Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 450264, 24 pages
doi:10.1155/2010/450264

Research Article

Oscillatory Behavior of Quasilinear Neutral Delay Dynamic Equations on Time Scales

Zhenlai Han,[1,2] Shurong Sun,[1,3] Tongxing Li,[1] and Chenghui Zhang[2]

[1] *School of Science, University of Jinan, Jinan, Shandong 250022, China*
[2] *School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China*
[3] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA*

Correspondence should be addressed to Zhenlai Han, hanzhenlai@163.com

Received 6 December 2009; Accepted 4 March 2010

Academic Editor: Leonid Berezansky

By means of the averaging technique and the generalized Riccati transformation technique, we establish some oscillation criteria for the second-order quasilinear neutral delay dynamic equations $[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)]^{\Delta} + q_1(t)|y(\delta_1(t))|^{\alpha-1}y(\delta_1(t)) + q_2(t)|y(\delta_2(t))|^{\beta-1}y(\delta_2(t)) = 0$, $t \in [t_0,\infty)_{\mathbb{T}}$, where $x(t) = y(t) + p(t)y(\tau(t))$, and the time scale interval is $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. Our results in this paper not only extend the results given by Agarwal et al. (2005) but also unify the oscillation of the second-order neutral delay differential equations and the second-order neutral delay difference equations.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis (see Hilger [1]). Several authors have expounded on various aspects of this new theory and references cited therein. A book on the subject of time scale, by Bohner and Peterson [2], summarizes and organizes much of the time scale calculus; we refer also the last book by Bohner and Peterson [3] for advances in dynamic equations on time scales.

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist (see Bohner and Peterson [2]).

In the last few years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales which attempts to harmonize the oscillation theory for the continuous and the discrete to include them in one comprehensive theory and to eliminate obscurity form both, for instance, the papers [4–20] and the reference cited therein.

For oscillation of delay dynamic equations on time scales, see recently papers [21–32]. However, there are very few results dealing with the oscillation of the solutions of neutral delay dynamic equations on time scales; we refer the reader to [33–44].

Agarwal et al. [33] and Saker [37] consider the second-order nonlinear neutral delay dynamic equations on time scales:

$$\left(r(t)\left((x(t)+p(t)x(t-\tau))^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,x(t-\delta))=0 \quad \text{for } t\in\mathbb{T}, \tag{1.1}$$

where $0 \le p(t) < 1$, $\gamma \ge 0$ is a quotient of odd positive integer, τ, δ are positive constants, $r, p \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$, $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty$, $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ such that $uf(t,u) > 0$ for all nonzero u, and there exists a nonnegative function $q(t)$ defined on $\mathbb{T}$ satisfing $|f(t,u)| \ge q(t)|u^{\gamma}|$.

Agwo [35] examines the oscillation of the second-order nonlinear neutral delay dynamic equations:

$$(x(t)-rx(\tau(t)))^{\Delta\Delta}+H\left(t,x(h_1(t)),x^{\Delta}(h_2(t))\right)=0 \quad \text{for } t\in\mathbb{T}. \tag{1.2}$$

Li et al. [36] discuss the existence of nonoscillatory solutions to the second-order neutral delay dynamic equation of the form

$$\left[x(t)+p(t)x(\tau_0(t))\right]^{\Delta\Delta}+q_1(t)x(\tau_1(t))-q_2(t)x(\tau_2(t))=e(t) \quad \text{for } t\in\mathbb{T}. \tag{1.3}$$

Saker et al. [38, 39], Sahíner [40], and Wu et al. [43] consider the second-order neutral delay and mixed-type dynamic equations on time scales:

$$\left(r(t)\left((x(t)+p(t)x(\tau(t)))^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,x(\delta(t)))=0 \quad \text{for } t\in\mathbb{T}, \tag{1.4}$$

where $0 \le p(t) < 1$, γ is a quotient of odd positive integer, $r, p \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$, $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty$, $\tau, \delta \in C_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t) \le t$, $\lim_{t\to\infty}\tau(t) = \infty$, $\lim_{t\to\infty}\delta(t) = \infty$, $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ such that $uf(t,u) > 0$ for all nonzero u, and there exists a nonnegative function q defined on $\mathbb{T}$ satisfing $|f(t,u)| \ge q(t)|u^{\gamma}|$.

Zhu and Wang [44] study existence of nonoscillatory solutions to neutral dynamic equations on time scales:

$$\left[x(t)+p(t)x(g(t))\right]^{\Delta}+f(t,x(h(t)))=0 \quad \text{for } t\in\mathbb{T}. \tag{1.5}$$

Recently, Tripathy [42] has established some new oscillation criteria for second-order nonlinear delay dynamic equations of the form

$$\left(r(t)\left(\left(x(t)+p(t)x(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+q(t)|x(t-\delta)|^{\gamma}\operatorname{sgn}x(t-\delta)=0,\quad \text{for } t\in\mathbb{T}, \tag{1.6}$$

where $0 \le p(t)$, $\gamma \ge 0$ is a quotient of odd positive integer, τ, δ are positive constants, $r, p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$, and $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty$.

To the best of our knowledge, there are no results regarding the oscillation of the solutions of the following second-order nonlinear neutral delay dynamic equations on time scales up to now:

$$\begin{aligned}\left[r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} &+ q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t))\\ &+ q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t)) = 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},\end{aligned} \tag{1.7}$$

where $x(t) = y(t) + p(t)y(\tau(t))$, and the time scale interval is $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.

In what follows we assume the following:

(A1) α, β, and γ are positive constants with $0 < \alpha < \gamma < \beta$;

(A2) $r, q_1, q_2 \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^{+}), \mathbb{R}^{+} = (0, \infty), \int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty, r^{\Delta}(t) \ge 0$;

(A3) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and $-1 < p_0 \le p(t) < 1, p_0$ constant;

(A4) $\tau, \delta_1, \delta_2 \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T}), \tau(t) \le t, \delta_1(t) \le t, \delta_2(t) \le t$, for $t \in [t_0, \infty)_{\mathbb{T}}$, and $\lim_{t\to\infty}\tau(t) = \infty, \lim_{t\to\infty}\delta_1(t) = \infty, \lim_{t\to\infty}\delta_2(t) = \infty$.

To develop the qualitative theory of delay dynamic equations on time scales, in this paper, by using the averaging technique and the generalized Riccati transformation, we consider the second-order nonlinear neutral delay dynamic equation on time scales (1.7) and establish several oscillation criteria. Our results in this paper not only extend the results given but also unify the oscillation of the second-order quasilinear delay differential equation and the second-order quasilinear delay difference equation. Applications to equations to which previously known criteria for oscillation are not applicable are given.

By a solution of (1.7), we mean a nontrivial real-valued function $y \in C^1_{rd}[t_y, \infty)_{\mathbb{T}}$, $t_y \in [t_0, \infty)_{\mathbb{T}}$, which has the property $r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t) \in C^1_{rd}[t_y, \infty)_{\mathbb{T}}$ and satisfying (1.7) for $t \in [t_y, \infty)_{\mathbb{T}}$. Our attention is restricted to those solutions y of (1.7) which exist on some half line $[t_y, \infty)_{\mathbb{T}}$ with $\sup\{|y(t)| : t \ge t_1\} > 0$ for any $t_1 \in [t_y, \infty)_{\mathbb{T}}$. A solution y of (1.7) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.7) is called oscillatory if all solutions are oscillatory.

Equation (1.7) includes many other special important equations; for example, if $q_1(t) \equiv 0$, (1.7) is the prototype of a wide class of nonlinear dynamic equations called Emden-Fowler neutral delay superlinear dynamic equation:

$$\left[r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta}+q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.8}$$

where $x(t) = y(t) + p(t)y(\tau(t))$.

If $q_2(t) \equiv 0$, (1.7) is the prototype of nonlinear dynamic equations called Emden-Fowler neutral delay sublinear dynamic equation:

$$\left[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} + q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.9}$$

where $x(t) = y(t) + p(t)y(\tau(t))$.

We note that if $\gamma = 1$, then (1.7) becomes second-order nonlinear delay dynamic equation on time scales:

$$\begin{aligned}\left[r(t)\left(y(t) + p(t)y(\tau(t))\right)^{\Delta}\right]^{\Delta} &+ q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) \\ &+ q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.\end{aligned} \tag{1.10}$$

If $p(t) \equiv 0$, then (1.7) becomes second-order nonlinear delay dynamic equation on time scales:

$$\begin{aligned}\left[r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\right]^{\Delta} &+ q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) \\ &+ q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.\end{aligned} \tag{1.11}$$

If $\gamma = 1, p(t) \equiv 0$, then (1.7) becomes second-order nonlinear delay dynamic equation on time scales:

$$\begin{aligned}\left[r(t)y^{\Delta}(t)\right]^{\Delta} + q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) + q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t)) = 0 \\ \text{for } t \in [t_0,\infty)_{\mathbb{T}}.\end{aligned} \tag{1.12}$$

If $\gamma = 1, r(t) \equiv 1, p(t) \equiv 0$, then (1.7) becomes second-order nonlinear delay dynamic equation on time scales:

$$\begin{aligned}y^{\Delta\Delta}(t) + q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) + q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t)) = 0 \\ \text{for } t \in [t_0,\infty)_{\mathbb{T}}.\end{aligned} \tag{1.13}$$

It is interesting to study (1.7) because the continuous version and its special cases have several physical applications, see [1] and when t is a discrete variable, and include its special cases also, are important in applications.

