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Research Article

Control of Oscillating Systems with a Single Delay

J. Diblík,[1,2] D. Ya. Khusainov,[3] J. Lukáčová,[4] and M. Růžičková[4]

[1] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 66237 Brno, Czech Republic*
[2] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 61600 Brno, Czech Republic*
[3] *Department of Complex System Modeling, Faculty of Cybernetics, Taras Shevchenko National University of Kyiv, 01033 Kyiv, Ukraine*
[4] *Department of Mathematics, The Faculty of Science, The University of Žilina, 01026 Žilina, Slovakia*

Correspondence should be addressed to J. Diblík, diblik.j@fce.vutbr.cz

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Systems are considered related to the control of processes described by oscillating second-order systems of differential equations with a single delay. An explicit representation of solutions with the aid of special matrix functions called a delayed matrix sine and a delayed matrix cosine is used to develop the conditions of relative controllability and to construct a specific control function solving the relative controllability problem of transferring an initial function to a prescribed point in the phase space.

1. Introduction

The problem of controllability of linear first-order autonomous systems without delay

$$\dot{x}(t) = Ax(t) + bu(t), \quad x \in \mathbb{R}^n, \ t \geq 0, \tag{1.1}$$

with an $n \times n$ constant matrix A, $b \in \mathbb{R}^n$ and $u : [0,\infty) \to \mathbb{R}$ is solved by the well-known Kalman criterion (e.g., [1–3]). According to this, for the control of a linear system, it is necessary and sufficient that the rank criterion

$$\text{rank } S_n = n \tag{1.2}$$

should be fulfilled where

$$S_n = \left\{b, Ab, A^2b, \ldots, A^{n-1}b\right\}. \tag{1.3}$$

A proof is based on two important results. The first is the formula for an integral representation of a solution of a Cauchy problem for the nonhomogeneous system

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}bu(s)ds, \tag{1.4}$$

where

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots \tag{1.5}$$

is the matrix exponential (throughout this paper, I stands for an $n\times n$ unit matrix). The second is the Cayley-Hamilton theorem saying that any power A^i, $i = n, n+1, \ldots$ of matrix A can be represented by a linear combination of powers A^i, $i = 0, 1, \ldots, n-1$ [4, 5]. We remark that the problem regarding the construction of a control function has a nonunique solution.

For control systems with delay, a solution to the controllability problem is considerably more complicated. The control function is a functional of a previous phase state. First results related to controllability of linear systems with constant coefficients and a constant delay have been formulated in [6, 7] and, for linear systems with variable coefficients and a variable delay, in [8]. Problems of optimal control of systems with delay are considered in [9, 10]. Recent results on controllability of systems with delay are collected in [11–14].

In this paper, we investigate systems related to control of processes, described by oscillating second-order systems of differential equations with a single delay, in the following form:

$$\ddot{x}(t) + \Omega^2 x(t-\tau) = bu(t), \tag{1.6}$$

where $t \geq 0$, $x : [0,\infty) \to \mathbb{R}^n$, Ω is an $n \times n$ constant regular matrix, $\tau > 0$, $\tau \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $u : [0,\infty) \to \mathbb{R}$.

One way to investigate such problem is to define additional dependent variables and, transforming initial system (1.6) into a system of first-order linear differential equations with constant coefficients and a constant delay, to get controllability criteria using the results in the above-mentioned sources. However, then the dimension of the auxiliary system equals $2n$ and the essential feature of the situation is that we lose an explicit form of influence of the matrix Ω when a control function is designed.

In the paper, special matrix functions, called a delayed matrix cosine and a delayed matrix sine, are utilized. As a motivation for the terminology used calling the analyzed systems "oscillating" served the formal similarity with the partial sums of the defining series for the usual matrix sine and matrix cosine together with the formal parallel between (1.6) and systems of ordinary differential equations describing oscillating processes ((1.6) with $\tau = 0$).

The main result is the construction of a control function (in terms of these matrix functions), solving the problem of a transferring of an initial function to a prescribed point in the phase space.

2. Preliminaries

For a solution to the control problem, we need formulas to represent the solutions of an oscillating system with a single delay. First we discuss a linear nonhomogeneous differential system with a single delay

$$\ddot{x}(t) + \Omega^2 x(t-\tau) = f(t), \tag{2.1}$$

where the meaning of t, x, τ, and Ω is the same as in (1.6), and $f : [0,\infty) \to \mathbb{R}^n$. Below we use the symbols Θ and θ. The symbol Θ stands for an $n \times n$ zero matrix and the symbol θ stands for the $n \times 1$ vector $(0,0,\ldots,0)^T$.

