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Research Article

Elementary Proof of Yu. V. Nesterenko Expansion of the Number Zeta(3) in Continued Fraction

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Yu. V. Nesterenko has proved that $\zeta(3) = b_0 + a_1|/|b_1 + \cdots + a_\nu|/|b_\nu + \cdots$, $b_0 = b_1 = a_2 = 2$, $a_1 = 1, b_2 = 4$, $b_{4k+1} = 2k+2$, $a_{4k+1} = k(k+1)$, $b_{4k+2} = 2k+4$, and $a_{4k+2} = (k+1)(k+2)$ for $k \in \mathbb{N}$; $b_{4k+3} = 2k+3$, $a_{4k+3} = (k+1)^2$, and $b_{4k+4} = 2k+2$, $a_{4k+4} = (k+2)^2$ for $k \in \mathbb{N}_0$. His proof is based on some properties of hypergeometric functions. We give here an elementary direct proof of this result.

1. Foreword

Applications of difference equations to the Number Theory have a long history. For example, one can find in this journal several articles connected with the mentioned applications (see [1–8]). The interest in this area increases after Apéry's discovery of irrationality of the number $\zeta(3)$. This paper is inspired by Yu. V. Nesterenko's work [9]. My goal is to give an elementary direct proof of his expansion of the number $\zeta(3)$ in continued fraction. Let us consider a difference equation

$$x_{\nu+1} - b_{\nu+1}x_\nu - a_{\nu+1}x_{\nu-1} = 0, \tag{1.1}$$

with $\nu \in \mathbb{N}_0$. We denote by

$$\{P_\nu(b_0, a_1, b_1, \ldots, a_\nu, b_\nu)\}_{\nu=-1}^{+\infty}, \qquad \{Q_\nu(b_0, a_1, b_1, \ldots, a_\nu, b_\nu)\}_{\nu=-1}^{+\infty} \tag{1.2}$$

the solutions of this equation with initial values

$$P_{-1} = 1, \quad Q_{-1} = 0, \quad P_0(b_0) = b_0, \quad Q_0(b_0) = 1. \tag{1.3}$$

Then

$$\left\{\frac{P_\nu(b_0, a_1, b_1, \ldots, a_\nu, b_\nu)}{Q_\nu(b_0, a_1, b_1, \ldots, a_\nu, b_\nu)}\right\}_{\nu=0}^{+\infty} \tag{1.4}$$

is a sequence of convergents of the continued fraction

$$b_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_\nu|}{|b_\nu} + \ldots \quad . \tag{1.5}$$

Accoding to the famous result of R. Apéry [10],

$$\zeta(3) = \lim_{\nu\to\infty}\frac{v_\nu}{u_\nu}, \tag{1.6}$$

where $\{u_\nu\}_{\nu=0}^{+\infty}$ and $\{v_\nu\}_{\nu=0}^{+\infty}$ are solutions of difference equation

$$(\nu+1)^3 x_{\nu+1} - \left(34\nu^3 + 51\nu^2 + 27\nu + 5\right)x_\nu + \nu^3 x_{\nu-1} = 0 \tag{1.7}$$

with initial values $u_0 = 1$, $u_1 = 5$, $v_1 = 0$, $v_1 = 6$. The equality (1.6) is equivalent to the equality

$$\zeta(3) = b_0^\vee + \frac{a_1^\vee|}{|b_1^\vee} + \frac{a_2^\vee|}{|b_2^\vee} + \cdots + \frac{a_\nu^\vee|}{|b_\nu^\vee} + \ldots \tag{1.8}$$

with

$$b_0^\vee = 0, \quad b_1^\vee = 5, \quad a_1^\vee = 6, \quad b_{\nu+1}^\vee = 34\nu^3 + 51\nu^2 + 27\nu + 5, \quad a_{\nu+1}^\vee = -\nu^6, \tag{1.9}$$

where $\nu \in \mathbb{N}$. Nesterenko in [9] has offered the following expansion of the number $2\zeta(3)$ in continued fraction:

$$2\zeta(3) = 2 + \frac{1|}{|2} + \frac{2|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}\ldots, \tag{1.10}$$

with

$$b_0 = b_1 = a_2 = 2, \quad a_1 = 1, \quad b_2 = 4, \tag{1.11}$$

$$b_{4k+1} = 2k+2, a_{4k+1} = k(k+1), \quad b_{4k+2} = 2k+4, \quad a_{4k+2} = (k+1)(k+2) \tag{1.12}$$

for $k \in \mathbb{N}$;

$$b_{4k+3} = 2k+3, \quad a_{4k+3} = (k+1)^2, \quad b_{4k+4} = 2k+2, \quad a_{4k+4} = (k+2)^2 \tag{1.13}$$

for $k \in \mathbb{N}_0$.

