2} F(\mathbf{0}, \mathbf{0}) = B$.

In particular, discrete-time delayed Hopfield-type neural networks described by

$$\mathbf{x}(n+1) = a\mathbf{x}(n) + Tg(\mathbf{x}(n-k)) \quad \forall n \geq k \tag{1.4}$$

belong to the class of nonlinear difference systems (1.3). In this context, $a \in (0,1)$ is the self-regulating parameter of the neurons, $T \in \mathbb{R}^{p \times p}$ is the interconnection matrix, $g_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2, \ldots, p\}$ are the neuron input-output activation functions satisfying $g_i(0) = 0$, and $g : \mathbb{R}^p \to \mathbb{R}^p$ is defined by $g(\mathbf{x}) = (g_1(x_1), g_2(x_2), \ldots, g_p(x_p))^T$. In this framework, stability and bifurcation results have been obtained in [10] for the two-dimensional case, in [11] for the case of a single-directional ring of four neurons, and in [12] for a bidirectional ring of p neurons. Moreover, coexistence of chaos and periodic orbits for a network of this type with two identical neurons and no self-connections has been observed in [13].

The stability of the null solution of system (1.1) will be investigated by analyzing the distribution of the roots of the corresponding characteristic equation with respect to the unit circle. Based on [2, Corollary 2.2], (1.1) is asymptotically stable if and only if all roots of the equation

$$\det\left[B - z^k(z-a)I\right] = 0 \tag{1.5}$$

lie inside the unit disk. This means that z is a root of the characteristic equation of (1.1) if and only if $z^k(z-a)$ is an eigenvalue of the matrix B.

In the followings, we will denote by λ_i, $i \in \{1, 2, \ldots, p\}$ the eigenvalues of B. Based on the previous remark, we obtain that the characteristic equation of (1.1) is

$$\prod_{i=1}^{p} \left(z^k(z-a) - \lambda_i\right) = 0. \tag{1.6}$$

The null solution of (1.1) is asymptotically stable if and only if all the roots of the characteristic equation (1.6) are inside the unit circle. Therefore, in order to characterize the asymptotic stability of the null solution of (1.1), we first need to analyze the distribution of the roots of the polynomial $P_\lambda(z) = z^k(z-a) - \lambda$, where λ is a complex parameter, with respect to the unit circle. This requires a generalization of the results first obtained by Kuruklis [6], for the case when λ is a real parameter.

2. The Roots of the Polynomial $P_\lambda(z) = z^k(z-a) - \lambda$, $\lambda \in \mathbb{C}$

In the followings, we will consider the following two important functions:

$$c_{a,k}(\theta) = \cos(k+1)\theta - a\cos k\theta,$$
$$s_{a,k}(\theta) = \sin(k+1)\theta - a\sin k\theta, \tag{2.1}$$

and the curve Γ in the complex plane given by the following parametric equation:

$$\Gamma : u(\theta) = c_{a,k}(\theta) + i s_{a,k}(\theta), \quad \theta \in [-\pi, \pi]. \tag{2.2}$$

Lemma 2.1. *Let $k \in \mathbb{N}$ and $a \in (0,1)$. The function $s_{a,k}$ has exactly $k+2$ roots in the interval $[0,\pi]$, more precisely, as follows:*

(i) $\theta^0_{a,k} = 0$ *is a root,*

(ii) *if $k \geq 1$, then there is one root $\theta^j_{a,k}$ in every interval $((2j-1)\pi/(2k+1), j\pi/(k+1)) \subset ((j-1)\pi/k, j\pi/k)$, $j \in \{1,2,\ldots,k\}$,*

(iii) $\theta^{k+1}_{a,k} = \pi$ *is a root.*

Moreover, $(-1)^j s_{a,k}(\theta) > 0$ for any $\theta \in (\theta^j_{a,k}, \theta^{j+1}_{a,k})$, $j \in \{0,1,\ldots,k\}$.

Proof. Obviously, 0 and π are solutions of the equation $s_{a,k}(\theta) = 0$ for any $k \in \mathbb{N}$.

Considering $k \geq 1$, on the interval $((j-1)\pi/k, j\pi/k)$, $j \in \{1,2,\ldots,k\}$, the equation $s_{a,k}(\theta) = 0$ becomes

$$\frac{\sin(k+1)\theta}{\sin k\theta} = a. \tag{2.3}$$

The function $h : ((j-1)\pi/k, j\pi/k) \to \mathbb{R}$ defined by $h(\theta) = \sin(k+1)\theta/\sin k\theta$ is differentiable and

$$\begin{aligned} h'(\theta) &= \frac{(k+1)\cos(k+1)\theta\sin k\theta - k\cos k\theta\sin(k+1)\theta}{\sin^2 k\theta} \\ &= \frac{(k+1)[\sin(2k+1)\theta - \sin\theta] - k[\sin(2k+1)\theta + \sin\theta]}{2\sin^2 k\theta} \\ &= \frac{\sin(2k+1)\theta - (2k+1)\sin\theta}{2\sin^2 k\theta}. \end{aligned} \tag{2.4}$$