In [15], system (2.1) was investigated and a representation of its solutions was derived using special matrix functions called a delayed matrix sine and a delayed matrix cosine. With their help, it was possible to derive a representation of the solutions of Cauchy problems. We state the basic definitions, formulated in [15], needed for a solution of the control problem described in Part 3.

Definition 2.1. The matrix function $\mathrm{Cos}_{\Omega,\tau} : \mathbb{R} \to \mathbb{R}^{n\times n}$, continuous on $\mathbb{R} \setminus \{-\tau\}$, and defined as

$$\mathrm{Cos}_{\Omega,\tau} t := \begin{cases} \Theta \quad \text{if} \quad -\infty < t < -\tau, \\ I \quad \text{if} \quad -\tau \le t < 0, \\ I - \Omega^2 \dfrac{t^2}{2!} \quad \text{if} \quad 0 \le t < \tau, \\ I - \Omega^2 \dfrac{t^2}{2!} + \Omega^4 \dfrac{(t-\tau)^4}{4!} \quad \text{if} \quad \tau \le t < 2\tau, \\ \cdots \\ I - \Omega^2 \dfrac{t^2}{2!} + \Omega^4 \dfrac{(t-\tau)^4}{4!} + \cdots + (-1)^k \Omega^{2k} \dfrac{(t-(k-1)\tau)^{2k}}{(2k)!} \\ \text{if} \quad (k-1)\tau \le t < k\tau, \quad k \ge 0, \\ \cdots \end{cases} \tag{2.2}$$

is called a delayed matrix cosine.

Definition 2.2. The matrix function $\mathrm{Sin}_{\Omega,\tau} : \mathbb{R} \to \mathbb{R}^{n\times n}$, continuous on $\mathbb{R}$, and defined as

$$\mathrm{Sin}_{\Omega,\tau}t := \begin{cases} \Theta \quad \text{if} \quad -\infty < t < -\tau, \\ \Omega(t+\tau) \quad \text{if} \quad -\tau \le t < 0, \\ \Omega(t+\tau) - \Omega^3\dfrac{t^3}{3!} \quad \text{if} \quad 0 \le t < \tau, \\ \Omega(t+\tau) - \Omega^3\dfrac{t^3}{3!} + \Omega^5\dfrac{(t-\tau)^5}{5!} \quad \text{if} \quad \tau \le t < 2\tau, \\ \cdots \\ \Omega(t+\tau) - \Omega^3\dfrac{t^3}{3!} + \Omega^5\dfrac{(t-\tau)^5}{5!} + \cdots + (-1)^k\Omega^{2k+1}\dfrac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!} \\ \text{if} \quad (k-1) \quad \le t < k\tau, \quad k \ge 0, \\ \cdots \end{cases} \tag{2.3}$$

is called a delayed matrix sine.

With the use of the above-defined special matrices, a solution of the Cauchy problem for nonhomogeneous system with a single delay can be written in an integral form. We recall the rules for computing the derivatives necessary for our investigation of $\mathrm{Sin}_{\Omega,\tau}t$ and $\mathrm{Cos}_{\Omega,\tau}t$ [15]. We remark that, in Definitions 2.1 and 2.2 as well as in formulas (2.4), (2.5) below, the matrix Ω can even be singular.

Lemma 2.3. *The following formulas are true for a delayed matrix cosine and a delayed matrix sine:*

$$\frac{d}{dt}\mathrm{Cos}_{\Omega,\tau}t = -\Omega\,\mathrm{Sin}_{\Omega,\tau}(t-\tau), \tag{2.4}$$

$$\frac{d}{dt}\mathrm{Sin}_{\Omega,\tau}t = \Omega\,\mathrm{Cos}_{\Omega,\tau}\,t. \tag{2.5}$$

The following theorem can be proved directly using formulas (2.4) and (2.5). A particular case of this result (when $\varphi' = \psi$) is given in [15]. Therefore, we omit the proof.

Theorem 2.4. *Let $\varphi, \psi : [-\tau, 0] \to \mathbb{R}^n$ be continuously differentiable vector functions and $f : [0,\infty) \to \mathbb{R}^n$ a locally integrable vector function. Then the solution $x : [-\tau,\infty] \to \mathbb{R}^n$ of the Cauchy problem*

$$x(t) \equiv \varphi(t), \quad x'(t) \equiv \psi(t), \quad t \in [-\tau, 0] \tag{2.6}$$

for the nonhomogeneous system (2.1) *has the form*

$$\begin{aligned} x(t) &= (\mathrm{Cos}_{\Omega,\tau} t)\varphi(-\tau) + \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau} t)\psi(-\tau) + \Omega^{-1}\int_{-\tau}^{0} (\mathrm{Sin}_{\Omega,\tau}(t-\tau-s))\psi'(s)\ ds \\ &\quad + \int_{-\tau}^{0} (\mathrm{Cos}_{\Omega,\tau}(t-\tau-s))\left(\varphi'(s)-\psi(s)\right)\ ds + \ \Omega^{-1}\int_{0}^{t} (\mathrm{Sin}_{\Omega,\tau}(t-\tau-s)) f(s)\ ds \end{aligned} \tag{2.7}$$

on $[0, \infty)$.