The halved convergents of continued fraction (1.10) compose a sequence containing convergents of continued fraction (1.8). I give an elementary proof of Yu. V. Nesterenko expansion in Section 2 .

2. Elementary Proof of Yu. V. Nesterenko Expansion

Instead of expansion (1.10) with (1.11), it is more convenient for us to prove the equivalent expansion

$$\zeta(3) = 1 + \frac{1|}{|4} + \frac{4|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}\ldots, \tag{2.1}$$

with

$$b_0 = 1, \quad a_1 = 1, \quad b_1 = a_2 = b_2 = 4. \tag{2.2}$$

Furthermore, to avoid confusion in notations, we denote below a_ν, b_ν for the fraction (2.1) by $a_\nu^\wedge, b_\nu^\wedge$. Let $P_{-1}^\vee = 1,\ Q_{-1}^\vee = 0$,

$$P_\nu^\vee = P_\nu\left(b_0^\vee, a_1^\vee, b_1^\vee, \ldots, a_\nu^\vee, b_\nu^\vee\right), \qquad Q_\nu^\vee = Q_\nu\left(b_0^\vee, a_1^\vee, b_1^\vee, \ldots, a_\nu^\vee, b_\nu^\vee\right), \tag{2.3}$$

where values $a_\nu^\vee, b_\nu^\vee$ are specified in (1.9), and $\nu \in \mathbb{N}_0$. Then

$$\begin{gathered} Q_0^\vee = 1, \quad P_0^\vee = b_0^\vee = 0, \quad Q_1^\vee = b_1^\vee = 5, \quad P_1^\vee = a_1^\vee = 6, \quad b_2^\vee = 117, \quad a_2^\vee = -1, \\ P_2^\vee = b_2^\vee P_1^\vee + a_2^\vee P_0^\vee = 702, \quad Q_2^\vee = b_2^\vee Q_1^\vee + a_2^\vee Q_0^\vee = 584. \end{gathered} \tag{2.4}$$

Let $P_{-1}^\wedge = 1,\ Q_{-1}^\wedge = 0$,

$$P_\nu^\wedge = P_\nu\left(b_0^\wedge, a_1^\wedge, b_1^\wedge, \ldots, a_\nu^\wedge, b_\nu^\wedge\right), \qquad Q_\nu^\wedge = Q_\nu\left(b_0^\wedge, a_1^\wedge, b_1^\wedge, \ldots, a_\nu^\wedge, b_\nu^\wedge\right), \tag{2.5}$$

where $\nu \in \mathbb{N}_0,\ a_\nu^\wedge := a_\nu,\ b_\nu^\wedge := b_\nu$, and values a_ν, b_ν are specified in (2.2), (1.12), and (1.13). We calculate first $P_k^\wedge$ and $Q_k^\wedge$ for $k = 0, \ldots, 6$.

Since $P^{\wedge}_{-1} = 1,\ Q^{\wedge}_{-1} = 0$, it follows from (2.2) that

$$P^{\wedge}_0 = b_0 = 1, \quad Q^{\wedge}_0 = 1,$$

$$P^{\wedge}_1 = b^{\wedge}_1 P^{\wedge}_0 + a^{\wedge}_1 P^{\wedge}_{-1} = 5, \quad Q^{\wedge}_1 = b^{\wedge}_1 Q^{\wedge}_0 + a^{\wedge}_1 Q^{\wedge}_{-1} = 4, \tag{2.6}$$

$$P^{\wedge}_2 = b^{\wedge}_2 P^{\wedge}_1 + a^{\wedge}_2 P^{\wedge}_0 = 24 = 4P^{\vee}_1,$$