Therefore, the sign of $h'(\theta)$ depends on the sign of $g(\theta) = \sin(2k+1)\theta - (2k+1)\sin\theta$. We have

$$g'(\theta) = (2k+1)[\cos(2k+1)\theta - \cos\theta] = -2(2k+1)\sin(k+1)\theta\sin k\theta. \tag{2.5}$$

One can easily verify that the only root of g' in the interval $((j-1)\pi/k, j\pi/k)$ is $\theta^\star = j\pi/(k+1)$. Moreover, $g'(\theta) < 0$ for any $\theta < \theta^\star$ and $g'(\theta) > 0$ for any $\theta > \theta^\star$. Therefore, the function g

is decreasing on the interval $((j-1)\pi/k, \theta^{\star})$ and increasing on $(\theta^{\star}, j\pi/k)$. As $g((j-1)\pi/k) = -2k\sin((j-1)\pi/k) \le 0$ and $g(j\pi/k) = -2k\sin(j\pi/k \le 0)$, it results that $g(\theta) < 0$ for any $\theta \in ((j-1)\pi/k, j\pi/k)$.

Hence, the function h is strictly decreasing on the interval $((j-1)\pi/k, j\pi/k)$. Hence, the equation $s_{a,k}(\theta) = 0$ has a single root $\theta^{j}_{a,k}$ in the interval $((j-1)\pi/k, j\pi/k)$. Moreover, as $h((2j-1)\pi/(2k+1)) = 1$, $h(j\pi/(k+1)) = 0$ and $a \in (0,1)$, we obtain that this single root $\theta^{j}_{a,k}$ belongs to the interval $((2j-1)\pi/(2k+1), j\pi/(k+1))$.

Moreover, for $j \in \{0, 1, \ldots, k\}$ we have

$$s_{a,k}\left(\frac{j\pi}{k+1}\right) = -a\sin\left(\frac{jk\pi}{k+1}\right) = -a\sin\left(j\pi - \frac{j\pi}{k+1}\right) = a(-1)^{j}\sin\left(\frac{j\pi}{k+1}\right). \tag{2.6}$$

Since $s_{a,k}$ has constant sign on the interval $(\theta^{j}_{a,k}, \theta^{j+1}_{a,k})$ and $\theta^{j}_{a,k} < j\pi/(k+1) < \theta^{j+1}_{a,k}$, it follows that, for any $\theta \in (\theta^{j}_{a,k}, \theta^{j+1}_{a,k})$, we have

$$\operatorname{sign}[s_{a,k}(\theta)] = \operatorname{sign}\left[s_{a,k}\left(\frac{j\pi}{k+1}\right)\right] = \operatorname{sign}\left[a(-1)^{j}\sin\left(\frac{j\pi}{k+1}\right)\right] = (-1)^{j}. \tag{2.7}$$

□

Lemma 2.2. *Let $k \in \mathbb{N}$ and $a \in (0,1)$. The function $c\prime_{a,k}$ has exactly $k+2$ roots in the interval $[0,\pi]$, namely*

(i) $\varphi^{0}_{a,k} = 0$ *is a root;*

(ii) *if $k \ge 1$, then there is one root $\varphi^{j}_{a,k}$ in every interval $(\theta^{j}_{a,k}, j\pi/(k+1)) \subset ((j-1)\pi/k, j\pi/k)$, $j \in \{1, 2, \ldots, k\}$;*

(iii) $\varphi^{k+1}_{a,k} = \pi$ *is a root.*

Moreover, the function $(-1)^{j}c_{a,k}(\theta)$ is strictly decreasing on the interval $(\varphi^{j}_{a,k}, \varphi^{j+1}_{a,k})$, $j \in \{0, 1, \ldots, k\}$, and

$$c_{a,k}\left(\theta^{j}_{a,k}\right) = (-1)^{j}\sqrt{1 + a^{2} - 2a\cos\theta^{j}_{a,k}} \quad \forall j \in \{0, 1, \ldots, k+1\}. \tag{2.8}$$

Proof. The first part of the proof is similar to the proof of Lemma 2.1. Since $c'_{a,k}(\theta) = -(k+1)\sin(k+1)\theta + ak\sin k\theta$, the equation $c'_{a,k}(\theta) = 0$ becomes

$$h(\theta) = \frac{ak}{k+1}, \tag{2.9}$$

where h is the function defined in the proof of Lemma 2.1. We have shown that h is strictly decreasing on the interval $((j-1)\pi/k, j\pi/k)$; hence, the equation $c'_{a,k}(\theta) = 0$ has a single root on the interval $((j-1)\pi/k, j\pi/k)$. Moreover, as $h(\theta^{j}_{a,k}) = a$, $h(j\pi/(k+1)) = 0$ and $ak/(k+1) \in (0,a)$, we obtain that this single root $\varphi^{j}_{a,k}$ belongs to the interval $(\theta^{j}_{a,k}, j\pi/(k+1))$.