3. Control of Oscillating Systems

In this part, we investigate the control problem and give the construction of a control function for oscillating systems with a single delay (1.6) within the meaning of the following definition. Since (1.6) is a second-order system, an initial Cauchy problem, in general, should fix $2n$ independent initial one-dimensional functions. For this reason, in the formulation of an initial Cauchy problem below, we prescribe initial vectors for the solution and its first derivative.

Definition 3.1. System (1.6) is relatively controllable if for any continuously differentiable initial vector functions $\varphi, \psi : [-\tau, 0] \to \mathbb{R}^n$, any finite terminal conditions $x_1, x'_1 \in \mathbb{R}^n$, and any sufficiently large terminal point $t_1 \in \mathbb{R}$, there exists a control $u^* : [0, t_1] \to \mathbb{R}$ such that the system (1.6) with the input $u = u^*$, that is, the system

$$\ddot{x}(t) + \Omega^2 x(t-\tau) = b u^*(t) \tag{3.1}$$

has a solution $x = x^* : [-\tau, t_1] \to \mathbb{R}^n$ such that

$$x^*(t) = \ \varphi(t), \quad -\tau \le t \le 0, \tag{3.2}$$

$$\frac{d}{dt} x^*(t) = \ \psi(t), \quad -\tau \le t \le 0, \tag{3.3}$$

$$x^*(t_1) = x_1, \tag{3.4}$$

$$\frac{d}{dt} x^*(t_1) = x'_1. \tag{3.5}$$

To investigate the problem (3.1)–(3.5), we need some auxiliary notions given below.

Definition 3.2. We call the pair (Ω^2, b) controllable if rank $S^*_n = n$, where

$$S^*_n := \left(b, \Omega^2 b, \Omega^4 b, \dots, \Omega^{2(n-1)} b\right). \tag{3.6}$$

Definition 3.3. Let a positive number r be given. We define the class of all uniformly bounded and piecewise-continuous functions on $[0,t_1]$, $t_1>0$ (called a control set) $U_r(0,t_1)$ as

$$U_r(0,t_1) := \left\{u : [0,t_1] \longrightarrow \mathbb{R}, \|u\|_{[0,t_1]} \le r\right\}, \tag{3.7}$$

where $r>0$ and

$$\|u\|_{[0,t_1]} := \sup_{t\in[0,t_1]} |u(t)|. \tag{3.8}$$

Definition 3.4. The domain

$$Q_{t_1,\varphi,\psi} := \{(x,x') \in \mathbb{R}^n \times \mathbb{R}^n : x = x(t_1), x' = x'(t_1)\}, \tag{3.9}$$

where $t_1>0$ and $x=x(t)$ is a solution of (1.6) corresponding to the fixed initial conditions

$$x(t) = \varphi(t), \quad x'(t) = \psi(t), \quad -\tau \le t \le 0 \tag{3.10}$$

and to an arbitrary control $u \in U_r(0,t_1)$, is called a domain of reachability (reachable set) with respect to the time $t=t_1$ and the functions φ, ψ.

We introduce a $2n$-dimensional auxiliary vector $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_{2n}(t))$:

$$\omega(t) := \begin{pmatrix} \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau} t)b \\ (\mathrm{Cos}_{\Omega,\tau} t)b \end{pmatrix} \tag{3.11}$$

and n-dimensional auxiliary vectors

$$\begin{aligned} \xi_1 = x_1 &- (\mathrm{Cos}_{\Omega,\tau} t_1)\varphi(-\tau) - \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau} t_1)\psi(-\tau) - \Omega^{-1}\int_{-\tau}^{0} (\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s))\psi'(s)\, ds \\ &- \int_{-\tau}^{0} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))\left(\varphi'(s)-\psi(s)\right) ds, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \xi_2 = x_2 &+ \Omega(\mathrm{Sin}_{\Omega,\tau}(t_1-\tau))\varphi(-\tau) - (\mathrm{Cos}_{\Omega,\tau} t_1)\psi(-\tau) \\ &- \int_{-\tau}^{0} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))\psi'(s)\, ds + \Omega\int_{-\tau}^{0} (\mathrm{Sin}_{\Omega,\tau}(t_1-2\tau-s))\left(\varphi'(s)-\psi(s)\right) ds. \end{aligned} \tag{3.13}$$

Before formulating the results on a relative controllability of (1.6), we present some auxiliary propositions.