$$Q^{\wedge}_2 = b^{\wedge}_2 Q_1 + a^{\wedge}_2 Q_0 = 20 = 4Q^{\vee}_1, \tag{2.7}$$

$$P^{\wedge}_3 = b^{\wedge}_3 P^{\wedge}_2 + a^{\wedge}_3 P^{\wedge}_1 = 77, \quad Q^{\wedge}_3 = b^{\wedge}_3 Q^{\wedge}_2 + a^{\wedge}_3 Q^{\wedge}_1 = 64,$$

$$P^{\wedge}_4 = b^{\wedge}_4 P^{\wedge}_3 + a^{\wedge}_4 P^{\wedge}_2 = 250, \quad Q^{\wedge}_4 = b^{\wedge}_4 Q^{\wedge}_3 + a^{\wedge}_4 Q^{\wedge}_2 = 208, \tag{2.8}$$

$$P^{\wedge}_5 = b^{\wedge}_5 P^{\wedge}_4 + a^{\wedge}_5 P^{\wedge}_3 = 1154, \quad Q^{\wedge}_5 = b^{\wedge}_5 Q^{\wedge}_4 + a^{\wedge}_5 Q^{\wedge}_3 = 960,$$

$$P^{\wedge}_6 = b^{\wedge}_6 P^{\wedge}_5 + a^{\wedge}_6 P^{\wedge}_4 = 12 \times 702 = 12P^{\vee}_2, \tag{2.9}$$

$$Q^{\wedge}_6 = b^{\wedge}_6 Q^{\wedge}_5 + a^{\wedge}_6 Q^{\wedge}_4 = 12 \times 584 = 12Q^{\vee}_2. \tag{2.10}$$

Let $k \in \mathbb{N},\ k \geq 2,$

$$P^*_k = \frac{P^{\wedge}_{4k-2}}{2(k+1)!}, \qquad Q^*_k = \frac{Q^{\wedge}_{4k-2}}{2(k+1)!}. \tag{2.11}$$

We want to to prove that if $k \in \mathbb{N}$, then

$$P^*_k = P^{\vee}_k, \qquad Q^*_k = Q^{\vee}_k. \tag{2.12}$$

Note that if $k = 1, 2$, then (2.12) follows from (2.6)–(2.10). Therefore, we can consider only $k \in [3, +\infty) \cap \mathbb{Z}$. Let us consider the following difference equations:

$$x_{\nu+1} - b^{\vee}_{\nu+1} x_\nu - a^{\vee}_{\nu+1} x_{\nu-1} = 0, \tag{2.13}$$

$$x_{\nu+1} - b^{\wedge}_{\nu+1} x_\nu - a^{\wedge}_{\nu+1} x_{\nu-1} = 0, \tag{2.14}$$

with $\nu \in \mathbb{N}_0$. Then $x_\nu = P^{\vee}_\nu$, $x_\nu = Q^{\vee}_\nu$, with $\nu \in (-1, +\infty) \cap \mathbb{Z}$ representing a fundamental system of solutions of (2.13), and $x_\nu = P^{\wedge}_\nu$, $x_\nu = Q^{\wedge}_\nu$ with $\nu \in (-1, +\infty) \cap \mathbb{Z}$ representing a fundamental system of solutions of (2.14). Making use of standard interpretation of a difference equation as a difference system, we rewrite the equalities (2.13) and (2.14), respectively in the form

$$X_{\nu+1} = A_\nu^\vee X_\nu, \tag{2.15}$$

$$X_{\nu+1} = A_\nu^\wedge X_\nu, \tag{2.16}$$

where

$$X_\nu = \begin{pmatrix} x_{\nu-1} \\ x_\nu \end{pmatrix}, \tag{2.17}$$

$$A_\nu^\vee = \begin{pmatrix} 0 & 1 \\ a_{1+\nu}^\vee & b_{1+\nu}^\vee \end{pmatrix}, \qquad A_\nu^\wedge = \begin{pmatrix} 0 & 1 \\ a_{1+\nu}^\wedge & b_{1+\nu}^\wedge \end{pmatrix}, \tag{2.18}$$