The paper is organized as follows: In the next section we present the basic definitions and apply a simple consequence of Keller's chain rule, Young's inequality:

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad a,b \in \mathbb{R},\ p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{1.14}$$

and the inequality

$$\lambda A B^{\lambda-1} - A^{\lambda} \le (\lambda - 1) B^{\lambda}, \quad \lambda \ge 1, \tag{1.15}$$

where A and B are nonnegative constants, devoted to the proof of the sufficient conditions for oscillation of all solutions of (1.7). In Section 3, we present some corollaries to illustrate our main results.

2. Main Results

In this section we shall give some oscillation criteria for (1.7) under the cases when $0 \le p(t) < 1$ and $-1 < p_0 \le p(t) < 0$. It will be convenient to make the following notations in the remainder of this paper. Define

$$\theta = \min\left\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\alpha}\right\}, \qquad \delta(t) = \min_{t\ge t_0}\{\delta_1(t), \delta_2(t)\}, \qquad \overline{\gamma} = \begin{cases} \gamma^2, & \gamma \ge 1, \\ \gamma, & 0 < \gamma < 1, \end{cases}$$

$$Q_1(t) = \theta\left(q_1(t)\left(1 - p(\delta_1(t))\right)^{\alpha}\right)^{\beta-\gamma/\beta-\alpha}\left(q_2(t)\left(1 - p(\delta_2(t))\right)^{\beta}\right)^{\gamma-\alpha/\beta-\alpha}\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma},$$

$$Q_2(t) = \theta\left(\left(q_1(t)\right)^{\alpha}\right)^{\beta-\gamma/\beta-\alpha}\left(\left(q_2(t)\right)^{\beta}\right)^{\gamma-\alpha/\beta-\alpha}\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma}. \tag{2.1}$$

We define the function space $\mathfrak{R}$ as follows: $H \in \mathfrak{R}$ provided that H is defined for $t_0 \le s \le \sigma(t), t, s \in [t_0, \infty)_{\mathbb{T}}, H(t,s) \ge 0, H(\sigma(t), t) = 0, H^{\Delta_s}(\sigma(t), s) = -h(t,s)H(\sigma(t), \sigma(s))$, and $h(t,s)$ is rd-continuous function and nonnegative. For given function $\rho, \eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, we set

$$\lambda(t,s) = h(\sigma(t), s) - \frac{\rho^{\Delta}(s)}{\rho(s)},$$
$$\Theta_i(t,s) = \left[Q_i(s) - \eta^{\Delta}(s)\right]\frac{\rho(\sigma(s))}{\rho(s)} + \lambda(t,s)\eta(s), \quad i = 1, 2. \tag{2.2}$$

In order to prove our main results, we will use the formula

$$\left((x(t))^{\gamma}\right)^{\Delta} = \gamma x^{\Delta}(t)\int_0^1 \left[h x(\sigma(t)) + (1-h)x(t)\right]^{\gamma-1} \mathrm{d}h, \tag{2.3}$$

where $x(t)$ is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see Bohner and Peterson [2, Theorem 1.90]).

Also, we assume the condition (H) $-1 < p_0 \le p(t) < 0$ and $\lim_{t\to\infty} p(t) = p > -1$, and there exists $\{c_k\}_{k\ge 0}$ such that $\lim_{k\to\infty} c_k = \infty$ and $\tau(c_{k+1}) = c_k$.

Lemma 2.1. *Assume* $0 \le p(t) < 1$*. If* $y(t)$ *is an eventually positive solution of* (1.7)*, then, there exists a* $t_* \in [t_0, \infty)_{\mathbb{T}}$ *such that* $x(t) > 0, x^{\Delta}(t) \ge 0$ *for* $t \in [t_*, \infty)_{\mathbb{T}}$*. Moreover,*

$$\begin{aligned}\left[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} \le &-q_1(t)\big((1-p(\delta_1(t)))x(\delta_1(t))\big)^{\alpha} \\ &- q_2(t)\big((1-p(\delta_2(t)))x(\delta_2(t))\big)^{\beta} < 0, \quad t \in [t_*, \infty)_{\mathbb{T}}.\end{aligned} \tag{2.4}$$

Lemma 2.2. *Assume* $0 \le p(t) < 1$*,*

$$\int_{t_0}^{\infty} q_2(t)\left[(1-p(\delta_2(t)))\delta_2(t)\right]^{\beta}\Delta t = \infty. \tag{2.5}$$

If $y(t)$ *is an eventually positive solution of* (1.7)*, then*

$$x^{\Delta\Delta}(t) < 0, x(t) \ge t x^{\Delta}(t), \quad \frac{x(t)}{t} \text{ is strictly decreasing.} \tag{2.6}$$

The proof of Lemmas 2.1 and 2.2 is similar to that of Saker et al. [39, Lemma 2.1]; so it is omitted.

Lemma 2.3. *Assume that the condition* (H) *holds:*

$$\int_{t_0}^{\infty} q_2(t)(\delta_2(t))^{\beta}\Delta t = \infty. \tag{2.7}$$

If $y(t)$ *is an eventually positive solution of* (1.7)*, then*

$$\left[r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} < -q_1(t)(x(\delta_1(t)))^{\alpha} - q_2(t)(x(\delta_2(t)))^{\beta} < 0, \quad t \in [t_*, \infty)_{\mathbb{T}}, \tag{2.8}$$

$$x^{\Delta\Delta}(t) < 0, x(t) \ge t x^{\Delta}(t), \quad \frac{x(t)}{t} \text{ is strictly decreasing,} \tag{2.9}$$

or $\lim_{t\to\infty} y(t) = 0$.

Proof. Since $y(t)$ is an eventually positive solution of (1.7), there exists a number $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_1(t)) > 0$ and $y(\delta_2(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. In view of (1.7), we have

$$\left[r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} = -q_1(t)\big(y(\delta_1(t))\big)^{\alpha} - q_2(t)\big(y(\delta_2(t))\big)^{\beta}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.10}$$

By $-1 < p_0 \le p(t) < 0$, then $y(t) > x(t)$, we get

$$\left[r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} < -q_1(t)(x(\delta_1(t)))^{\alpha} - q_2(t)(x(\delta_2(t)))^{\beta} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.11}$$

Let $t_* \geq t_1$, then (2.8) holds, and $r(t)|x^{\Delta}(t)|^{\gamma-1}\, x^{\Delta}(t)$ is an eventually decreasing function. It follows that

$$\lim_{t\to\infty} r(t)\left|x^{\Delta}(t)\right|^{\gamma-1} x^{\Delta}(t) = l \geq -\infty. \tag{2.12}$$

We prove that $r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)$ is eventually positive.

Otherwise, there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $r(t_2)|x^{\Delta}(t_2)|^{\gamma-1}x^{\Delta}(t_2) = c < 0$, then we have $r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t) \leq r(t_2)|x^{\Delta}(t_2)|^{\gamma-1}x^{\Delta}(t_2) = c$ for $t \in [t_2,\infty)_{\mathbb{T}}$, and hence $x^{\Delta}(t) \leq -(-c)^{1/\gamma}(1/r(t))^{1/\gamma}$, which implies that

$$x(t) \leq x(t_2) - (-c)^{1/\gamma}\int_{t_2}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty. \tag{2.13}$$

Therefore, there exists $d > 0$, and $t_3 \geq t_2$ such that

$$y(t) \leq -d - p(t)y(\tau(t)) \leq -d + p_0 y(\tau(t)), \quad t \geq t_3. \tag{2.14}$$

We can choose some positive integer k_0 such that $c_k \geq t_3$, for $k \geq k_0$. Thus, we obtain

$$\begin{aligned} y(c_k) &\leq -d + p_0 y(\tau(c_k)) = -d + p_0 y(c_{k-1}) \leq -d - r_0 d + p_0^2 y(\tau(c_{k-1})) \\ &= -d - p_0 d + p_0^2 x(c_{k-2}) \leq \cdots \leq -d - p_0 d - \cdots - p_0^{k-k_0-1}d + p_0^{k-k_0}y(\tau(c_{k_0+1})) \\ &= -d - p_0 d - \cdots - p_0^{k-k_0-1}d + p_0^{k-k_0}y(c_{k_0}). \end{aligned} \tag{2.15}$$

The above inequality implies that $y(c_k) < 0$ for sufficiently large k, which contradicts the fact that $y(t) > 0$ eventually. Hence $x^{\Delta}(t) > 0$ eventually. Consequently, there are two possible cases:

(i) $x(t) > 0$, eventually;

(ii) $x(t) < 0$, eventually.