Moreover, for $j \in \{0, 1, \ldots, k\}$ we have

$$c'_{a,k}\left(\frac{j\pi}{k+1}\right) = ak\sin\left(\frac{jk\pi}{k+1}\right) = ak\sin\left(j\pi - \frac{j\pi}{k+1}\right) = ak(-1)^{j+1}\sin\left(\frac{j\pi}{k+1}\right). \tag{2.10}$$

Since $c'_{a,k}$ has constant sign on the interval $(\varphi^j_{a,k}, \varphi^{j+1}_{a,k})$ and $\varphi^j_{a,k} < j\pi/(k+1) < \varphi^{j+1}_{a,k}$, it follows that, for any $\theta \in (\varphi^j_{a,k}, \varphi^{j+1}_{a,k})$, we have

$$\operatorname{sign}\left[c'_{a,k}(\theta)\right] = \operatorname{sign}\left[c'_{a,k}\left(\frac{j\pi}{k+1}\right)\right] = \operatorname{sign}\left[ak(-1)^{j+1}\sin\left(\frac{j\pi}{k+1}\right)\right] = (-1)^{j+1}, \tag{2.11}$$

and hence, the function $(-1)^j c_{a,k}(\theta)$ is strictly decreasing on the interval $(\varphi^j_{a,k}, \varphi^{j+1}_{a,k})$.

From $c_{a,k}(\theta)^2 + s_{a,k}(\theta)^2 = 1 + a^2 - 2a\cos\theta$, we easily obtain that $c_{a,k}(\theta^j_{a,k})^2 = 1 + a^2 - 2a\cos\theta^j_{a,k}$. We also observe that $c_{a,k}(\theta)^2 + s_{a,k}(\theta)^2$ is strictly increasing on $[0, \pi]$, and hence,

$$c_{a,k}(\theta)c'_{a,k}(\theta) + s_{a,k}(\theta)s'_{a,k}(\theta) > 0 \quad \forall\theta \in (0, \pi). \tag{2.12}$$

It follows that $c_{a,k}(\theta^j_{a,k})c'_{a,k}(\theta^j_{a,k}) > 0$. Since $\theta^j_{a,k} \in (\varphi^{j-1}_{a,k}, \varphi^j_{a,k})$ for any $j \in \{1, 2, \ldots, k+1\}$ and $(-1)^{j-1}c_{a,k}(\theta)$ is strictly decreasing on the interval $(\varphi^{j-1}_{a,k}, \varphi^j_{a,k})$, we obtain that $(-1)^{j-1}c'_{a,k}(\theta^j_{a,k}) < 0$, and hence, $(-1)^j c_{a,k}(\theta^j_{a,k}) > 0$. Taking into account that $c_{a,k}(\theta^j_{a,k})^2 = 1 + a^2 - 2a\cos\theta^j_{a,k}$, it follows that $c_{a,k}(\theta^j_{a,k}) = (-1)^j\sqrt{1 + a^2 - 2a\cos\theta^j_{a,k}}$. □

Remark 2.3. Properties of the curve Γ defined by (2.2):

(a) we can easily see that

$$|u(\theta)|^2 = c_{a,k}(\theta)^2 + s_{a,k}(\theta)^2 = 1 + a^2 - 2a\cos\theta \tag{2.13}$$

and hence, $|u(\theta)|$ is strictly decreasing on the interval $[-\pi, 0]$ and increasing on the interval $[0, \pi]$. The curve pieces $\Gamma|_{[-\pi,0]}$ and $\Gamma|_{[0,\pi]}$ are, therefore, simple curves;

(b) moreover

$$\frac{d}{d\theta}\left[\tan(\arg(u(\theta)))\right] = \frac{d}{d\theta}\left[\frac{s_{a,k}(\theta)}{c_{a,k}(\theta)}\right] = \frac{k(1 + a^2 - 2a\cos\theta) + 1 - a\cos\theta}{c_{a,k}(\theta)^2} > 0. \tag{2.14}$$

Therefore, as θ increases from $-\pi$ to π, the corresponding point $u(\theta)$ from the curve Γ moves anticlockwise around the origin;

(c) the curve pieces $\Gamma|_{[-\pi,0]}$ and $\Gamma|_{[0,\pi]}$ are symmetrical with respect to the real axis, that is, $u(\theta) \in \Gamma|_{[0,\pi]}$ if and only if $\overline{u(\theta)} = u(-\theta) \in \Gamma|_{[-\pi,0]}$;

(d) the curve Γ intersects the real axis at the points

$$u\left(\theta^{j}_{a,k}\right) = c_{a,k}\left(\theta^{j}_{a,k}\right) = (-1)^{j}\sqrt{1 + a^{2} - 2a\cos\theta^{j}_{a,k}}, \quad j \in \{0, 1, \dots, k+1\}, \tag{2.15}$$

which are at the same time the intersection points of the curve pieces $\Gamma|_{[-\pi,0]}$ and $\Gamma|_{[0,\pi]}$.

In what follows, we will consider the following curve pieces:

$$\Gamma^{j}_{a,k} = \Gamma|_{[-\theta^{j}_{a,k}, -\theta^{j-1}_{a,k}] \cup [\theta^{j-1}_{a,k}, \theta^{j}_{a,k}]}, \quad j \in \{1, 2, \dots, k+1\}. \tag{2.16}$$

Based on the previous remarks, one can easily see that these are closed curves.