and $\nu \in \mathbb{N}_0$. Let

$$U_\nu^\vee = \begin{pmatrix} P_{\nu-1}^\vee & Q_{\nu-1}^\vee \\ P_\nu^\vee & Q_\nu^\vee \end{pmatrix}, \tag{2.19}$$

$$U_\nu^\wedge = \begin{pmatrix} P_{\nu-1}^\wedge & Q_{\nu-1}^\wedge \\ P_\nu^\wedge & Q_\nu^\wedge \end{pmatrix}, \tag{2.20}$$

with $\nu \in \mathbb{N}_0$ be fundamental matrices of solutions of systems (2.15) and (2.16), respectively. Therefore,

$$U_\nu^\wedge = A_{\nu-1}^\wedge U_{\nu-1}^\wedge, \qquad U_\nu^\vee = A_{\nu-1}^\vee U_{\nu-1}^\vee \tag{2.21}$$

for $\nu \in \mathbb{N}$. In view of (2.18) and (2.21), $\det(U_\nu) = -a_\nu \det(U_{\nu-1})$, and therefore,

$$\det\left(U_\nu^\wedge\right) = (-1)^\nu \det\left(U_0^\wedge\right) \prod_{k=1}^{\nu} a_k^\wedge = (-1)^\nu \prod_{k=1}^{\nu} a_k^\wedge. \tag{2.22}$$

Hence

$$\frac{P_{\nu-1}^\wedge}{Q_{\nu-1}^\wedge} - \frac{P_\nu^\wedge}{Q_\nu^\wedge} = (-1)^\nu \frac{\prod_{k=1}^{\nu} a_k^\wedge}{Q_\nu^\wedge Q_{\nu-1}^\wedge} \tag{2.23}$$

(see [11]).

Further, we have

$$U_0^{\vee} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1^{\vee} = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix}, \quad U_2^{\vee} = \begin{pmatrix} 6 & 5 \\ 702 & 584 \end{pmatrix},$$
$$U_0^{\wedge} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1^{\wedge} = \begin{pmatrix} 1 & 1 \\ 5 & 4 \end{pmatrix}, \quad U_2^{\wedge} = \begin{pmatrix} 5 & 4 \\ 24 & 20 \end{pmatrix},$$
$$U_3^{\wedge} = \begin{pmatrix} 24 & 20 \\ 77 & 64 \end{pmatrix}, \quad U_4^{\wedge} = \begin{pmatrix} 77 & 64 \\ 250 & 208 \end{pmatrix}, \tag{2.24}$$
$$U_5^{\wedge} = \begin{pmatrix} 250 & 208 \\ 1154 & 960 \end{pmatrix}, \quad U_6^{\wedge} = \begin{pmatrix} 1154 & 960 \\ 8424 & 7008 \end{pmatrix},$$

$$\left(U_1^{\vee}\right)\left(U_2^{\wedge}\right)^{-1} = \frac{1}{4}\begin{pmatrix} -24 & 5 \\ 0 & 1 \end{pmatrix}, \tag{2.25}$$

$$\left(U_2^{\vee}\right)\left(U_6^{\wedge}\right)^{-1} = \frac{1}{96}\begin{pmatrix} -36 & 5 \\ 0 & 8 \end{pmatrix}. \tag{2.26}$$

Let $k \in \mathbb{N}, k \geq 2$. Then, in view of (2.20),

$$A_{4k-6}^{\wedge} = \begin{pmatrix} 0 & 1 \\ a_{4(k-2)+3}^{\wedge} & b_{4(k-2)+3}^{\wedge} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (k-1)^2 & 2k-1 \end{pmatrix},$$
$$A_{4k-5}^{\wedge} = \begin{pmatrix} 0 & 1 \\ a_{4(k-2)+4}^{\wedge} & b_{4(k-2)+4}^{\wedge} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k^2 & 2k-2 \end{pmatrix}, \tag{2.27}$$
$$A_{4k-4}^{\wedge} = \begin{pmatrix} 0 & 1 \\ a_{4(k-1)+1}^{\wedge} & b_{4(k-1)+1}^{\wedge} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k^2-k & 2k \end{pmatrix},$$
$$A_{4k-3}^{\wedge} = \begin{pmatrix} 0 & 1 \\ a_{4(k-1)+2}^{\wedge} & b_{4(k-1)+2}^{\wedge} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k^2+k & 2k+2 \end{pmatrix}.$$