If Case (i) holds, we can get

$$z(t) \geq tz^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) < 0, \quad \frac{z(t)}{t} \text{ is nonincreasing.} \tag{2.16}$$

Actually, by $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} = r^{\Delta}(t)(x^{\Delta}(t))^{\gamma} + r(\sigma(t))((x^{\Delta}(t))^{\gamma})^{\Delta} < 0, r^{\Delta}(t) \geq 0$, we can easily verify that $((x^{\Delta}(t))^{\gamma})^{\Delta} < 0$. Using (2.3), we get

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} = \gamma x^{\Delta\Delta}(t)\int_0^1\left[h\left(x^{\Delta}\right)^{\sigma} + (1-h)x^{\Delta}\right]^{\gamma-1}\mathrm{d}h. \tag{2.17}$$

From $\int_0^1[h(x^{\Delta})^{\sigma} + (1-h)x^{\Delta}]^{\gamma-1}\mathrm{d}h > 0$, we have that $x^{\Delta\Delta}(t)$ is eventually negative.

Let $X(t) := x(t) - tx^{\Delta}(t)$, since $X^{\Delta}(t) = x^{\Delta}(t) - (x^{\Delta}(t) + \sigma(t)x^{\Delta\Delta}(t)) = -\sigma(t)x^{\Delta\Delta}(t)$ is eventually positive, so $X(t)$ is eventually increasing. Therefore, $X(t)$ is either eventually positive or eventually negative. If $X(t)$ is eventually negative, then there is a $t_3 \in [t_*, \infty)_{\mathbb{T}}$ such that $X(t) < 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. So,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} > 0, \quad t \in [t_3, \infty)_{\mathbb{T}}, \tag{2.18}$$

which implies that $x(t)/t$ is strictly increasing for $t \in [t_3, \infty)_{\mathbb{T}}$. Pick $t_4 \in [t_3, \infty)_{\mathbb{T}}$ so that $\delta_i(t) \geq \delta_i(t_*)$ for $t \in [t_4, \infty)_{\mathbb{T}}, i = 1, 2$. Then

$$\frac{x(\delta_i(t))}{\delta_i(t)} \geq \frac{x(\delta_i(t_*))}{\delta_i(t_*)} := d_i > 0, \tag{2.19}$$

so that $x(\delta_i(t)) \geq d_i\delta_i(t)$ for $t \in [t_4, \infty)_{\mathbb{T}}$.

By (2.8), we have

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} - r(t_4)\left(x^{\Delta}(t_4)\right)^{\gamma} < -\int_{t_4}^{t}\left[q_1(s)x(\delta_1(s))^{\alpha} + q_2(s)x(\delta_2(s))^{\beta}\right]\Delta s, \tag{2.20}$$

which implies that

$$\begin{aligned} r(t_4)\left(x^{\Delta}(t_4)\right)^{\gamma} &> r(t)\left(x^{\Delta}(t)\right)^{\gamma} + \int_{t_4}^{t}\left[q_1(s)x(\delta_1(s))^{\alpha} + q_2(s)x(\delta_2(s))^{\beta}\right]\Delta s \\ &\geq {d_1}^{\alpha}\int_{t_4}^{t} q_1(s)(\delta_1(s))^{\alpha}\Delta s + {d_2}^{\beta}\int_{t_4}^{t} q_2(s)(\delta_2(s))^{\beta}\Delta s, \end{aligned} \tag{2.21}$$

which contradicts (2.7). Hence, without loss of generality, there is a $t_* \geq t_0$ such that $X(t) > 0$, that is, $x(t) > tx^{\Delta}(t)$ for $t \geq t_*$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} < 0, \quad t \geq t_*, \tag{2.22}$$

and we have that $x(t)/t$ is strictly decreasing for $t \geq t_*$.

If there exists a $t_4 \geq t_1$ such that Case (ii) holds, then $\lim_{t\to\infty}x(t)$ exists, $\lim_{t\to\infty}x(t) = l \leq 0$, and we claim that $\lim_{t\to\infty}x(t) = 0$. Otherwise, $\lim_{t\to\infty}x(t) < 0$. We can choose some positive integer k_0 such that $c_k \geq t_4$, for $k \geq k_0$. Thus, we obtain

$$\begin{aligned} y(c_k) &\leq p_0 y(\tau(c_k)) = p_0 y(c_{k-1}) \leq p_0^2 y(\tau(c_{k-1})) \\ &= p_0^2 y(c_{k-2}) \leq \cdots \leq p_0^{k-k_0} y(\tau(c_{k_0+1})) = p_0^{k-k_0} y(c_{k_0}), \end{aligned} \tag{2.23}$$

which implies that $\lim_{k\to\infty}y(c_k) = 0$, and $\lim_{k\to\infty}x(c_k) = 0$, which contradicts $\lim_{t\to\infty}x(t) < 0$.

Now, we assert that $y(t)$ is bounded. If it is not true, there exists $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that

$$y(t_k) = \sup_{t_0 \le s \le t_k} y(s), \qquad \lim_{k\to\infty} y(t_k) = \infty. \tag{2.24}$$

From $\tau(t) \le t$ and

$$x(t_k) = y(t_k) + p(t_k)y(\tau(t_k)) \ge (1-p_0)y(t_k), \tag{2.25}$$

which implies that $\lim_{k\to\infty}x(t_k) = \infty$, it contradicts the existence of $\lim_{t\to\infty}x(t)$. Therefore, we can assume that

$$\limsup_{t\to\infty} y(t) = y_1, \qquad \liminf_{t\to\infty} y(t) = y_2. \tag{2.26}$$

By $-1 < p < 0$, we get

$$y_1 + py_1 \le 0 \le y_2 + py_2, \tag{2.27}$$

thus $y_1 \le y_2$, and $y_1 = y_2$. Hence, $\lim_{t\to\infty}y(t) = 0$. The proof is complete. □

Theorem 2.4. *Assume that* (2.5) *holds,* $0 \le p(t) < 1$,

$$\limsup_{t\to\infty}\left[t\left(\frac{1}{r(t)}\int_t^\infty q_2(s)(1-p(\delta_2(s)))^\beta\left(\frac{\delta_2(s)}{s}\right)^\beta \Delta s\right)^{1/\gamma}\right] = \infty. \tag{2.28}$$

Then (1.7) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. We shall consider only this case, since the proof when $y(t)$ is eventually negative is similar. In view of Lemmas 2.1 and 2.2, there exists a $t_* \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x^\Delta(t) \ge 0, x^{\Delta\Delta}(t) < 0, x(t) \ge tx^\Delta(t)$, and $x(t)/t$ is strictly decreasing for $t \in [t_*,\infty)_{\mathbb{T}}$.

From (2.4) we have for $T \ge t, T, t \in [t_*,\infty)_{\mathbb{T}}$,

$$\begin{aligned}\int_t^T q_2(s)(1-p(\delta_2(s)))^\beta (x(\delta_2(s)))^\beta \Delta s &\le -\int_t^T \left[r(s)\left(x^\Delta(s)\right)^\gamma\right]^\Delta \Delta s \\ &= r(t)\left(x^\Delta(t)\right)^\gamma - r(T)\left(x^\Delta(T)\right)^\gamma \\ &\le r(t)\left(x^\Delta(t)\right)^\gamma,\end{aligned} \tag{2.29}$$

and hence

$$\left(\frac{1}{r(t)}\int_t^\infty q_2(s)(1-p(\delta_2(s)))^\beta (x(\delta_2(s)))^\beta \Delta s\right)^{1/\gamma} \le x^\Delta(t). \tag{2.30}$$

So,

$$x(t) \geq tx^{\Delta}(t) \geq t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)(1-p(\delta_2(s)))^{\beta}(x(\delta_2(s)))^{\beta}\Delta s\right)^{1/\gamma}$$
$$\geq t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)(1-p(\delta_2(s)))^{\beta}\left(\frac{\delta_2(s)}{s}x(s)\right)^{\beta}\Delta s\right)^{1/\gamma} \tag{2.31}$$
$$\geq x^{\beta/\gamma}(t)t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)(1-p(\delta_2(s)))^{\beta}\left(\frac{\delta_2(s)}{s}\right)^{\beta}\Delta s\right)^{1/\gamma}.$$

So,

$$t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)(1-p(\delta_2(s)))^{\beta}\left(\frac{\delta_2(s)}{s}\right)^{\beta}\Delta s\right)^{1/\gamma} \leq \left(\frac{1}{x(t)}\right)^{(\beta/\gamma)-1}. \tag{2.32}$$

Now note that $\beta/\gamma > 1$ imply

$$t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)(1-p(\delta_2(s)))^{\beta}\left(\frac{\delta_2(s)}{s}\right)^{\beta}\Delta s\right)^{1/\gamma} \leq \left(\frac{1}{x(t_*)}\right)^{(\beta/\gamma)-1}. \tag{2.33}$$

This contradicts (2.28). The proof is complete. □

Remark 2.5. Theorem 2.4 includes results of Agarwal et al. [21, Theorem 4.4] and Han et al. [25, Theorem 3.1].