For every $j \in \{1, 2, \dots, k+1\}$, let $\Delta^{j}_{a,k}$ denote the domain (open and connected set, containing the origin) of the complex plane inclosed by the curve $\Gamma^{j}_{a,k}$.

Remark 2.4. For the curves $\Gamma^{j}_{a,k}$ and the inclosed domains $\Delta^{j}_{a,k}$, the following properties hold:

(a) $\Gamma^{j}_{a,k} \cap \Gamma^{j+1}_{a,k} = \{c_{a,k}(\theta^{j}_{a,k})\}$ for any $j \in \{1, 2, \dots, k\}$,

(b) $\partial\Delta^{j}_{a,k} = \Gamma^{j}_{a,k}$ for any $j \in \{1, 2, \dots, k+1\}$ (here, ∂S denotes the boundary of the set S);

(c) $\Delta^{1}_{a,k} \subset \Delta^{2}_{a,k} \subset \cdots \subset \Delta^{k+1}_{a,k}$.

Using all these preliminary notations and results, the following proposition is obtained.

Proposition 2.5. *Considering the polynomial $P_{\lambda}(z) = z^{k}(z-a) - \lambda$, $\lambda \in \mathbb{C}$, the following hold.*

(a) *If $\lambda \in \Delta^{1}_{a,k}$, then all roots of the polynomial $P_{\lambda}(z)$ are inside the unit circle.*

(b) *If $\lambda \in \Delta^{j}_{a,k} \setminus \overline{\Delta^{j-1}_{a,k}}$ (with $j \in \{2, 3, \dots, k+1\}$), then the polynomial $P_{\lambda}(z)$ has exactly $k-j+2$ roots inside the unit circle and $j-1$ roots outside the unit circle.*

(c) *If $\lambda \in \mathbb{C} \setminus \overline{\Delta^{k+1}_{a,k}}$, then all roots of the polynomial $P_{\lambda}(z)$ are outside the unit circle.*

(d) *If $\lambda \in \Gamma^{j}_{a,k} \setminus \{c_{a,k}(\theta^{j-1}_{a,k}), c_{a,k}(\theta^{j}_{a,k})\}$ (with $j \in \{1, 2, \dots, k+1\}$), then the polynomial $P_{\lambda}(z)$ has exactly one simple root on the unit circle, $k-j+1$ roots inside the unit circle and $j-1$ roots outside the unit circle.*

(e) *If $\lambda = c_{a,k}(\theta^{j}_{a,k})$, $j \in \{1, 2, \dots, k\}$, then the polynomial $P_{\lambda}(z)$ has exactly two simple roots on the unit circle, $k-j$ roots inside the unit circle, and $j-1$ roots outside the unit circle.*

(f) *If $\lambda = c_{a,k}(\theta^{0}_{a,k}) = 1-a$, then the polynomial $P_{\lambda}(z)$ has the simple root $z=1$ on the unit circle and k roots inside the unit circle.*

(g) *If $\lambda = c_{a,k}(\theta^{k+1}_{a,k}) = (-1)^{k+1}(1+a)$, then the polynomial $P_{\lambda}(z)$ has the simple root $z=-1$ on the unit circle and k roots outside the unit circle.*

Here, $\overline{S}$ denotes the closure of the set S.

Proof. The polynomial $P_\lambda(z)$ has a root $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$ on the unit circle if and only if

$$\lambda = e^{ik\theta}\left(e^{i\theta} - a\right) = c_{a,k}(\theta) + is_{a,k}(\theta), \tag{2.17}$$

that is, if and only if $\lambda \in \Gamma$.

Furthermore, due to the properties of the curve Γ stated in Remark 2.3, we easily obtain that the polynomial $P_\lambda(z)$ has a unique root on the unit circle if and only if $\lambda \in \Gamma \setminus \{c_{a,k}(\theta^1_{a,k}), c_{a,k}(\theta^2_{a,k}), \ldots, c_{a,k}(\theta^k_{a,k})\}$. We also obtain that if $\lambda = c_{a,k}(\theta^j_{a,k})$, $j \in \{1, 2, \ldots, k\}$, then the polynomial $P_\lambda(z)$ has exactly two roots on the unit circle, namely, $z = e^{i\theta^j_{a,k}}$ and $\overline{z} = e^{-i\theta^j_{a,k}}$.

Moreover, if the polynomial $P_\lambda(z)$ has a root on the unit circle, then this root is simple. Indeed, assuming that there exists $\theta \in [-\pi, \pi]$ such that $z = e^{i\theta}$ is a root of $P_\lambda(z)$ and $P'_\lambda(z) = (k+1)z^k - akz^{k-1} = 0$, we obtain that $s'_{a,k}(\theta) = c'_{a,k}(\theta) = 0$. But we can easily see that

$$\begin{aligned} s'_{a,k}(\theta)^2 + c'_{a,k}(\theta)^2 &= (k+1)^2 + a^2k^2 - 2ak(k+1)\cos\theta \geq (k+1)^2 + a^2k^2 - 2ak(k+1) \\ &= (k+1-ak)^2 > 0, \end{aligned} \tag{2.18}$$

and hence, a contradiction is obtained.