Lemma 3.5. *Let the pair* (Ω^2, b) *be controllable. Then, on an arbitrary interval* $[-\tau, t^*]$ *with* $t^* > (n-2)\tau$*, the coordinates of the vector function* $\omega(t)$ *are linearly independent, that is, there is no nonzero constant vector*

$$l^T = \left(l_1^T, l_2^T\right), \quad l_1^T = (l_1, l_2, \dots, l_n), \quad l_2^T = (l_{n+1}, l_{n+2} \dots, l_{2n}) \tag{3.14}$$

such that

$$l^T \omega(t) \equiv 0 \tag{3.15}$$

for every $t \in [-\tau, t^*]$.

Proof. Suppose, on the contrary, that there exists a nonzero vector (3.14) such that (3.15) holds for every $t \in [-\tau, t^*]$, that is,

$$l_1^T \Omega^{-1} (\mathrm{Sin}_{\Omega,\tau} t) b + l_2^T (\mathrm{Cos}_{\Omega,\tau} t) b \equiv 0, \quad t \in [-\tau, t^*]. \tag{3.16}$$

We will analyse the identity (3.16). Using Definition 2.1 of a delayed matrix cosine and Definition 2.2 of a delayed matrix sine, we obtain

$$\begin{aligned} &\Bigg(l_1^T \left[I(t+\tau) - \Omega^2 \frac{t^3}{3!} + \Omega^4 \frac{(t-\tau)^5}{5!} + \dots + (-1)^k \Omega^{2k} \frac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!} \right] \\ &\quad + l_2^T \left[I - \Omega^2 \frac{t^2}{2!} + \Omega^4 \frac{(t-\tau)^4}{4!} + \dots + (-1)^k \Omega^{2k} \frac{(t-(k-1)\tau)^{2k}}{(2k)!} \right] \Bigg) b \equiv 0 \\ &\text{if} \quad (k-1)\tau \le t < k\tau, \quad k = 1, 2, \dots . \end{aligned} \tag{3.17}$$

Considering identities (3.17) for $k = 1, 2, \dots, n-1$ and taking into account the fact that the left-hand side of (3.17) is, on every interval $(k-1)\tau \le t < k\tau$, $k = 1, 2, \dots$, a polynomial in t having only a finite number of zero points, we conclude that identity (3.17) is only true in the case of polynomials having all their coefficients equal to zero. In other words, we get the conditions

$$l_1^T S_n^* = \theta^T, \tag{3.18}$$

$$l_2^T S_n^* = \theta^T. \tag{3.19}$$

The homogeneous systems (3.18), (3.19) have a nonzero solution if and only if their determinants are equal to zero, that is, $\det S_n^* = 0$ or $\operatorname{rank} S_n^* < n$. This contradicts the assumption of controllability of the pair (Ω^2, b). □

Lemma 3.6. *If a vector function $\kappa^T(t) = (\kappa_1(t), \kappa_2(t), \ldots, \kappa_{2n}(t))$ consists of linearly independent elements on the interval $[-\tau, t^*]$ where $t^* > -\tau$, then*

$$\det\left(\int_{-\tau}^{t^*} \kappa(t)\kappa^T(t)\ dt\right) \neq 0. \tag{3.20}$$

Proof. The statement is a consequence of the fact that the symmetric matrix

$$\int_{-\tau}^{t^*} \kappa(t)\kappa^T(t)dt \tag{3.21}$$

is positively definite and thus regular. □

Remark 3.7. Note that it is easy to see that the matrix $\kappa(t)\kappa^T(t)$ is (unlike the matrix (3.21)) singular for every $t \in [-\tau, \infty)$.

Now we are able to present a result on the relative controllability of system (1.6), and give an inequality for the value t_1, mentioned in Definition 3.1.

Theorem 3.8. *System* (1.6) *is relatively controllable if and only if $t_1 > (n-1)\tau$ and the pair (Ω^2, b) is controllable.*

