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Research Article

Existence of Homoclinic Solutions for a Class of Nonlinear Difference Equations

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By using the critical point theory, we establish some existence criteria to guarantee that the nonlinear difference equation $\Delta[p(n)(\Delta x(n-1))^{\delta}] - q(n)(x(n))^{\delta} = f(n, x(n))$ has at least one homoclinic solution, where $n \in \mathbb{Z}$, $x(n) \in \mathbb{R}$, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is non periodic in n. Our conditions on the nonlinear term $f(n, x(n))$ are rather relaxed, and we generalize some existing results in the literature.

1. Introduction

Consider the nonlinear difference equation of the form

$$\Delta\left[p(n)(\Delta u(n-1))^{\delta}\right] - q(n)(x(n))^{\delta} = f(n, u(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, $\Delta^2 u(n) = \Delta(\Delta u(n))$, $\delta > 0$ is the ratio of odd positive integers, $\{p(n)\}$ and $\{q(n)\}$ are real sequences, $\{p(n)\} \neq 0$. $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$, then $u(n)$ is called a nontrivial homoclinic solution.

Difference equations have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, neural network, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equation, such as stability, attractiveness, periodicity, oscillation, and boundary value problem. Recently, there are some new results on periodic solutions of nonlinear difference equations by using the critical point theory in the literature; see [1–3].

In general, (1.1) may be regarded as a discrete analogue of a special case of the following second-order differential equation:

$$(p(t)\varphi(x'))' + f(t,x) = 0, \tag{1.2}$$

which has arose in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor (see, e.g., [4–6] and their references). In the case of $\varphi(x) = |x|^{\delta-2}x$, (1.2) has been discussed extensively in the literature; we refer the reader to the monographs [7–10].

It is well known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been already recognized from Poincaré; homoclinic orbits play an important role in analyzing the chaos of dynamical system. In the past decade, this problem has been intensively studied using critical point theory and variational methods.

In some recent papers [1–3, 11–14], the authors studied the existence of periodic solutions, subharmonic solutions, and homoclinic solutions of some special forms of (1.1) by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions for difference equations. Along this direction, Ma and Guo [13] applied the critical point theory to prove the existence of homoclinic solutions of the following special form of (1.1):

$$\Delta\left[p(n)\Delta u(n-1)\right] - q(n)u(n) + f(n,u(n)) = 0, \tag{1.3}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}$, $p, q : \mathbb{Z} \to \mathbb{R}$, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$.

Theorem A (see [13]). *Assume that p, q, and f satisfy the following conditions:*

(p) $p(n) > 0$ *for all* $n \in \mathbb{Z}$;

(q) $q(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n|\to+\infty} q(n) = +\infty$;

(f1) *there is a constant* $\mu > 2$ *such that*

$$0 < \mu \int_0^x f(n,s)ds \le xf(n,x), \quad \forall (n,x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\}); \tag{1.4}$$

(f2) $\lim_{x\to 0} f(n,x)/x = 0$ *uniformly with respect to* $n \in \mathbb{Z}$.

Then (1.3) *possesses a nontrivial homoclinic solution.*

It is worth pointing out that to establish the existence of homoclinic solutions of (1.3), condition (f1) is the special form (with $N = 1$) of the following so-called global Ambrosetti-Rabinowitz condition on W; see [15].

(AR) For every $n \in \mathbb{Z}$, W is continuously differentiable in x, and there is a constant $\mu > 2$ such that

$$0 < \mu W(n,x) \leq (\nabla W(n,x), x), \quad \forall (n,x) \in \mathbb{Z} \times \left(\mathbb{R}^N \setminus \{0\}\right). \tag{1.5}$$

However, it seems that results on the existence of homoclinic solutions of (1.1) by critical point method have not been considered in the literature. The main purpose of this paper is to develop a new approach to the above problem by using critical point theory.

Motivated by the above papers [13, 14], we will obtain some new criteria for guaranteeing that (1.1) has one nontrivial homoclinic solution without any periodicity and generalize Theorem A. Especially, $F(n,x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known literature.