To prove (a), we will use the argument principle for the investigation of the roots of the polynomial $P_\lambda(z) = z^k(z-a) - \lambda$. In other words, we will study the increase of the argument of $P_\lambda(z)$ along the unit circle. Consider the function

$$G(\theta, \lambda) = P_\lambda\left(e^{i\theta}\right) = e^{ik\theta}\left(e^{i\theta} - a\right) - \lambda = u(\theta) - \lambda, \tag{2.19}$$

where $u(\theta) = c_{a,k}(\theta) + is_{a,k}(\theta)$. We will estimate the increase of the argument of $G(\theta, \lambda)$ as θ increases from $-\pi$ to π.

From Remark 2.3, we know that $|u(\theta)|$ is strictly decreasing on the interval $[-\pi, 0]$ and increasing on the interval $[0, \pi]$, and as θ increases from $-\pi$ to π, the corresponding point $u(\theta)$ moves anticlockwise around the origin. Moreover, Remark 2.3 provides that the locus of $u(\theta)$ intersects the real axis $2(k+1)$ times as θ increases on the interval $(-\pi, \pi]$. Hence, the increase of the argument of $u(\theta)$ as θ increases from $-\pi$ to π is $2(k+1)\pi$ (see Figure 1).

The locus of $G(\theta, \lambda)$ is obtained by the translation of the locus of $u(\theta)$ by the vector $(-\operatorname{Re}(\lambda), -\operatorname{Im}(\lambda))$. If λ lies inside the domain $\Delta^1_{a,k}$, then the increase of the argument of $G(\theta, \lambda)$ is the same as the increase of the argument of $u(\theta)$, that is, it is equal to $2(k+1)\pi$. The argument principle provides that all the roots of the polynomial $P_\lambda(z)$ are inside the unit circle, and (a) is proved.

Let $j \in \{1, 2, \ldots, k+1\}$. The next step of the proof is to show that when the complex parameter $\lambda = \lambda_1 + i\lambda_2$ leaves the domain $\Delta^j_{a,k}$ by crossing its boundary $\Gamma^j_{a,k}$ at a value $\lambda^\star = c_{a,k}(\theta) + is_{a,k}(\theta)$, with $\theta \in [-\theta^j_{a,k}, -\theta^{j-1}_{a,k}] \cup [\theta^{j-1}_{a,k}, \theta^j_{a,k}]$, the root $z = z(\lambda)$ of the polynomial $P_\lambda(z)$, which is equal to $e^{i\theta}$ when $\lambda = \lambda^\star$, crosses the unit circle.

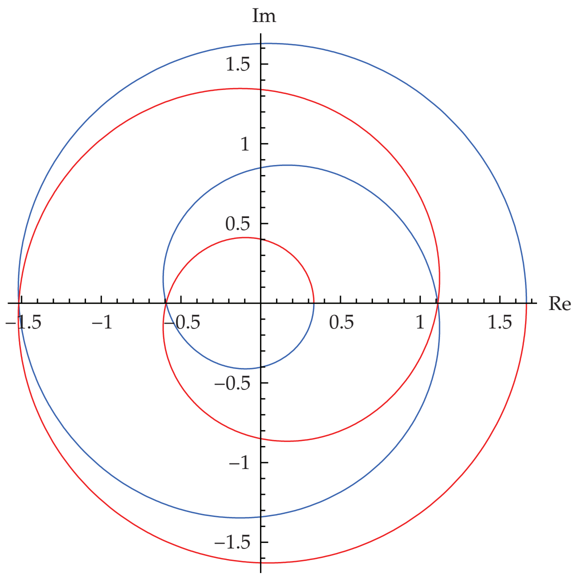

Figure 1: The curve Γ defined by (2.2) for $k = 3$ and $a = 2/3$. The blue part represents the curve piece $\Gamma|_{[-\pi,0]}$ and the red part represents the curve piece $\Gamma|_{[0,\pi]}$.

In the following computations, we rely on the fact that the roots of the polynomial $P_\lambda(z)$ which lie on the unit circle are simple.

As $z^{k+1} - az^k = \lambda_1 + i\lambda_2$ and $\overline{z}^{k+1} - a\overline{z}^k = \lambda_1 - i\lambda_2$, differentiating with respect to λ_1 and then with respect to λ_2, we obtain

$$\frac{\partial z}{\partial \lambda_1} = \frac{1}{P'_\lambda(z)}; \qquad \frac{\partial \overline{z}}{\partial \lambda_1} = \frac{1}{\overline{P'_\lambda(z)}}; \qquad \frac{\partial z}{\partial \lambda_2} = \frac{i}{P'_\lambda(z)}; \qquad \frac{\partial \overline{z}}{\partial \lambda_2} = \frac{-i}{\overline{P'_\lambda(z)}}. \tag{2.20}$$