Proof (Necessity). Let the system (1.6) be relatively controllable within the meaning of Definition 3.1. We use the representation formula for a solution of the Cauchy problem for nonhomogeneous equation in the form (2.7) for the control u^* (i.e. $f := bu^*$) and time $t = t_1$. We obtain

$$\begin{aligned} x_1 = {}& (\mathrm{Cos}_{\Omega,\tau} t_1)\varphi(-\tau) + \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau} t_1)\varphi(-\tau) + \Omega^{-1}\int_{-\tau}^{0} (\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s))\varphi'(s)ds \\ &+ \int_{-\tau}^{0} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))\left(\varphi'(s)-\varphi(s)\right)\ ds + \Omega^{-1}\int_{0}^{t_1} (\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s))bu^*(s)ds, \end{aligned} \tag{3.22}$$

$$\begin{aligned} x_1' = {}& -\Omega(\mathrm{Sin}_{\Omega,\tau}(t_1-\tau))\varphi(-\tau) + (\mathrm{Cos}_{\Omega,\tau} t_1)\varphi(-\tau) \\ &+ \int_{-\tau}^{0} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))\varphi'(s)ds - \Omega\int_{-\tau}^{0} (\mathrm{Sin}_{\Omega,\tau}(t_1-2\tau-s))\left(\varphi'(s)-\varphi(s)\right)ds \\ &+ \int_{0}^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))bu^*(s)ds. \end{aligned} \tag{3.23}$$

Equation (3.22) and (3.23) can be rewritten as a system of the two following equations:

$$\int_{0}^{t_1} (\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s))bu^*(s)ds = \Omega\xi_1, \tag{3.24}$$

$$\int_{0}^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))bu^*(s)ds = \xi_2, \tag{3.25}$$

where ξ_1, ξ_2 are defined by (3.12) and (3.13).

We will investigate the system (3.24). Let, for an integer $k \geq 0$, $t_1 \in [(k-1)\tau, k\tau)$. We use the representation of a delayed matrix sine and, after putting $t_1 - \tau - s = \eta$, the left-hand side of (3.24) equals

$$\begin{aligned}
&\int_0^{t_1} (\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)) bu^*(s)ds = \int_{-\tau}^{t_1-\tau} (\mathrm{Sin}_{\Omega,\tau}\eta) bu^*(t_1 - \tau - \eta)d\eta \\
&\quad = \int_{-\tau}^{0} \Omega(\eta + \tau) bu^*(t_1 - \tau - \eta)d\eta + \int_0^{\tau} \left[\Omega(\eta + \tau) - \Omega^3 \frac{\eta^3}{3!}\right] bu^*(t_1 - \tau - \eta)d\eta \\
&\qquad + \int_{\tau}^{2\tau} \left[\Omega(\eta + \tau) - \Omega^3 \frac{\eta^3}{3!} + \Omega^5 \frac{(\eta - \tau)^5}{5!}\right] bu^*(t_1 - \tau - \eta)d\eta \\
&\qquad + \dots + \int_{(k-2)\tau}^{t_1-\tau} \left[\Omega(\eta + \tau) - \Omega^3 \frac{\eta^3}{3!} + \Omega^5 \frac{(\eta - \tau)^5}{5!} + \dots \right. \\
&\qquad\qquad \left. + (-1)^{k-1}\Omega^{2k-1} \frac{(\eta - (k-2)\tau)^{2k-1}}{(2k-1)!}\right] bu^*(t_1 - \tau - \eta)d\eta.
\end{aligned} \tag{3.26}$$

We denote

$$\begin{aligned}
\psi_1(t_1) &= \int_{-\tau}^{t_1-\tau} (\eta + \tau)u^*(t_1 - \tau - \eta)d\eta, \\
\psi_3(t_1) &= -\frac{1}{3!}\int_0^{t_1-\tau} \eta^3 u^*(t_1 - \tau - \eta)d\eta, \\
\psi_5(t_1) &= \frac{1}{5!}\int_{\tau}^{t_1-\tau} (\eta - \tau)^5 u^*(t_1 - \tau - \eta)d\eta, \\
&\dots \\
\psi_{2k-1}(t_1) &= \frac{(-1)^{k-1}}{(2k-1)!}\int_{(k-2)\tau}^{t_1-\tau} (\eta - (k-2)\tau)^{2k-1} u^*(t_1 - \tau - \eta)d\eta.
\end{aligned} \tag{3.27}$$

Using (3.27) and the regularity of the matrix Ω, we rewrite the system (3.24) in the form

$$b\psi_1(t_1) + \Omega^2 b\psi_3(t_1) + \Omega^4 b\psi_5(t_1) + \dots + \Omega^{2(k-1)} b\psi_{2k-1}(t_1) = \xi_1. \tag{3.28}$$

Now we go on analysing the system (3.25). Let, as in the previous case, for an integer $k \geq 0$, $t_1 \in [(k-1)\tau, k\tau)$. We use the representation of a delayed matrix cosine and, after putting