Let $Y_k = X_{4k-6}$ for $k \in [2, +\infty) \cap \mathbb{Z}$. In view of (2.16) and (2.18),

$$Y_{k+1} = B_k^{\wedge} Y_k, \tag{2.28}$$

$$U_{4k-2}^{\wedge} = B_k^{\wedge} U_{4k-6}^{\wedge}, \tag{2.29}$$

where, as before, $k \in [2, +\infty) \cap \mathbb{Z}$,

$$B_k^{\wedge} = A_{4k-3}^{\wedge} A_{4k-4}^{\wedge} A_{4k-5}^{\wedge} A_{4k-6} = \begin{pmatrix} 5k(k-1)^3 & k\left(12k^2-15k+5\right) \\ 12k(k+1)(k-1)^3 & k(k+1)\left(29k^2-36k+12\right) \end{pmatrix}. \tag{2.30}$$

In view of (2.22), (2.2), (1.12), (1.13), (2.29), and (2.28), the matrix $U^{\wedge}_{4k-6}$ is a fundamental matrix of solutions of system (2.28). The substitution $Z_k = C_k Y_k$, with $\det(C_k) \neq 0$ for $k \in [2, +\infty) \cap \mathbb{Z}$, transforms the system (2.28) into the system

$$Z_{k+1} = D_k Z_k, \tag{2.31}$$

with $D_k = C_{k+1} B^{\wedge}_k (C_k)^{-1}$ for $k \in [2, +\infty) \cap \mathbb{Z}$. We prove now that if we take $k \in [3, +\infty) \cap \mathbb{Z}$, and $C_k = H_{k-1}$, where

$$H_1 = \frac{1}{4}\begin{pmatrix} -24 & 5 \\ 0 & 1 \end{pmatrix}, \tag{2.32}$$

$$H_k = \begin{pmatrix} 12(k+2)(k+1)c(k+1) & -5(k+2)c(k+1) \\ 0 & -(k-1)^3 c(k) \end{pmatrix}, \tag{2.33}$$

with $k \in [2, +\infty) \cap \mathbb{Z}$ and $c(k) = \left(-2(k-1)^3 (k+1)!\right)^{-1}$, then we obtain the equality $D_k = A^{\vee}_{k-1}$. So, let $k \in [3, +\infty) \cap \mathbb{Z}$. Then, in view of (2.33),

$$H_{k-1} = \begin{pmatrix} 12(k+1)kc(k) & -5(k+1)c(k) \\ 0 & -(k-2)^3 c(k-1) \end{pmatrix}. \tag{2.34}$$

In view of(1.9)

$$b^{\vee}_k = 34(k-1)^3 + 51(k-1)^2 + 27(k-1) + 5 = 34k^3 - 51k^2 + 27k - 5, \quad a^{\vee}_k = -(k-1)^6, \tag{2.35}$$

where $k \in [3, +\infty) \cap \mathbb{Z}$. Hence, in view of (2.19),

$$A^{\vee}_{k-1} = \begin{pmatrix} 0 & 1 \\ -(k-1)^6 & 34k^3 - 51k^2 + 27k - 5 \end{pmatrix}. \tag{2.36}$$

In view of (2.34)–(2.36),

$$\begin{aligned} A^{\vee}_{k-1} H_{k-1} &= \begin{pmatrix} 0 & 1 \\ -(k-1)^6 & 34k^3 - 51k^2 + 27k - 5 \end{pmatrix} \times \begin{pmatrix} 12(k+1)kc(k) & -5(k+1)c(k) \\ 0 & -(k-2)^3 c(k-1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(k-2)^3 c(k-1) \\ -(k-1)^6 12(k+1)kc(k) & (k-1)^6 5(k+1)c(k) - b^{\vee}_k (k-2)^6 c(k-1). \end{pmatrix}. \end{aligned} \tag{2.37}$$