Now we can evaluate that

$$\begin{aligned} \frac{\partial |z|^2}{\partial \lambda_1} &= z\frac{\partial \overline{z}}{\partial \lambda_1} + \overline{z}\frac{\partial z}{\partial \lambda_1} = \frac{z}{\overline{P'_\lambda(z)}} + \frac{\overline{z}}{P'_\lambda(z)} = \frac{2\,\mathrm{Re}\left(zP'_\lambda(z)\right)}{\left|P'_\lambda(z)\right|^2}, \\ \frac{\partial |z|^2}{\partial \lambda_2} &= z\frac{\partial \overline{z}}{\partial \lambda_2} + \overline{z}\frac{\partial z}{\partial \lambda_2} = \frac{-iz}{\overline{P'_\lambda(z)}} + \frac{i\overline{z}}{P'_\lambda(z)} = \frac{2\mathrm{Im}\left(zP'_\lambda(z)\right)}{\left|P'_\lambda(z)\right|^2}. \end{aligned} \tag{2.21}$$

Given the positive parametrization of the curve Γ, we obtain that $T(\theta) = c'_{a,k}(\theta) + is'_{a,k}(\theta)$ is tangent to $\Gamma^j_{a,k}$ at the point $\lambda^\star = c_{a,k}(\theta) + is_{a,k}(\theta)$ in the counterclockwise direction. Hence, $N(\theta) = s'_{a,k}(\theta) - ic'_{a,k}(\theta)$ is an outward-pointing normal vector to $\Gamma^j_{a,k}$ at the point $\lambda^\star = c_{a,k}(\theta) + is_{a,k}(\theta)$.

We compute the directional derivative of $|z|^2$ at $\lambda^\star = c_{a,k}(\theta) + is_{a,k}(\theta)$ in the direction $w = w_1 + iw_2$ ($|w| = 1$) as

$$\begin{aligned}
\nabla_w |z|^2(\lambda^\star) &= w_1 \frac{\partial |z|^2}{\partial \lambda_1}(\lambda^\star) + w_2 \frac{\partial |z|^2}{\partial \lambda_2}(\lambda^\star) \\
&= w_1 \frac{2\,\mathrm{Re}\left(e^{i\theta} P'_\lambda\left(e^{i\theta}\right)\right)}{\left|P'_\lambda\left(e^{i\theta}\right)\right|^2} + w_2 \frac{2\mathrm{Im}\left(e^{i\theta} P'_\lambda\left(e^{i\theta}\right)\right)}{\left|P'_\lambda\left(e^{i\theta}\right)\right|^2} \\
&= \frac{2\left(w_1 s'_{a,k}(\theta) - w_2 c'_{a,k}(\theta)\right)}{\left|P'_\lambda\left(e^{i\theta}\right)\right|^2} \\
&= \frac{2\,\mathrm{Re}(\overline{w} \cdot N(\theta))}{\left|P'_\lambda\left(e^{i\theta}\right)\right|^2} \\
&= \frac{2|N(\theta)| \cos(\measuredangle(w, N(\theta)))}{\left|P'_\lambda\left(e^{i\theta}\right)\right|^2}.
\end{aligned} \tag{2.22}$$

When λ crosses the curve $\Gamma^j_{a,k}$ through the point $\lambda^\star$, from the inside of the domain $\Delta^j_{a,k}$ to the outside, in the direction w, we have $\cos(\measuredangle(w, N(\theta))) > 0$, and hence,

$$\nabla_w |z|^2(\lambda^\star) > 0, \tag{2.23}$$

that is, $|z(\lambda)|$ increases and the root $z(\lambda)$ crosses the unit circle. This, together with the continuous dependence of the roots of the polynomial $P_\lambda(z)$ on the parameter λ, guarantees the validity of the statements (b)–(g) and completes the proof. □

3. Stability Results

3.1. Characterization of the Stability Domain

Based on the results presented in the previous section and the characteristic equation (1.6), the following main result is obtained.

Proposition 3.1. *The null solution of system* (1.1) *is asymptotically stable if and only if all eigenvalues of matrix B belong to the domain* $\Delta^1_{a,k}$ *of the complex plane inclosed by the closed curve* $\Gamma^1_{a,k}$ *given by the parametric equation*

$$\Gamma^1_{a,k}: \; u(\theta) = c_{a,k}(\theta) + is_{a,k}(\theta), \quad \theta \in \left[-\theta^1_{a,k}, \theta^1_{a,k}\right]. \tag{3.1}$$

We know from Lemma 2.2 that the function $c_{a,k}$ is strictly decreasing on the interval $[0, \theta^1_{a,k}] \subset [0, \varphi^1_{a,k}]$, and hence, invertible, allowing us to state the following remark.

Remark 3.2. The domain $\Delta^1_{a,k}$ can be expressed as

$$\Delta^1_{a,k} = \left\{\lambda \in \mathbb{C} \ : \ c_{a,k}\left(\theta^1_{a,k}\right) < \mathrm{Re}(\lambda) < 1 - a \ , |\mathrm{Im}(\lambda)| < h_{a,k}(\mathrm{Re}(\lambda))\right\}, \tag{3.2}$$

where $h_{a,k} = s_{a,k} \circ c^{-1}_{a,k}$ and $c^{-1}_{a,k}$ denotes the inverse of the restriction of $c_{a,k}$ to the interval $[0, \theta^1_{a,k}]$.