Theorem 2.6. *Assume that the condition* (H) *and* (2.7) *hold:*

$$\limsup_{t\to\infty}\left[t\left(\frac{1}{r(t)}\int_t^{\infty} q_2(s)\left(\frac{\delta_2(s)}{s}\right)^{\beta}\Delta s\right)^{1/\gamma}\right] = \infty. \tag{2.34}$$

Then every solution of (1.7) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. In view of Lemma 2.3, either $\lim_{t\to\infty} y(t) = 0$ or there exists a $t_* \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0,\ x^{\Delta}(t) \geq 0,\ x^{\Delta\Delta}(t) < 0,\ x(t) \geq tx^{\Delta}(t)$, and $x(t)/t$ is strictly decreasing for $t \in [t_*,\infty)_{\mathbb{T}}$.

Then from (2.8), we have for $T \geq t, T, t \in [t_*,\infty)_{\mathbb{T}}$,

$$\int_t^{T} q_2(s)(x(\delta_2(s)))^{\beta}\Delta s \leq -\int_t^{T}\left[r(s)\left(x^{\Delta}(s)\right)^{\gamma}\right]^{\Delta}\Delta s$$
$$= r(t)\left(x^{\Delta}(t)\right)^{\gamma} - r(T)\left(x^{\Delta}(T)\right)^{\gamma} \leq r(t)\left(x^{\Delta}(t)\right)^{\gamma}, \tag{2.35}$$

Since the rest of the proof is similar to Theorem 2.4, so we omit the detail. The proof is complete. □

Theorem 2.7. *Assume that* (2.5) *holds.* $0 \le p(t) < 1$, *let* $\rho, \eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(\sigma(s))\left(Q_1(s)-\eta^{\Delta}\right)-\rho^{\Delta}\eta-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{\left((\rho^{\Delta}(s))_+\right)^{\gamma+1}}{(\rho(\sigma(s)))^{\gamma}}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}\right]\Delta s=\infty, \tag{2.36}$$

then (1.7) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$, *where* $(\rho^{\Delta}(s))_+ = \max\{\rho^{\Delta}(s), 0\}$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. In view of Lemmas 2.1 and 2.2, there exists a $t_* \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x^{\Delta}(t) \ge 0, x^{\Delta\Delta}(t) < 0, x(t) \ge t x^{\Delta}(t)$, and $x(t)/t$ is strictlydecreasing for $t \in [t_*,\infty)_{\mathbb{T}}$. Define the function $\omega(t)$ by

$$\omega(t)=\rho(t)\left[\frac{r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)}+\eta(t)\right],\quad t\in[t_*,\infty)_{\mathbb{T}}. \tag{2.37}$$

We get

$$\omega^{\Delta}(t)=\frac{\rho^{\Delta}(t)}{\rho(t)}\omega(t)+\rho(\sigma(t))\left[\frac{\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta}}{(x(\sigma(t)))^{\gamma}}-\frac{r(t)(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x(\sigma(t)))^{\gamma}}+\eta^{\Delta}(t)\right]. \tag{2.38}$$

If $\gamma \ge 1$, by (2.3), we get

$$\left((x(t))^{\gamma}\right)^{\Delta}=\gamma x^{\Delta}(t)\int_0^1[hx(\sigma(t))+(1-h)x(t)]^{\gamma-1}\mathrm{d}h\ge\gamma(x(t))^{\gamma-1}x^{\Delta}(t), \tag{2.39}$$

So, from (2.4) we have

$$\begin{aligned}\omega^{\Delta}(t)\le{}&\frac{\rho^{\Delta}(t)}{\rho(t)}\omega(t)-\rho(\sigma(t))q_1(t)(1-p(\delta_1(t)))^{\alpha}\frac{(x(\delta_1(t)))^{\alpha}}{(x(\sigma(t)))^{\gamma}}\\&-\rho(\sigma(t))q_2(t)(1-p(\delta_2(t)))^{\beta}\frac{(x(\delta_2(t)))^{\beta}}{(x(\sigma(t)))^{\gamma}}-\gamma\rho(\sigma(t))\frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x(t)(x(\sigma(t)))^{\gamma}}+\rho(\sigma(t))\eta^{\Delta}(t).\end{aligned} \tag{2.40}$$

By Young's inequality (1.14), we obtain that

$$
\begin{aligned}
&\frac{\beta-\gamma}{\beta-\alpha} q_1(t)(1-p(\delta_1(t)))^{\alpha}\frac{(x(\delta_1(t)))^{\alpha}}{(x(\sigma(t)))^{\gamma}} + \frac{\gamma-\alpha}{\beta-\alpha} q_2(t)(1-p(\delta_2(t)))^{\beta}\frac{(x(\delta_2(t)))^{\beta}}{(x(\sigma(t)))^{\gamma}}\\
&\geq \left[q_1(t)(1-p(\delta_1(t)))^{\alpha}\frac{(x(\delta_1(t)))^{\alpha}}{(x(\sigma(t)))^{\gamma}}\right]^{\beta-\gamma/\beta-\alpha}\left[q_2(t)(1-p(\delta_2(t)))^{\beta}\frac{(x(\delta_2(t)))^{\beta}}{(x(\sigma(t)))^{\gamma}}\right]^{\gamma-\alpha/\beta-\alpha}\\
&= \left(q_1(t)(1-p(\delta_1(t)))^{\alpha}\right)^{\beta-\gamma/\beta-\alpha}\left(q_2(t)(1-p(\delta_2(t)))^{\beta}\right)^{\gamma-\alpha/\beta-\alpha}\\
&\quad\times\left(\frac{(x(\delta_1(t)))^{\alpha}}{(x(\sigma(t)))^{\gamma}}\right)^{\beta-\gamma/\beta-\alpha}\left(\frac{(x(\delta_2(t)))^{\beta}}{(x(\sigma(t)))^{\gamma}}\right)^{\gamma-\alpha/\beta-\alpha}\\
&\geq \left(q_1(t)(1-p(\delta_1(t)))^{\alpha}\right)^{\beta-\gamma/\beta-\alpha}\left(q_2(t)(1-p(\delta_2(t)))^{\beta}\right)^{\gamma-\alpha/\beta-\alpha}\left(\frac{x(\delta(t))}{x(\sigma(t))}\right)^{\gamma}.
\end{aligned}
\tag{2.41}
$$

From $x(t)/t$ being strictly decreasing, $x(\delta(t))/x(\sigma(t)) \geq \delta(t)/\sigma(t), x(t)/x(\sigma(t)) \geq t/\sigma(t)$, by (2.37) and (2.40), we get that

$$
\begin{aligned}
\omega^{\Delta}(t) \leq{}& \frac{\rho^{\Delta}(t)}{\rho(t)}\omega(t) - \mu\rho(\sigma(t))\left(q_1(t)(1-p(\delta_1(t)))^{\alpha}\right)^{\beta-\gamma/\beta-\alpha}\left(q_2(t)(1-p(\delta_2(t)))^{\beta}\right)^{\gamma-\alpha/\beta-\alpha}\\
&\times\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma} - \gamma\rho(\sigma(t))\frac{1}{(r(t))^{1/\gamma}}\left(\frac{t}{\sigma(t)}\right)^{\gamma}\left(\frac{\omega(t)}{\rho(t)}-\eta(t)\right)^{\gamma+1/\gamma} + \rho(\sigma(t))\eta^{\Delta}(t),
\end{aligned}
\tag{2.42}
$$

that is,

$$
\omega^{\Delta}(t) \leq -\rho(\sigma(t))\left[Q_1(t)-\eta^{\Delta}(t)\right] + \frac{\rho^{\Delta}(t)}{\rho(t)}\omega(t) - \gamma\rho(\sigma(t))\frac{1}{(r(t))^{1/\gamma}}\left(\frac{t}{\sigma(t)}\right)^{\gamma}\left|\frac{\omega(t)}{\rho(t)}-\eta(t)\right|^{\gamma+1/\gamma}.
\tag{2.43}
$$