$t_1 - \tau - s = \eta$, we obtain

$$\begin{aligned}
\int_0^{t_1} \mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s) b u^*(s) ds &= \int_{-\tau}^{t_1-\tau} (\mathrm{Cos}_{\Omega,\tau}\eta) b u^*(t_1 - \tau - \eta) d\eta \\
&= \int_{-\tau}^{0} b u^*(t_1 - \tau - \eta) d\eta + \int_0^{\tau} \left[I - \Omega^2 \frac{\eta^2}{2!} \right] b u^*(t_1 - \tau - \eta) d\eta \\
&\quad + \int_{\tau}^{2\tau} \left[I - \Omega^2 \frac{\eta^2}{2!} + \Omega^4 \frac{(\eta-\tau)^4}{4!} \right] b u^*(t_1 - \tau - \eta) d\eta \\
&\quad + \dots + \int_{(k-2)\tau}^{t_1-\tau} \left[I - \Omega^2 \frac{\eta^2}{2!} + \Omega^4 \frac{(\eta-\tau)^4}{4!} \right. \\
&\quad\quad \left. - \dots + (-1)^{k-1} \Omega^{2(k-1)} \frac{(\eta - (k-2)\tau)^{2(k-1)}}{(2k-2)!} \right] b u^*(t_1 - \tau - \eta) d\eta.
\end{aligned} \tag{3.29}$$

We denote

$$\begin{aligned}
\psi_0(t_1) &= \int_{-\tau}^{t_1-\tau} u^*(t_1 - \tau - \eta) d\eta, \\
\psi_2(t_1) &= -\frac{1}{2!} \int_0^{t_1-\tau} \eta^2 u^*(t_1 - \tau - \eta) d\eta, \\
\psi_4(t_1) &= \frac{1}{4!} \int_{\tau}^{t_1-\tau} (\eta-\tau)^4 u^*(t_1 - \tau - \eta) d\eta, \\
&\dots \\
\psi_{2k-2}(t_1) &= (-1)^{k-1} \int_{(k-2)\tau}^{t_1-\tau} \frac{(\eta - (k-2)\tau)^{2(k-1)}}{(2k-2)!} u^*(t_1 - \tau - \eta) d\eta.
\end{aligned} \tag{3.30}$$

Using (3.30) and the regularity of the matrix Ω, we rewrite the system (3.25) in the form

$$b\psi_0(t_1) + \Omega^2 b\psi_2(t_1) + \dots + \Omega^{2(k-1)} b\psi_{2k-2}(t_1) = \xi_2. \tag{3.31}$$

It was assumed that the system (1.6) is relatively controllable. Consequently, systems (3.28) and (3.31) have solutions for arbitrary vectors ξ_1, ξ_2. If $k < n$ or $k = n$ and $t_1 = (n-1)\tau$, then both systems are overdetermined and the existence of a solution is not guaranteed. Therefore, for (1.6) to be relatively controllable, it is necessary that $k \geq n$ and, if $k = n$, then $t_1 \neq (n-1)\tau$, that is,

$$t_1 > (k-1)\tau \geq (n-1)\tau. \tag{3.32}$$

A simple consequence of Cayley-Hamilton theorem (e.g. [5]) is that an arbitrary degree $(\Omega^2)^i$, $i \geq n$ of the matrix Ω^2 can be represented as a linear combination of matrices

$$I, \Omega^2, \Omega^4, \dots, \Omega^{2(n-1)}. \tag{3.33}$$

Then, for $k \geq n$, both systems (3.28) and (3.31) can be replaced by the systems

$$b\varphi_1^*(t_1) + \Omega^2 b\varphi_3^*(t_1) + \dots + \Omega^{2(n-1)} b\varphi_{2n-1}^*(t_1) = \xi_1, \tag{3.34}$$

$$b\varphi_0^*(t_1) + \Omega^2 b\varphi_2^*(t_1) + \dots + \Omega^{2(n-1)} b\varphi_{2n-2}^*(t_1) = \xi_2, \tag{3.35}$$

where $\varphi_i^*(t_1)$, $i = 0, 1, \dots, 2n-1$ are some new functions depending on t_1. If systems (3.34) and (3.35) have solutions

$$\begin{aligned} &\varphi_1^*(t_1), \varphi_3^*(t_1), \dots, \varphi_{2n-1}^*(t_1), \\ &\varphi_0^*(t_1), \varphi_2^*(t_1), \dots, \varphi_{2n-2}^*(t_1) \end{aligned} \tag{3.36}$$

for an arbitrary choice of ξ_1, ξ_2, then $\det S_n^* \neq 0$, that is, the pair (Ω^2, b) is controllable. The necessity is proved.

Sufficiency. The proof almost fully copies a known proof of sufficiency for linear systems without delay. Due to the linearity of the problem considered, we can assume, without loss of generality, that the initial functions are zero vector-functions, that is,

$$\varphi(t) = \psi(t) = \theta, \quad -\tau \leq t \leq 0. \tag{3.37}$$

In addition to this, $t_1 > (n-1)\tau$ and the controllability of (Ω^2, b) is assumed. We prove that the system (1.6), is relatively controllable.