3.2. *Dependence of the Stability Domain on the Delay*

Proposition 3.3. *As the delay k increases, the stability domain becomes smaller, that is, the domains $\Delta^1_{a,k}$ satisfy*

$$\Delta^1_{a,k+1} \subset \Delta^1_{a,k} \quad \text{for any } k \in \mathbb{N}. \tag{3.3}$$

If $D(0, 1 - a)$ denotes the open disk of the complex plane, centered at the origin, of radius $1 - a$, one has

$$\bigcap_{k=0}^{\infty} \Delta^1_{a,k} = D(0, 1 - a). \tag{3.4}$$

Proof. Let $k \in \mathbb{N}$. Based on Remark 3.2, the proof of the fact that $\Delta^1_{a,k+1} \subset \Delta^1_{a,k}$ will consist of the following steps.

Step 1. We will prove that $c_{a,k}(\theta^1_{a,k}) < c_{a,k+1}(\theta^1_{a,k+1})$. Since $c_{a,k}(\theta^1_{a,k}) = -\sqrt{1 + a^2 - 2a\cos\theta^1_{a,k}}$, this reduces to show that $\theta^1_{a,k+1} < \theta^1_{a,k}$.

Indeed, assuming the contrary that is, $\theta^1_{a,k} < \theta^1_{a,k+1}$ and taking into account that $s_{a,k+1}(\theta) > 0$ for any $\theta \in (0, \theta^1_{a,k+1})$, it follows that $s_{a,k+1}(\theta^1_{a,k}) > 0$. On the other hand, we can easily see that

$$s_{a,k+1}(\theta) = \sin(k+2)\theta - a\sin(k+1)\theta = s_{a,k}(\theta)\cos\theta + c_{a,k}(\theta)\sin\theta \tag{3.5}$$

and hence,

$$s_{a,k+1}\left(\theta^1_{a,k}\right) = c_{a,k}\left(\theta^1_{a,k}\right)\sin\theta^1_{a,k} < 0. \tag{3.6}$$

So the assumption is contradicted. Therefore, it follows that $c_{a,k}(\theta^1_{a,k}) < c_{a,k+1}(\theta^1_{a,k+1})$.

Step 2. We will prove that $h_{a,k+1}(t) < h_{a,k}(t)$ for any $t \in (c_{a,k+1}(\theta^1_{a,k+1}), 1-a)$. Since

$$c_{a,k}(\theta)^2 + s_{a,k}(\theta)^2 = 1 + a^2 - 2a\cos\theta, \tag{3.7}$$

making $\theta = c^{-1}_{a,k}(t)$, it follows that

$$h_{a,k}(t)^2 = 1 + a^2 - t^2 - 2a\cos c^{-1}_{a,k}(t). \tag{3.8}$$

Therefore, in order to prove that $h_{a,k+1}(t) < h_{a,k}(t)$, it is sufficient to prove that $c^{-1}_{a,k+1}(t) < c^{-1}_{a,k}(t)$ for any $t \in (c_{a,k+1}(\theta^1_{a,k+1}), 1-a)$. Since $c_{a,k}$ is decreasing on $(0, \theta^1_{a,k})$, this is equivalent to show that $c_{a,k+1}(\theta) < c_{a,k}(\theta)$ for any $\theta \in (0, \theta^1_{a,k+1})$. Indeed, for any $\theta \in (0, \theta^1_{a,k+1})$ we have

$$c_{a,k+1}(\theta) = \cos(k+2)\theta - a\cos(k+1)\theta = c_{a,k}(\theta)\cos\theta - s_{a,k}(\theta)\sin\theta < c_{a,k}(\theta). \tag{3.9}$$

Finally, we will prove that $\bigcap_{k=0}^{\infty}\Delta^1_{a,k} = D(0, 1-a)$.

We remark that $D(0,1-a) \subset \Delta^1_{a,k}$, for any $k \in \mathbb{N}$. Indeed, it is easy to see that if $\lambda \in \partial\Delta^1_{a,k} = \Gamma^1_{a,k}$, there exists $\theta \in [-\theta^1_{a,k}, \theta^1_{a,k}]$ such that $\lambda = c_{a,k}(\theta) + is_{a,k}(\theta)$, and hence, $|\lambda| = \sqrt{1 + a^2 - 2a\cos\theta} \geq 1 - a$. Therefore $\lambda \notin D(0, 1-a)$ and it follows that $D(0,1-a) \subset \Delta^1_{a,k}$. We obtain that $D(0,1-a) \subset \bigcap_{k=0}^{\infty}\Delta^1_{a,k}$.