In this paper, we always assume that $F(n,x) = \int_0^x f(n,s)ds$, $F_1(n,x) = \int_0^x f_1(n,s)ds$, $F_2(n,x) = \int_0^x f_2(n,s)ds$. Our main results are the following theorems.

Theorem 1.1. *Assume that p, q, and f satisfy the following conditions:*

(p) $p(n) > 0$ *for all* $n \in \mathbb{Z}$;

(q) $q(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n| \to +\infty} q(n) = +\infty$;

(F1) $F(n,x) = F_1(n,x) - F_2(n,x)$, *for every* $n \in \mathbb{Z}$, F_1 *and* F_2 *are continuously differentiable in* x, *and there is a bounded set* $J \subset \mathbb{Z}$ *such that*

$$\begin{gathered} F_2(n,x) \geq 0, \quad \forall (n,x) \in J \times \mathbb{R}, \ |x| \leq 1, \\ \frac{1}{q(n)} |f(n,x)| = o\left(|x|^{\delta}\right) \quad \text{as } x \longrightarrow 0 \end{gathered} \tag{1.6}$$

uniformly in $n \in \mathbb{Z} \setminus J$;

(F2) *there is a constant* $\mu > \delta + 1$ *such that*

$$0 < \mu F_1(n,x) \leq x f_1(n,x), \quad \forall (n,x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\}); \tag{1.7}$$

(F3) $F_2(n,0) \equiv 0$, *and there is a constant* $\varrho \in (\delta + 1, \mu)$ *such that*

$$x f_2(n,x) \leq \varrho F_2(n,x), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}. \tag{1.8}$$

Then (1.1) *possesses a nontrivial homoclinic solution.*

Theorem 1.2. *Assume that p, q, and F satisfy (p), (q), (F_2), (F_3), and the following assumption:*

(F1′) *$F(n,x) = F_1(n,x) - F_2(n,x)$, for every $n \in \mathbb{Z}$, F_1 and F_2 are continuously differentiable in x, and*

$$\frac{1}{q(n)}\left|f(n,x)\right| = o\left(|x|^{\delta}\right) \quad \text{as } x \longrightarrow 0 \tag{1.9}$$

uniformly in $n \in \mathbb{Z}$. Then (1.1) *possesses a nontrivial homoclinic solution.*

Remark 1.3. *Obviously, both conditions (F_1) and (F_1') are weaker than (f_1). Therefore, both Theorems 1.1 and 1.2 generalize Theorem A by relaxing conditions (f_1) and (f_2).*

When $F(n,x)$ is subquadratic at infinity, as far as the authors are aware, there is no research about the existence of homoclinic solutions of (1.1). Motivated by the paper [16], the intention of this paper is that, under the assumption that $F(n,x)$ is indefinite sign and subquadratic as $|n| \to +\infty$, we will establish some existence criteria to guarantee that (1.1) has at least one homoclinic solution by using minimization theorem in critical point theory.

Now we present the basic hypothesis on p, q, and F in order to announce the results in this paper.

(F4) *For every $n \in \mathbb{Z}$, F is continuously differentiable in x, and there exist two* constants $1 < \gamma_1 < \gamma_2 < \delta + 1$ *and two functions $a_1, a_2 \in l^{(\delta+1)/(\delta+1-\gamma_1)}(\mathbb{Z}, [0,+\infty))$ such that*

$$\begin{aligned} |F(n,x)| &\le a_1(n)|x|^{\gamma_1}, \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R},\ |x| \le 1, \\ |F(n,x)| &\le a_2(n)|x|^{\gamma_2}, \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R},\ |x| \ge 1. \end{aligned} \tag{1.10}$$

(F5) *There exist two functions $b \in l^{(\delta+1)/(\delta+1-\gamma_1)}(\mathbb{Z}, [0,+\infty))$ and $\varphi \in C([0,+\infty), [0,+\infty))$ such that*

$$\left|f(n,x)\right| \le b(n)\varphi(|x|), \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R} \tag{1.11}$$

where $\varphi(s) = O(s^{\gamma_1 - 1})$ as $|s| \le c$, c is a positive constant.