First we prove that the domain of reachability $Q_{t_1,\theta,\theta}$ has a dimension of $2n$ if $u \in \Omega_1(0, t_1)$. Let, on the contrary, $\dim Q_{t_1,\theta,\theta} < 2n$. Then, there exists a fixed vector $l \in \mathbb{R}^{2n}$,

$$l^T = \left(l_1^T, l_2^T\right), \qquad l_1^T = (l_1, l_2, \dots, l_n), \qquad l_2^T = (l_{n+1}, l_{n+2} \dots, l_{2n}) \tag{3.38}$$

such that, for an arbitrary $u \in \Omega_1(0, t_1)$, for the relevant solution $x = x(t)$ of (1.6) we have

$$l_1^T x(t_1) + l_2^T x'(t_1) = 0. \tag{3.39}$$

Since the initial functions are zero vector-functions, condition (3.39) turns, due to formula (2.7) with $f := bu$, into

$$l_1^T \Omega^{-1} \left[\int_0^{t_1} (\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)) bu(s) ds \right] + l_2^T \left[\int_0^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s)) bu(s) ds \right] = 0. \tag{3.40}$$

Because (3.40) is satisfied for an arbitrary function $u \in \Omega_1(0, t_1)$, we deduce, by the fundamental lemma of the calculus of variations, that

$$l_1^T \Omega^{-1} (\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s))b + l_2^T (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))b \equiv 0, \quad 0 \le s \le t_1, \tag{3.41}$$

or, after putting $t_1 - \tau - s = \xi$, we have

$$\left[l_1^T \Omega^{-1} \mathrm{Sin}_{\Omega,\tau} \xi + l_2^T \mathrm{Cos}_{\Omega,\tau} \xi \right] b \equiv 0, \quad -\tau \le \xi \le t_1 - \tau. \tag{3.42}$$

This contradicts the statement of Lemma 3.5 with $t^* = t_1 - \tau$. Thus, the assumption that the dimension of $Q_{t_1,\theta,\theta}$ is smaller than $2n$ is false.

Since the domain of reachability $Q_{t_1,\theta,\theta}$ together with a point $(x(t_1), x'(t_1))$ corresponding to a control $u \in \Omega_1(0, t_1)$ also contains a point $(-x(t_1), -x'(t_1))$ (corresponding to a control $-u \in \Omega_1(0, t_1)$), we conclude that $Q_{t_1,\theta,\theta}$ is symmetric. Due to the linearity of the problem considered, it is also a convex domain. Consequently, it contains a ball with a radius of $\delta > 0$.

Obviously, if we consider the control set $U_r(0, t_1)$ instead of $\Omega_1(0, t_1)$ and $r \to \infty$, then $\delta \to \infty$, that is, $Q_{t_1,\theta,\theta} = \mathbb{R}^{2n}$. Simultaneously, it says that, for every point $(x, x') \in \mathbb{R}^{2n}$, there exists a control $u = u^* : [0, t_1] \to \mathbb{R}$ such that the solution $x = x^*$ of (3.1) satisfies (3.2)–(3.5).

This conclusion remains valid even in the case of any nonidentically zero initial functions. Indeed, a simple transformation $x(t) = x_{\varphi,\psi}(t) + z(t)$, where $z(t)$ is a new dependent function and $x_{\varphi,\psi}(t)$ is a solution of a homogeneous problem

$$\begin{gathered} \ddot{x}(t) + \Omega^2 x(t - \tau) = \theta, \quad t \ge 0, \\ x(t) = \varphi(t), \quad -\tau \le t \le 0, \\ \frac{d}{dt} x(t) = \psi(t), \quad -\tau \le t \le 0 \end{gathered} \tag{3.43}$$

leads to the same problem with respect to z with zero initial vector-functions. Thus, the system (1.6) is relatively controllable. □

Now we give the formula for a relevant control function. An advantage of the result obtained is an explicit dependence of the control function on the delayed matrix cosine and delayed matrix sine.