In view of (2.30) and (2.33),

$$
\begin{aligned}
H_k B_k^{\wedge} &= \begin{pmatrix} 12(k+2)(k+1)c(k+1) & -5(k+2)c(k+1) \\ 0 & -(k-1)^3 c(k) \end{pmatrix} \\
&\quad \times \begin{pmatrix} 5k(k-1)^3 & k\left(12k^2-15k+5\right) \\ 12k(k+1)(k-1)^3 & k(k+1)\left(29k^2-36k+12\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & (k+2)c(k+1)k(k+1)\left(-k^2\right) \\ -c(k)12k(k+1)(k-1)^6 & -(k-1)^3 c(k)k(k+1)\left(29k^2-36k+12\right) \end{pmatrix}.
\end{aligned} \tag{2.38}
$$

Since

$$
\begin{aligned}
&-(k+2)(k+1)c(k+1)k^3 = -c(k-1)(k-2)^3, \\
&-(k-1)^3 c(k)k(k+1)\left(29k^2-36k+12\right) - (k-1)^6 5(k+1)c(k) \\
&\quad = -\left(34k^3-51k^2+27k-5\right)(k-1)^3(k+1)c(k) \\
&\quad = -\left(34k^3-51k^2+27k-5\right)(k-2)^3 c(k-1),
\end{aligned} \tag{2.39}
$$

it follows from (2.35), (2.37), and (2.38) that

$$
A_{k-1}^{\vee} H_{k-1} = H_k B_k^{\wedge} \tag{2.40}
$$

for $k \in [3, +\infty) \cap \mathbb{Z}$. We prove by induction now the following equality:

$$
U_k^{\vee} = H_k U_{4k-2}^{\wedge}, \tag{2.41}
$$

for any $k \in \mathbb{N}$. In view of (2.25) and (2.32), the equality (2.41) holds for $k = 1$. In view of (2.26) and (2.33), the equality (2.41) hold for $k = 2$. Let $k \in [3, +\infty) \cap \mathbb{Z}$ and (2.41) holds for $k - 1$. Then, in view of (2.29), (2.40), and (2.21),

$$
H_k U_{4k-2}^{\wedge} = H_k B_k U_{4k-6}^{\wedge} = A_{k-1}^{\vee} H_{k-1} U_{4k-6}^{\wedge} = A_{k-1}^{\vee} U_{k-1}^{\vee} = U_k^{\vee}. \tag{2.42}
$$

So, the equality (2.41) holds for any $k \in \mathbb{N}$. In view of (2.41),

$$
P_k^{\vee} = (2(k+1)!)^{-1} P_{4k-2}^{\wedge}, \qquad Q_k^{\vee} = (2(k+1)!)^{-1} Q_{4k-2}^{\wedge} \tag{2.43}
$$

for $k \in [2, +\infty) \cap \mathbb{Z}$. Since

$$
P_{\nu}^{\vee} = (\nu!)^3 v_{\nu}, \quad Q_{\nu}^{\vee} = (\nu!)^3 u_{\nu} \tag{2.44}
$$

for v_ν and u_ν in (1.6) and $\nu \in \mathbb{N}_0$, it follows from (2.43) and (2.44), that

$$P^{\wedge}_{4k-2} = 2(k+1)(k!)^4 v_k, \qquad Q^{\wedge}_{4k-2} = 2(k+1)(k!)^4 u_k. \tag{2.45}$$

As it is well known, for any $\varepsilon > 0$ there exist $C_1(\varepsilon) > 0$ and $C_2(\varepsilon) > 0$ such that

$$C_1(\varepsilon)\left(1+\sqrt{2}\right)^{4k(1-\varepsilon)} < |u_k| < C_2(\varepsilon)\left(1+\sqrt{2}\right)^{4k(1+\varepsilon)}, \tag{2.46}$$

$$C_1(\varepsilon)\left(1+\sqrt{2}\right)^{4k(1-\varepsilon)} < |v_k| < C_2(\varepsilon)\left(1+\sqrt{2}\right)^{4k(1+\varepsilon)}, \tag{2.47}$$

$$\frac{C_1(\varepsilon)}{\left(1+\sqrt{2}\right)^{8k(1+\varepsilon)}} < \left|\zeta(3) - \frac{v_k}{u_k}\right| < \frac{C_2(\varepsilon)}{\left(1+\sqrt{2}\right)^{8k(1-\varepsilon)}}. \tag{2.48}$$