So,

$$
\begin{aligned}
\omega^{\Delta}(t) \leq{}& -\rho(\sigma(t))\left[Q_1(t)-\eta^{\Delta}(t)\right] + \rho^{\Delta}(t)\eta(t) + \left(\rho^{\Delta}(t)\right) + \left|\frac{\omega(t)}{\rho(t)}-\eta(t)\right|\\
&- \gamma\rho(\sigma(t))\frac{1}{(r(t))^{1/\gamma}}\left(\frac{t}{\sigma(t)}\right)^{\gamma}\left|\frac{\omega(t)}{\rho(t)}-\eta(t)\right|^{\gamma+1/\gamma}.
\end{aligned}
\tag{2.44}
$$

Using the inequality (1.15) we have

$$
\omega^{\Delta}(t) \leq -\rho(\sigma(t))\left[Q_1(t)-\eta^{\Delta}(t)\right] + \rho^{\Delta}(t)\eta(t) + \frac{r(t)}{(\gamma+1)^{\gamma+1}}\frac{\left(\left(\rho^{\Delta}(t)\right)_{+}\right)^{\gamma+1}}{(\rho(\sigma(t)))^{\gamma}}\left(\frac{\sigma(t)}{t}\right)^{\gamma^2}.
\tag{2.45}
$$

Integrating the inequality above from t_* to t we obtain

$$\omega(t)-\omega(t_*) \le -\int_{t_*}^{t}\left(\rho(\sigma(s))\left[Q_1(s)-\eta^{\Delta}(s)\right]-\rho^{\Delta}(s)\eta(s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{\left((\rho^{\Delta}(s))_+\right)^{\gamma+1}}{(\rho(\sigma(s)))^{\gamma}}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}\right)\Delta s. \tag{2.46}$$

Therefore,

$$\int_{t_*}^{t}\left(\rho(\sigma(s))\left[Q_1(s)-\eta^{\Delta}(s)\right]-\rho^{\Delta}(s)\eta(s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{\left((\rho^{\Delta}(s))_+\right)^{\gamma+1}}{(\rho(\sigma(s)))^{\gamma}}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}\right)\Delta s \le \omega(t_*)-\omega(t)\le\omega(t_*), \tag{2.47}$$

which contradicts (2.36).

If $0<\gamma<1$, proceeding as the proof of above, we have (2.37) and (2.38). By (2.3), we get that

$$\left((x(t))^{\gamma}\right)^{\Delta}=\gamma x^{\Delta}(t)\int_0^1\left[hx(\sigma(t))+(1-h)x(t)\right]^{\gamma-1}\mathrm{d}h\ge\gamma(x(\sigma(t)))^{\gamma-1}x^{\Delta}(t). \tag{2.48}$$

So, from (2.4) we have

$$\begin{aligned}\omega^{\Delta}(t)\le{}&\frac{\rho^{\Delta}(t)}{\rho(t)}\omega(t)-\rho(\sigma(t))q_1(t)(1-p(\delta_1(t)))^{\alpha}\frac{(x(\delta_1(t)))^{\alpha}}{(x(\sigma(t)))^{\gamma}}\\&-\rho(\sigma(t))q_2(t)(1-p(\delta_2(t)))^{\beta}\frac{(x(\delta_2(t)))^{\beta}}{(x(\sigma(t)))^{\gamma}}-\gamma\rho(\sigma(t))\frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x(\sigma(t))x^{\gamma}(t)}+\rho(\sigma(t))\eta^{\Delta}(t).\end{aligned} \tag{2.49}$$

Since the rest of the proof is similar to that of above, so we omit the detail. The proof is complete. □

Theorem 2.8. *Assume that the condition* (H) *and* (2.7) *hold, let* $\rho,\eta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(\sigma(s))\left(Q_2(s)-\eta^{\Delta}\right)-\rho^{\Delta}\eta-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\frac{\left((\rho^{\Delta}(s))_+\right)^{\gamma+1}}{(\rho(\sigma(s)))^{\gamma}}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}\right]\Delta s=\infty, \tag{2.50}$$

then every solution of (1.7) *either oscillates or tends to zero as* $t\to\infty$, *where* $(\rho^{\Delta}(s))_+=\max\{\rho^{\Delta}(s),0\}$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. In view of Lemma 2.3, either $\lim_{t\to\infty} y(t) = 0$ or there exists a $t_* \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x^{\Delta}(t) \geq 0, x^{\Delta\Delta}(t) < 0, x(t) \geq t x^{\Delta}(t)$, and $x(t)/t$ is strictly decreasing for $t \in [t_*, \infty)_{\mathbb{T}}$.

Since the rest of the proof is similar to Theorem 2.7, so we omit the detail. The proof is complete. □

Theorem 2.9. *Assume that* (2.5) *holds.* $0 \leq p(t) < 1$, *let* $\rho, \eta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), t_0)} \times \int_{t_0}^{t} H(\sigma(t), \sigma(s))\rho(s)\left(\Theta_1(t,s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}} |\lambda(t,s)|^{\gamma+1}\right)\Delta s = \infty, \tag{2.51}$$

then (1.7) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. We prove only case $\gamma \geq 1$. The proof of case $0 < \gamma < 1$ is similar. Proceeding as the proof of Theorem 2.7, we have (2.37) and (2.43). Replacing t in (2.43) by s, then multiplying (2.43) by $H(\sigma(t), \sigma(s))$, and integrating from T to t, $t > T, t, T \in [t_*, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_T^t H(\sigma(t), \sigma(s))\rho(\sigma(s))\left[Q_1(s) - \eta^{\Delta}(s)\right]\Delta s \\
&\leq -\int_T^t H(\sigma(t), \sigma(s))\omega^{\Delta}(s)\Delta s + \int_T^t H(\sigma(t), \sigma(s))\rho^{\Delta}(s)\frac{\omega(s)}{\rho(s)}\Delta s \\
&\quad - \gamma\int_T^t H(\sigma(t), \sigma(s))\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left(\frac{\omega(s)}{\rho(s)} - \eta(s)\right)^{\gamma+1/\gamma}\Delta s.
\end{aligned} \tag{2.52}$$

Integrating by parts and using the fact that $H(\sigma(t), t) = 0$, we get

$$\int_T^t H(\sigma(t), \sigma(s))\omega^{\Delta}(s)\Delta s = -H(\sigma(t), T)\omega(T) - \int_T^t H^{\Delta_s}(\sigma(t), s)\omega(s)\Delta s. \tag{2.53}$$

So,

$$\begin{aligned}
&\int_T^t H(\sigma(t), \sigma(s))\rho(\sigma(t))\left[Q_1(s) - \eta^{\Delta}(s)\right]\Delta s \leq H(\sigma(t), T)\omega(T) \\
&\quad - \int_T^t h(\sigma(t), s)H(\sigma(t), \sigma(s))\omega(s)\Delta s + \int_T^t H(\sigma(t), \sigma(s))\rho^{\Delta}(s)\frac{\omega(s)}{\rho(s)}\Delta s \\
&\quad - \gamma\int_T^t H(\sigma(t), \sigma(s))\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left(\frac{\omega(s)}{\rho(s)} - \eta(s)\right)^{\gamma+1/\gamma}\Delta s,
\end{aligned} \tag{2.54}$$

that is,

$$\begin{aligned}
\int_T^t H(\sigma(t),\sigma(s))\rho(\sigma(s))\left[Q_1(s)-\eta^{\Delta}(s)\right]\Delta s &\le H(\sigma(t),T)\omega(T) \\
&- \int_T^t \left[h(\sigma(t),s)-\frac{\rho^{\Delta}(s)}{\rho(s)}\right]H(\sigma(t),\sigma(s))\omega(s)\Delta s \\
&- \gamma\int_T^t H(\sigma(t),\sigma(s))\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left(\frac{\omega(s)}{\rho(s)}-\eta(s)\right)^{\gamma+1/\gamma}\Delta s.
\end{aligned} \tag{2.55}$$

Hence,

$$\begin{aligned}
\int_T^t H(\sigma(t),\sigma(s))\rho(t)\Theta_1(t,s)\Delta s &\le H(\sigma(t),T)\omega(T) \\
&+ \int_T^t H(\sigma(t),\sigma(s))\rho(s)|\lambda(t,s)|\left|\frac{\omega(s)}{\rho(s)}-\eta(s)\right|\Delta s \\
&- \gamma\int_T^t H(\sigma(t),\sigma(s))\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left|\frac{\omega(s)}{\rho(s)}-\eta(s)\right|^{\gamma+1/\gamma}\Delta s.
\end{aligned} \tag{2.56}$$

Using the inequality (1.15) we have

$$\begin{aligned}
&\frac{1}{H(\sigma(t),T)} \\
&\times\int_T^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_1(t,s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \le \omega(T).
\end{aligned} \tag{2.57}$$

Set $T=t_*$, so,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_*)} \\
&\times\int_{t_*}^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_1(t,s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \le \omega(t_*).
\end{aligned} \tag{2.58}$$

This contradicts (2.51) and finishes the proof. □

Theorem 2.10. *Assume that the condition* (H) *and* (2.7) *hold, let* $\rho, \eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \times \int_{t_0}^{t} H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_2(t,s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\right)\Delta s = \infty, \tag{2.59}$$

then every solution of (1.7) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. In view of Lemma 2.3, either $\lim_{t\to\infty} y(t) = 0$ or there exists a $t_* \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x^{\Delta}(t) \geq 0, x^{\Delta\Delta}(t) < 0, x(t) \geq t x^{\Delta}(t)$, and $x(t)/t$ is strictly decreasing for $t \in [t_*,\infty)_{\mathbb{T}}$.