Theorem 3.9. *Let $t_1 > (n-1)\tau$ and let the pair (Ω^2, b) be controllable. Then, a control function relevant to the problem* (3.1)–(3.5) *can be thought of as having the form*

$$u^*(t) = b^T \left(\Omega^{-1} \mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - t) \right)^T C_1^0 + b^T (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - t))^T C_2^0, \tag{3.44}$$

where $t \in [0, t_1]$, $C_1^0 = (c_1^0, c_2^0, \dots, c_n^0)^T$ and $C_2^0 = (c_{n+1}^0, c_{n+2}^0, \dots, c_{2n}^0)^T$ are the solutions of a system of linear nonhomogeneous algebraic equations

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} C_1^0 \\ C_2^0 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{3.45}$$

where $n \times n$ matrices D_{ij}, $i, j = 1, 2$, are defined as

$$
\begin{aligned}
D_{11} &= \int_0^{t_1} \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s))bb^T \left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)\right)^T ds, \\
D_{12} &= \int_0^{t_1} \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s))bb^T (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))^T ds, \\
D_{21} &= \int_0^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))bb^T \left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)\right)^T ds, \\
D_{22} &= \int_0^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))bb^T (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))^T ds,
\end{aligned}
\tag{3.46}
$$

and vectors ξ_1, ξ_2 are defined by (3.13).

Proof. Let $t_1 > (n-1)\tau$. Since the pair (Ω^2, b) is controllable, the vectors

$$
b, \Omega^2 b, \Omega^4 b, \ldots, \Omega^{2(n-1)} b \tag{3.47}
$$

are linearly independent. From the integral representation (2.7), it follows that the terminal values of a solution $x = x(t)$ of system (1.6) at $t = t_1$ satisfying initial conditions (3.2) and (3.3) have the form (3.22) and (3.23). Therefore, a suitable control function u should satisfy the system of integral equations (3.24) and (3.25). Let us try to find the control function in the form of a linear combination

$$
u^*(t) = b^T \left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - t)\right)^T C_1 + b^T (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - t))^T C_2, \tag{3.48}
$$

where $C_1 = (c_1, c_2, \ldots, c_n)^T$ and $C_2 = (c_{n+1}, c_{n+2}, \ldots, c_{2n})^T$ are unknown constant vectors. We apply representation (3.48) to (3.24), (3.25). Then, we get

$$
\begin{aligned}
&\int_0^{t_1} \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s))bb^T \\
&\qquad \times \left[\left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)\right)^T C_1 + (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))^T C_2\right] ds = \xi_1, \\
&\int_0^{t_1} (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))bb^T \\
&\qquad \times \left[\left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1 - \tau - s)\right)^T C_1 + (\mathrm{Cos}_{\Omega,\tau}(t_1 - \tau - s))^T C_2\right] ds = \xi_2,
\end{aligned}
\tag{3.49}
$$

or

$$
\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{3.50}
$$

with the above defined matrices D_{ij}, $i, j = 1, 2$. The determinant Δ of the system (3.50) can be written in the form

$$\Delta = \det\left(\int_0^{t_1} \begin{pmatrix} \Omega^{-1}(\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s))b \\ (\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))b \end{pmatrix} \right.$$
$$\left. \times \left(b^T\left(\Omega^{-1}\mathrm{Sin}_{\Omega,\tau}(t_1-\tau-s)\right)^T, b^T(\mathrm{Cos}_{\Omega,\tau}(t_1-\tau-s))^T \right) ds \right). \tag{3.51}$$

Using a transformation $t_1 - \tau - s = t$ in the integral and the denotation (3.11), we get

$$\Delta = \det\left(\int_{-\tau}^{t_1-\tau} \omega(t)\omega_\tau^T(t)dt\right). \tag{3.52}$$

By Lemma 3.5, the coordinates of $\omega(t)$ are linearly independent on $[-\tau, t^*]$ where $t^* > (n-2)\tau$. Then, by Lemma 3.6 (with $t^* > (n-2)\tau$), $\Delta \neq 0$. Consequently, the system (3.50) has a unique solution $C_1 = C_1^0$, $C_2 = C_2^0$, and the control (3.48) coincides with (3.44). □

4. Conclusions and Future Directions

The paper studied the problem of the relative controllability of oscillating systems (1.6) within the meaning of Definition 3.1. An explicit representation of solutions of (1.6) with the aid of special matrix functions called a delayed matrix sine and a delayed matrix cosine was used to solve this problem. The necessary and sufficient conditions of relative controllability were derived and a specific control function was constructed in terms of these matrix functions, solving the relative controllability problem of transferring an initial function to a prescribed point in the phase space. Some previous results of investigating the controllability problems using special matrix functions were derived for linear delayed systems with a single delay in [16] (the case of continuous systems) and in [17] (the case of discrete systems) where representations of solutions of linear discrete systems [18, 19] are used. It is an open problem how to extend the results derived to systems of discrete equations with a single delay

$$x(n+2) + \Omega^2 x(n-m) = bu(n), \tag{4.1}$$

where n is an independent variable, m is a positive integer, and (4.1) is a discrete analogy of (1.6). Another open problem is how to extend the results derived to fractional systems (see, e.g., [20]).

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