On the other hand, if $\lambda \in \bigcap_{k=0}^{\infty}\Delta^1_{a,k}$, then it follows from Remark 3.2 that

$$c_{a,k}\left(\theta^1_{a,k}\right) < \mathrm{Re}(\lambda) < 1 - a, \quad |\mathrm{Im}(\lambda)| < h_{a,k}(\mathrm{Re}(\lambda)) \quad \forall k \in \mathbb{N}. \tag{3.10}$$

From the first inequality, since $c^{-1}_{a,k}$ is decreasing (see Lemma 2.2), it follows that $0 < c^{-1}_{a,k}(\mathrm{Re}(\lambda)) < \theta^1_{a,k}$. From the second inequality, we obtain

$$\begin{aligned} |\lambda|^2 &= \mathrm{Re}\,(\lambda)^2 + \mathrm{Im}(\lambda)^2 < Re(\lambda)^2 + h_{a,k}(\mathrm{Re}(\lambda))^2 = 1 + a^2 - 2a\cos\left[c^{-1}_{a,k}(\mathrm{Re}(\lambda))\right] \\ &< 1 + a^2 - 2a\cos\theta^1_{a,k}, \end{aligned} \tag{3.11}$$

and hence

$$|\lambda| < \sqrt{1 + a^2 - 2a\cos\theta^1_{a,k}} \quad \forall k \in \mathbb{N}. \tag{3.12}$$

From Lemma 2.1 we know that $0 < \theta^1_{a,k} < \pi/(k+1)$, and hence, $\lim_{k\to\infty}\theta^1_{a,k} = 0$. Passing to the limit when $k \to \infty$ in the previous inequality, we obtain that $|\lambda| < 1 - a$, and therefore, $\lambda \in D(0, 1-a)$. It results that $\bigcap_{k=0}^{\infty}\Delta^1_{a,k} \subset D(0,1-a)$ and the proof is complete. □

The results presented in Proposition 3.3 are exemplified in Figure 2.

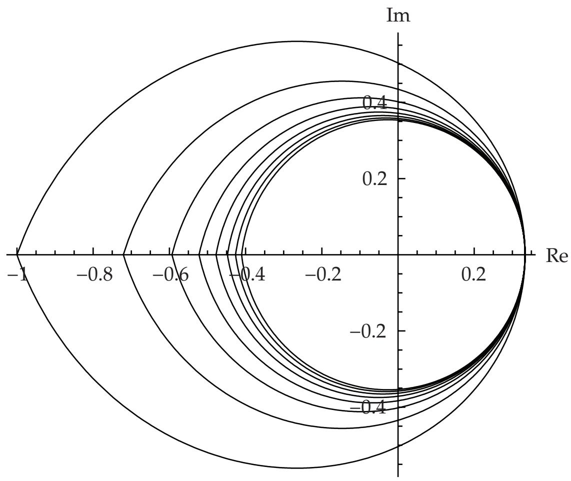

Figure 2: The stability domains for $a = 2/3$ and $k \in \{1, 2, \ldots, 8\}$. We have $\Delta^1_{a,8} \subset \Delta^1_{a,7} \subset \cdots \subset \Delta^1_{a,2} \subset \Delta^1_{a,1}$.

3.3. Some Particular Cases

Corollary 3.4. *If all the eigenvalues of the matrix B are real, then the null solution of system* (1.1) *is asymptotically stable if and only if all eigenvalues of matrix B belong to the interval* $(-\sqrt{1+a^2-2a\cos\theta^1_{a,k}}, 1-a)$.

For example, the previous corollary covers the case when B is a symmetric matrix. In particular, for the 1-dimensional case ($p = 1$) we obtain the result of Kuruklis [6] and Papanicolaou [7], when $a \in (0,1)$. For the 2-dimensional case ($p = 2$), if the matrix B has two real eigenvalues, then we obtain the result of Matsunaga [5], when $a \in (0,1)$. On the other hand, if the matrix $B \in \mathbb{R}^{2\times 2}$ has two complex eigenvalues, then we obtain the following simple formulation.

Corollary 3.5. *In the case of a 2-dimensional system of the form* (1.1)*, where the matrix B has two complex conjugated eigenvalues* $\lambda_{1,2} = \beta_1 \pm i\beta_2$*, the null solution is asymptotically stable if and only if*

$$-\sqrt{1+a^2-2a\cos\theta^1_{a,k}} < \beta_1 < 1-a, \qquad |\beta_2| < h_{a,k}(\beta_1), \tag{3.13}$$

where $h_{a,k} = s_{a,k} \circ c^{-1}_{a,k}$ *and* $c^{-1}_{a,k}$ *denotes the inverse of the restriction of* $c_{a,k}$ *to the interval* $[0, \theta^1_{a,k}]$.

4. Conclusions and Future Directions

In this paper, we have characterized the stability domain of the null solution of the linear delay difference system (1.1), in terms of the eigenvalues of the matrix B. We have also studied the dependence of the stability domain on the delay, showing that the stability domain becomes smaller as the delay increases. These results have potential applications in the stability analysis of many nonlinear discrete-time dynamical systems arising from practical problems, such as discrete-time Hopfield neural networks. Investigating the bifurcations occurring in such nonlinear dynamical systems at the boundary of the stability domain may constitute a direction for future research.

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