Since the rest of the proof is similar to Theorem 2.9, so we omit the detail. The proof is complete. □

Following the procedure of the proof of Theorem 2.7, we can also prove the following theorem.

Theorem 2.11. *Assume that* (2.5) *holds.* $0 \leq p(t) < 1$, *let* $H \in \mathfrak{R}, \rho, \eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$,

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty} \frac{H(\sigma(t),s)}{H(\sigma(t),t_0)}\right] \leq \infty, \tag{2.60}$$

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{t} H(\sigma(t),\sigma(s))\rho(s) r(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\Delta s < \infty, \tag{2.61}$$

there exists $\varphi \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\int_{t_0}^{\infty} \rho(\sigma(s)) \frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left(\frac{\varphi(s)}{\rho(s)} - \eta(s)\right)_{+}^{\gamma+1/\gamma}\Delta s = \infty, \quad \text{if } \gamma \geq 1, \tag{2.62}$$

or

$$\int_{t_0}^{\infty} \rho(\sigma(s)) \frac{1}{(r(s))^{1/\gamma}} \frac{s}{\sigma(s)}\left(\frac{\varphi(s)}{\rho(s)} - \eta(s)\right)_{+}^{\gamma+1/\gamma}\Delta s = \infty, \quad \text{if } 0 < \gamma < 1, \tag{2.63}$$

and for any $T \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \times \int_T^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_1(t,s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \geq \varphi(T), \tag{2.64}$$

and then (1.7) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$, *where* $(\varphi(t))_+ = \max\{\varphi(t), 0\}$.

Proof. We prove only case $\gamma \geq 1$. The proof of case $0 < \gamma < 1$ is similar. Proceeding as the proof of Theorem 2.9, we have (2.56) and (2.57). So we have for all $t > T, t, T \in [t_*, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \times \int_T^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_1(t,s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \leq \omega(T). \tag{2.65}$$

By (2.64), we obtain

$$\varphi(T) \leq \omega(T), \quad T \in [t_*, \infty)_{\mathbb{T}}. \tag{2.66}$$

Define

$$\begin{aligned} P(t) &= \frac{1}{H(\sigma(t),t_*)}\int_{t_*}^t H(\sigma(t),\sigma(s))\rho(s)\left|\lambda(t,s)\right|\left|\frac{\omega(s)}{\rho(s)} - \eta(s)\right|\Delta s, \\ Q(t) &= \frac{\gamma}{H(\sigma(t),t_*)}\int_{t_*}^t H(\sigma(t),\sigma(s))\rho(\sigma(s))\left(\frac{s}{\sigma(s)}\right)^{\gamma}\frac{1}{(r(s))^{1/\gamma}}\left|\frac{\omega(s)}{\rho(s)} - \eta(s)\right|^{\gamma+1/\gamma}\Delta s. \end{aligned} \tag{2.67}$$

Then, by (2.56) and (2.64), we have that

$$\begin{aligned} &\liminf_{t\to\infty}[Q(t) - P(t)] \leq \omega(t_*) \\ &\quad - \limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_*)}\int_{t_*}^t H(\sigma(t),\sigma(s))\rho(s)\Theta_1(t,s)\Delta s \leq \omega(t_*) - \varphi(t_*) < \infty. \end{aligned} \tag{2.68}$$

We claim that

$$\int_{t_*}^t \rho(\sigma(s))\left(\frac{s}{\sigma(s)}\right)^{\gamma}\frac{1}{(r(s))^{1/\gamma}}\left|\frac{\omega(s)}{\rho(s)} - \eta(s)\right|^{\gamma+1/\gamma}\Delta s < \infty. \tag{2.69}$$

Suppose, to the contrary, that

$$\int_{t_*}^{t} \rho(\sigma(s))\left(\frac{s}{\sigma(s)}\right)^{\gamma} \frac{1}{(r(s))^{1/\gamma}}\left|\frac{\omega(s)}{\rho(s)}-\eta(s)\right|^{\gamma+1/\gamma} \Delta s=\infty. \tag{2.70}$$

By (2.60), there exists a positive constant k_1 such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(\sigma(t),s)}{H(\sigma(t),t_0)}\right]\geq k_1. \tag{2.71}$$

Let k_2 be an arbitrary positive number, then it follows from (2.70) that there exists a $T_1\in[t_*,\infty)_{\mathbb{T}}$ such that

$$\int_{t_*}^{t} \rho(\sigma(s))\left(\frac{s}{\sigma(s)}\right)^{\gamma} \frac{1}{(r(s))^{1/\gamma}}\left|\frac{\omega(s)}{\rho(s)}-\eta(s)\right|^{\gamma+1/\gamma} \Delta s\geq\frac{k_2}{\gamma k_1},\quad t\in[T_1,\infty)_{\mathbb{T}}. \tag{2.72}$$

Hence,

$$\begin{aligned}
Q(t)&=\frac{\gamma}{H(\sigma(t),t_*)}\\
&\quad\times\int_{t_*}^{t} H(\sigma(t),\sigma(s))\left(\int_{t_*}^{s}\rho(\sigma(\tau))\left(\frac{\tau}{\sigma(\tau)}\right)^{\gamma}\frac{1}{(r(\tau))^{1/\gamma}}\left|\frac{\omega(\tau)}{\rho(\tau)}-\eta(\tau)\right|^{\gamma+1/\gamma}\Delta\tau\right)^{\Delta}\Delta s\\
&=\frac{\gamma}{H(\sigma(t),t_*)}\\
&\quad\times\int_{t_*}^{t}\left(-H^{\Delta_s}(\sigma(t),s)\right)\left(\int_{t_*}^{s}\rho(\sigma(\tau))\left(\frac{\tau}{\sigma(\tau)}\right)^{\gamma}\frac{1}{(r(\tau))^{1/\gamma}}\left|\frac{\omega(\tau)}{\rho(\tau)}-\eta(\tau)\right|^{\gamma+1/\gamma}\Delta\tau\right)\Delta s\\
&\geq\frac{\gamma}{H(\sigma(t),t_*)}\\
&\quad\times\int_{T_1}^{t}\left(-H^{\Delta_s}(\sigma(t),s)\right)\left(\int_{t_*}^{s}\rho(\sigma(\tau))\left(\frac{\tau}{\sigma(\tau)}\right)^{\gamma}\frac{1}{(r(\tau))^{1/\gamma}}\left|\frac{\omega(\tau)}{\rho(\tau)}-\eta(\tau)\right|^{\gamma+1/\gamma}\Delta\tau\right)\Delta s\\
&\geq\frac{k_2}{k_1}\frac{1}{H(\sigma(t),t_*)}\int_{T_1}^{t}\left(-H^{\Delta_s}(\sigma(t),s)\right)\Delta s=\frac{k_2}{k_1}\frac{H(\sigma(t),T_1)}{H(\sigma(t),t_*)}.
\end{aligned} \tag{2.73}$$

By (2.71), there exists a $T_2\in[T_1,\infty)_{\mathbb{T}}$ such that $H(\sigma(t),T_1)/H(\sigma(t),t_*)\geq k_1, t\in[T_2,\infty)_{\mathbb{T}}$, which implies that $Q(t)\geq k_2$. Since k_2 is arbitrary, then

$$\lim_{t\to\infty}Q(t)=\infty. \tag{2.74}$$

In view of (2.68), we consider a sequence $\{T_n\}_{n=1}^{\infty} \subset [t_*, \infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ satisfying

$$\lim_{n\to\infty} (Q(T_n) - P(T_n)) = \liminf_{t\to\infty} [Q(t) - P(t)] < \infty. \tag{2.75}$$

Then, there exists a constant M such that $Q(T_n) - P(T_n) \le M$ for all sufficiently large $n \in \mathbb{N}$. Since (2.70) ensures that

$$\lim_{n\to\infty} Q(T_n) = \infty, \lim_{n\to\infty} P(T_n) = \infty \tag{2.76}$$

so,

$$\frac{P(T_n)}{Q(T_n)} - 1 \ge \frac{-M}{Q(T_n)} > -\frac{1}{2} \qquad \text{or } \frac{P(T_n)}{Q(T_n)} \ge \frac{1}{2} \tag{2.77}$$

hold for all sufficiently large $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} \frac{P^{\gamma+1}(T_n)}{Q^{\gamma}(T_n)} = \infty. \tag{2.78}$$