We apply (2.23) now. Let $k \in [2,+\infty)\cap\mathbb{Z}$. In view of (2.2), (1.12)–(1.13), and (2.45), if $\eta = 1,2,3$, then

$$\begin{aligned} 0 &\le \prod_{\kappa=1}^{4k-2+\eta} a_\kappa \le \prod_{\kappa=1}^{4k+1} a_\kappa \le a_{4k-1}a_{4k}a_{4k+1} \times k^3(k+1)^3 \prod_{\kappa=1}^{4k-2} a_\kappa \\ &= 4k^3(k+1)^3 \prod_{\kappa=2}^{k} a_{4\kappa-5}a_{4\kappa-4}a_{4\kappa-3}a_{4\kappa-2} \\ &= 4k^3(k+1)^3 \prod_{\kappa=2}^{k} (\kappa-1)^2\kappa^2(\kappa-1)\kappa\kappa(\kappa+1) = 2(k!)^8(k+1)^4, \end{aligned} \tag{2.49}$$

$$4(k+1)^2(k!)^8 u_k^2 = (Q_{4k-2})^2 < Q_{4k-3+\eta}Q_{4k-2+\eta}. \tag{2.50}$$

In view of (2.23), (2.50), and (2.49), if $\theta = 1,2,3$

$$\begin{aligned} \left|\frac{P_{4k-2}}{Q_{4k-2}} - \frac{P_{4k-2+\theta}}{Q_{4k-1+\theta}}\right| &\le \sum_{\eta=1}^{\theta}\left|\frac{P_{4k-3+\eta}}{Q_{4k-3+\eta}} - \frac{P_{4k-2+\eta}}{Q_{4k-2+\eta}}\right| \\ &\le \sum_{\eta=1}^{3}\left|\frac{P_{4k-3+\eta}}{Q_{4k-3+\eta}} - \frac{P_{4k-2+\eta}}{Q_{4k-2+\eta}}\right| \le 3\frac{(k+1)^2}{2u_k^2} \le \left(1+\sqrt{2}\right)^{8k(-1+o(1))}, \end{aligned} \tag{2.51}$$

when $k \to +\infty$. In view of (2.45), (2.48), and (2.51), there exist $C_3(\varepsilon) > 0$ and $C_4(\varepsilon) > 0$ such that

$$\frac{C_3(\varepsilon)}{\left(1+\sqrt{2}\right)^{8k(1+\varepsilon)}} < \left|\zeta(3) - \frac{P^{\wedge}_{4k-2+\theta}}{Q^{\wedge}_{4k-2}}\right| < \frac{C_4(\varepsilon)}{\left(1+\sqrt{2}\right)^{8k(1-\varepsilon)}}, \tag{2.52}$$

where $\theta = 0, 1, 2, 3$. So, the equality (2.1) is proved. In view of (2.23),

$$\zeta(3) - \frac{P^{\wedge}_0}{Q^{\wedge}_0} = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}d_\nu, \tag{2.53}$$

where

$$0 < d_\nu = \frac{\prod_{k=1}^{\nu} a^{\wedge}_k}{\left(Q^{\wedge}_\nu Q_{\nu-1}\right)}^{\wedge}. \tag{2.54}$$

Further, we have

$$\frac{d_{\nu+1}}{d_\nu} = \frac{a_{\nu+1} \wedge Q_{\nu-1} \wedge}{b^{\wedge}_{\nu+1} Q^{\wedge}_\nu + a^{\wedge}_{\nu+1} Q^{\wedge}_{\nu-1}} < 1. \tag{2.55}$$

Hence, the series (2.53) is the series of Leibnitz type. Therefore, $P^{\wedge}_{2k-1}/Q^{\wedge}_{2k-1}$ decreases, when k increases in $\mathbb{N}$, and $P^{\wedge}_{2k}/Q^{\wedge}_{2k}$ increases, when k increases in $\mathbb{N}$.

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