On the other hand, from the definition of P we can obtain, by Hölder's inequality,

$$\begin{aligned} P(T_n) \le{} & \frac{1}{H(\sigma(T_n), t_*)} \\ & \times \left[\int_{t_*}^{T_n} H(\sigma(T_n), \sigma(s))\rho(\sigma(s))\left(\frac{s}{\sigma(s)}\right)^{\gamma} \frac{1}{(r(s))^{1/\gamma}} \left|\frac{\omega(s)}{\rho(s)} - \eta(s)\right|^{\gamma+1/\gamma} \Delta s\right]^{\gamma/(\gamma+1)} \\ & \times \left[\int_{t_*}^{T_n} H(\sigma(T_n), \sigma(s))\rho(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma} \left(\frac{\sigma(s)}{s}\right)^{\gamma^2} r(s)|\lambda(T_n, s)|^{\gamma+1} \Delta s\right]^{1/(\gamma+1)}, \end{aligned} \tag{2.79}$$

and, accordingly,

$$\frac{P^{\gamma+1}(T_n)}{Q^{\gamma}(T_n)} \le \frac{1}{\gamma^{\gamma}} \frac{1}{H(\sigma(T_n), t_*)} \int_{t_*}^{T_n} H(\sigma(T_n), \sigma(s))\rho(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma} \left(\frac{\sigma(s)}{s}\right)^{\gamma^2} r(s)|\lambda(T_n, s)|^{\gamma+1} \Delta s. \tag{2.80}$$

So, because of (2.78), we get

$$\lim_{n\to\infty} \frac{1}{\gamma^{\gamma}} \frac{1}{H(\sigma(T_n), t_*)} \int_{t_*}^{T_n} H(\sigma(T_n), \sigma(s))\rho(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma} \left(\frac{\sigma(s)}{s}\right)^{\gamma^2} r(s)|\lambda(T_n, s)|^{\gamma+1} \Delta s = \infty, \tag{2.81}$$

which gives that

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_*)}\int_{t_*}^{t}H(\sigma(t),\sigma(s))\rho(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\gamma^2}r(s)|\lambda(t,s)|^{\gamma+1}\Delta s=\infty \tag{2.82}$$

contradicting (2.61). Hence, (2.69) holds. In view of (2.66), from (2.69), we have

$$\begin{aligned}&\int_{t_*}^{\infty}\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left(\frac{\varphi(s)}{\rho(s)}-\eta(s)\right)_+^{\gamma+1/\gamma}\Delta s\\&\quad\le\int_{t_*}^{\infty}\rho(\sigma(s))\frac{1}{(r(s))^{1/\gamma}}\left(\frac{s}{\sigma(s)}\right)^{\gamma}\left|\frac{\omega(s)}{\rho(s)}-\eta(s)\right|^{\gamma+1/\gamma}\Delta s<\infty,\end{aligned} \tag{2.83}$$

which contradicts (2.62). The proof is complete. □

Theorem 2.12. *Assume that the condition* (H) *and* (2.7) *hold, let* $H\in\Re,\rho,\eta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, *there exists* $\varphi\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.60) *and* (2.61), *and either* (2.62) *or* (2.63) *hold, and for any* $T\in[t_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(\sigma(t),T)}\\&\times\int_{T}^{t}H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_2(t,s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\right)\Delta s\ge\varphi(T),\end{aligned} \tag{2.84}$$

and then every solution of (1.7) *either oscillates or tends to zero as* $t\to\infty$.

Proof. Suppose that (1.7) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is eventually positive. In view of Lemma 2.3, either $\lim_{t\to\infty}y(t)=0$ or there exists a $t_*\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0,x^{\Delta}(t)\ge 0,x^{\Delta\Delta}(t)<0,x(t)\ge tx^{\Delta}(t)$, and $x(t)/t$ is strictly decreasing for $t\in[t_*,\infty)_{\mathbb{T}}$.

Since the rest of the proof is similar to Theorem 2.11, so we omit the detail. The proof is complete. □

Following the procedure of the proof of Theorem 2.8, we can also prove the following theorem.

Theorem 2.13. *Assume that* (2.5) *and* (2.60) *hold,* $0\le p(t)<1$, *let* $H\in\Re,\rho,\eta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$,

$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{t}H(\sigma(t),\sigma(s))\rho(s)r(s)\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\Delta s<\infty, \tag{2.85}$$

there exists $\varphi \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.62) *or* (2.63) *holds, and for any* $T \in [t_0,\infty)_{\mathbb{T}}$,

$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),T)} \times \int_T^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_1(t,s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \geq \varphi(T), \tag{2.86}$$

and then (1.7) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Theorem 2.14. *Assume that the condition* (H), (2.7), *and* (2.60) *hold, let* $H \in \Re, \rho, \eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, *there exists* $\varphi \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.62) *or* (2.63) *holds, and for any* $T \in [t_0,\infty)_{\mathbb{T}}$,

$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),T)} \times \int_T^t H(\sigma(t),\sigma(s))\rho(s)\left(\Theta_2(t,s)-\frac{r(s)}{(\gamma+1)^{\gamma+1}}\left(\frac{\rho(s)}{\rho(\sigma(s))}\right)^{\gamma}\left(\frac{\sigma(s)}{s}\right)^{\overline{\gamma}}|\lambda(t,s)|^{\gamma+1}\right)\Delta s \geq \varphi(T), \tag{2.87}$$

and then every solution of (1.7) *either oscillates or tends to zero as* $t \to \infty$.

Remark 2.15. As Theorem 2.7–Theorem 2.14 are rather general, it is convenient for applications to derive a number of oscillation criteria with the appropriate choice of the functions H, ρ and η.

3. Examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the following delay dynamic equations on time scales:

$$\begin{gathered}\left[t^{\gamma}|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta}+\left(\frac{\sigma(t)}{\delta_1(t)}\right)^{\alpha}\frac{1}{t}\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t)) \\ +\left(\frac{t}{\delta_2(t)}\right)^{\beta}\frac{1}{t}\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t))=0, \quad t\in[1,\infty)_{\mathbb{T}},\end{gathered} \tag{3.1}$$

where

$$\begin{gathered}r(t)=t^{\gamma}, \quad q_1(t)=\left(\frac{\sigma(t)}{\delta_1(t)}\right)^{\alpha}\frac{1}{t}, \quad q_2(t)=\left(\frac{t}{\delta_2(t)}\right)^{\beta}\frac{1}{t}, \quad x(t)=y(t)+p_0y(\tau(t)), \\ 0\leq p_0<1, 0<\alpha<\gamma<\beta\end{gathered} \tag{3.2}$$

with $\gamma = (\alpha+\beta)/2$, $\tau, \delta_1, \delta_2 \in C_{rd}([1,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \le t$, $\delta_1(t) \le t$, $\delta_2(t) \le t$, for $t \in [1,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty}\tau(t) = \infty$, $\lim_{t\to\infty}\delta_1(t) = \infty$, $\lim_{t\to\infty}\delta_2(t) = \infty$. Then, by Theorem 2.4, we have

$$\int_1^\infty \left(\frac{1}{r(t)}\right)^{1/\gamma}\Delta t = \int_1^\infty \frac{1}{t}\Delta t = \infty,$$

$$\int_1^\infty q_1(t)\left((1-p_0)\delta_1(t)\right)^\alpha \Delta t = \int_1^\infty (1-p_0)^\alpha \frac{(\sigma(t))^\alpha}{t}\Delta t \ge \int_1^\infty (1-p_0)^\alpha \frac{t^\alpha}{t}\Delta t = \infty,$$

$$\limsup_{t\to\infty}\left[t\left(\frac{1}{r(t)}\int_t^\infty q_2(s)(1-p_0)^\beta\left(\frac{\delta_2(s)}{s}\right)^\beta\Delta s\right)^{1/\gamma}\right] = \limsup_{t\to\infty}\left(\int_t^\infty (1-p_0)^\beta\frac{1}{s}\Delta s\right)^{1/\gamma} = \infty. \tag{3.3}$$

Then (3.1) is oscillatory on $[1,\infty)_{\mathbb{T}}$.

Example 3.2. Consider the second-order delay dynamic equations on time scales:

$$\begin{aligned}&\left[t^\gamma\left|x^\Delta(t)\right|^{\gamma-1}x^\Delta(t)\right]^\Delta + \left(\frac{\sigma(t)}{\delta(t)}\right)^\alpha\frac{1}{t}\left|y(\delta(t))\right|^{\alpha-1}y(\delta(t))\\ &\quad+\left(\frac{\sigma(t)}{\delta(t)}\right)^\beta\frac{1}{t}\left|y(\delta(t))\right|^{\beta-1}y(\delta(t)) = 0, \quad t\in[1,\infty)_{\mathbb{T}},\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}&r(t) = t^\gamma, \quad q_1(t) = \left(\frac{\sigma(t)}{\delta(t)}\right)^\alpha\frac{1}{t}, \quad q_2(t) = \left(\frac{\sigma(t)}{\delta(t)}\right)^\beta\frac{1}{t}, \quad x(t) = y(t) + p_0 y(\tau(t)),\\ &\qquad 0\le p_0 < 1, \quad 0<\alpha<\gamma<\beta\end{aligned} \tag{3.5}$$

with $\gamma = (\alpha+\beta)/2$, $\tau, \delta \in C_{rd}([1,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t)\le t$, $\delta(t)\le t$, for $t\in[1,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty}\tau(t) = \infty$, $\lim_{t\to\infty}\delta(t) = \infty$. Then,

$$\theta = \min\left\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\alpha}\right\} = 2,$$

$$Q_1(t) = \mu\left(q_1(t)\left(1-p(\delta_1(t))\right)^\alpha\right)^{\beta-\gamma/\beta-\alpha}\left(q_2(t)\left(1-p(\delta_2(t))\right)^\beta\right)^{\gamma-\alpha/\beta-\alpha}\left(\frac{\delta(t)}{\sigma(t)}\right)^\gamma = 2(1-p_0)^\gamma\frac{1}{t}, \tag{3.6}$$

let $\rho = 1$, $\eta = 1$, and by Theorem 2.7, we have

$$\limsup_{t\to\infty}\int_1^t Q_1(s)\Delta s = \limsup_{t\to\infty}\int_1^t 2(1-p_0)^\gamma\frac{1}{s}\Delta s = \infty. \tag{3.7}$$

Then (3.4) is oscillatory on $[1,\infty)$.

Acknowledgments

This research is supported by the Natural Science Foundation of China (60774004, 60904024), China Postdoctoral Science Foundation funded project (20080441126, 200902564), Shandong Postdoctoral funded project (200802018), the Natural Science Foundation of Shandong (Y2008A28, ZR2009AL003), and also supported by University of Jinan Research Funds for Doctors (B0621, XBS0843).

References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[4] D. R. Anderson, "Interval criteria for oscillation of nonlinear second-order dynamic equations on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4614–4623, 2008.

[5] M. Bohner and S. H. Saker, "Oscillation of second order nonlinear dynamic equations on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 34, no. 4, pp. 1239–1254, 2004.

[6] E. Akin-Bohner and J. Hoffacker, "Oscillation properties of an Emden-Fowler type equation on discrete time scales," *Journal of Difference Equations and Applications*, vol. 9, no. 6, pp. 603–612, 2003.

[7] E. Akin-Bohner, M. Bohner, and S. H. Saker, "Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations," *Electronic Transactions on Numerical Analysis*, vol. 27, pp. 1–12, 2007.

[8] L. Erbe, "Oscillation results for second-order linear equations on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1061–1071, 2002.

[9] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear dynamic equations on time scales," *Journal of the London Mathematical Society*, vol. 67, no. 3, pp. 701–714, 2003.

[10] L. Erbe, T. S. Hassan, and A. Peterson, "Oscillation criteria for nonlinear damped dynamic equations on time scales," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 343–357, 2008.

[11] L. Erbe, J. Baoguo, and A. Peterson, "Nonoscillation for second order sublinear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 232, no. 2, pp. 594–599, 2009.

[12] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, and S. Wichuta, "On the oscillation of certain second order nonlinear dynamic equations," *Mathematical and Computer Modelling*, vol. 50, no. 1-2, pp. 273–286, 2009.

[13] S. R. Grace, R. P. Agarwal, M. Bohner, and D. O'Regan, "Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3463–3471, 2009.

[14] T. S. Hassan, "Oscillation criteria for half-linear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 176–185, 2008.

[15] T. S. Hassan, "Oscillation of third order nonlinear delay dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1573–1586, 2009.

[16] B. Jia, L. Erbe, and A. Peterson, "New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2744–2756, 2008.

[17] T. Li, Z. Han, S. Sun, and C. Zhang, "Forced oscillation of second-order nonlinear dynamic equations on time scales," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 60, pp. 1–8, 2009.

[18] S. H. Saker, "Oscillation of nonlinear dynamic equations on time scales," *Applied Mathematics and Computation*, vol. 148, no. 1, pp. 81–91, 2004.

[19] S. H. Saker, "Oscillation criteria of second-order half-linear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 177, no. 2, pp. 375–387, 2005.

[20] Z.-H. Yu and Q.-R. Wang, "Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 531–540, 2009.

[21] R. P. Agarwal, M. Bohner, and S. H. Saker, "Oscillation of second order delay dynamic equations," *The Canadian Applied Mathematics Quarterly*, vol. 13, no. 1, pp. 1–18, 2005.

[22] D. R. Anderson, "Oscillation of second-order forced functional dynamic equations with oscillatory potentials," *Journal of Difference Equations and Applications*, vol. 13, no. 5, pp. 407–421, 2007.
[23] M. Bohner, "Some oscillation criteria for first order delay dynamic equations," *Far East Journal of Applied Mathematics*, vol. 18, no. 3, pp. 289–304, 2005.
[24] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 505–522, 2007.
[25] Z. Han, S. Sun, and B. Shi, "Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 847–858, 2007.
[26] Z. Han, B. Shi, and S. Sun, "Oscillation criteria for second-order delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2007, Article ID 70730, 16 pages, 2007.
[27] Z. L. Han, B. Shi, and S. R. Sun, "Oscillation of second-order delay dynamic equations on time scales," *Acta Scientiarum Naturalium Universitatis Sunyatseni*, vol. 46, no. 6, pp. 10–13, 2007.
[28] Z. Han, T. Li, S. Sun, and C. Zhang, "Oscillation for second-order nonlinear delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2009, Article ID 756171, 13 pages, 2009.
[29] Y. Şahiner, "Oscillation of second-order delay differential equations on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e1073–e1080, 2005.
[30] S. Sun, Z. Han, and C. Zhang, "Oscillation of second-order delay dynamic equations on time scales," *Journal of Applied Mathematics and Computing*, vol. 30, no. 1-2, pp. 459–468, 2009.
[31] B. G. Zhang and X. Deng, "Oscillation of delay differential equations on time scales," *Mathematical and Computer Modelling*, vol. 36, no. 11–13, pp. 1307–1318, 2002.
[32] B. G. Zhang and Z. Shanliang, "Oscillation of second-order nonlinear delay dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 49, no. 4, pp. 599–609, 2005.
[33] R. P. Agarwal, D. O'Regan, and S. H. Saker, "Oscillation criteria for second-order nonlinear neutral delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 203–217, 2004.
[34] Z. Han, T. Li, S. Sun, and C. Zhang, "Oscillation behavior of third order neutral Emden-Fowler delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2010, Article ID 586312, 23 pages, 2010.
[35] H. A. Agwo, "Oscillation of nonlinear second order neutral delay dynamic equations on time scales," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 2, pp. 299–312, 2008.
[36] T. Li, Z. Han, S. Sun, and D. Yang, "Existence of nonoscillatory solutions to second-order neutral delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2009, Article ID 562329, 10 pages, 2009.
[37] S. H. Saker, "Oscillation of second-order nonlinear neutral delay dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 187, no. 2, pp. 123–141, 2006.
[38] S. H. Saker, R. P. Agarwal, and D. O'Regan, "Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales," *Applicable Analysis*, vol. 86, no. 1, pp. 1–17, 2007.
[39] S. H. Saker, D. O'Regan, and R. P. Agarwal, "Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales," *Acta Mathematica Sinica*, vol. 24, no. 9, pp. 1409–1432, 2008.
[40] Y. Şahíner, "Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales," *Advances in Difference Equations*, vol. 2006, Article ID 65626, 9 pages, 2006.
[41] Y. Sun, Z. Han, T. Li, and G. Zhang, "Oscillation criteria for second order quasi-linear neutral delay dynamic equations on time scales," *Advances in Difference Equations*. In press.
[42] A. K. Tripathy, "Some oscillation results for second order nonlinear dynamic equations of neutral type," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e1727–e1735, 2009.
[43] H.-W. Wu, R.-K. Zhuang, and R. M. Mathsen, "Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations," *Applied Mathematics and Computation*, vol. 178, no. 2, pp. 321–331, 2006.
[44] Z.-Q. Zhu and Q.-R. Wang, "Existence of nonoscillatory solutions to neutral dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 751–762, 2007.