(F6) *There exist $n_0 \in \mathbb{Z}$ and two constants $\eta > 0$ and $\gamma_3 \in (1, \delta+1)$ such that*

$$F(n_0, x) \ge \eta |x|^{\gamma_3}, \quad \forall x \in \mathbb{R},\ |x| \le 1. \tag{1.12}$$

Up to now, we can state our main results.

Theorem 1.4. *Assume that p, q, and F satisfy (p), (q), $(F4)$, $(F5)$, and $(F6)$. Then* (1.1) *possesses at least one nontrivial homoclinic solution.*

By Theorem 1.4, we have the following corollary.

Corollary 1.5. *Assume that p, q, and F satisfy (p), (q), and the following conditions:*

(F7) *$F(n,x) = a(n)V(x)$, where $V \in C^1(\mathbb{R},\mathbb{R})$ and $a \in l^{(\delta+1)/(\delta+1-\gamma_1)}(\mathbb{Z},[0,+\infty))$, $\gamma_1 \in (1,\delta+1)$ is a constant such that $a(n_0) > 0$ for some $n_0 \in \mathbb{Z}$.*

(F8) *There exist constants $M, M' > 0$, $\gamma_2 \in [\gamma_1, \delta+1)$, and $\gamma_3 \in (1,\delta+1)$ such that*

$$\begin{gathered} M'|x|^{\gamma_3} \le V(x) \le M|x|^{\gamma_1}, \quad \forall x \in \mathbb{R},\ |x| \le 1, \\ 0 < V(x) \le M|x|^{\gamma_2}, \quad \forall x \in \mathbb{R},\ |x| \ge 1, \end{gathered} \tag{1.13}$$

(F9) *$V'(x) = O(|x|^{\gamma_1 - 1})$ as $|x| \le c$, c is a positive constant.*

Then (1.1) *possesses at least one nontrivial homoclinic solution.*

2. Preliminaries

Let

$$\begin{gathered} S = \{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R},\ n \in \mathbb{Z}\}, \\ E = \left\{u \in S : \sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u(n-1))^{\delta+1} + q(n)(u(n))^{\delta+1}\right] < +\infty\right\}, \end{gathered} \tag{2.1}$$

and for $u \in E$, let

$$\|u\| = \left\{\sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u(n-1))^{\delta+1} + q(n)(u(n))^{\delta+1}\right] < +\infty\right\}^{1/\delta+1}, \quad u \in E. \tag{2.2}$$

Then E is a uniform convex Banach space with this norm.

As usual, for $1 \le p < +\infty$, let

$$\begin{gathered} l^p(\mathbb{Z},\mathbb{R}) = \left\{u \in S : \sum_{n\in\mathbb{Z}} |u(n)|^p < +\infty\right\}, \\ l^\infty(\mathbb{Z},\mathbb{R}) = \left\{u \in S : \sup_{n\in\mathbb{Z}} |u(n)| < +\infty\right\}, \end{gathered} \tag{2.3}$$

and their norms are defined by

$$\|u\|_p = \left(\sum_{n\in\mathbb{Z}} |u(n)|^p\right)^{1/p}, \quad \forall u \in l^p(\mathbb{Z},\mathbb{R}); \qquad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \quad \forall u \in l^\infty(\mathbb{Z},\mathbb{R}), \tag{2.4}$$

respectively.

For any $n_1, n_2 \in \mathbb{Z}$ with $n_1 < n_2$, we let $\mathbb{Z}(n_1, n_2) = [n_1, n_2] \cap \mathbb{Z}$, and for function $f : \mathbb{Z} \to \mathbb{R}$ and $a \in \mathbb{R}$, we set

$$\mathbb{Z}(f(n) \geq a) = \{n \in \mathbb{Z} : f(n) \geq a\}, \qquad \mathbb{Z}(f(n) \leq a) = \{n \in \mathbb{Z} : f(n) \leq a\}. \tag{2.5}$$

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n, u(n)). \tag{2.6}$$

If (p), (q), and (F1), (F1′), or (F4) holds, then $I \in C^1(E,\mathbb{R})$, and one can easily check that

$$\begin{aligned}&\langle I'(u), v\rangle \\ &= \sum_{n\in\mathbb{Z}} \Big[\Big(p(n)(\Delta u(n-1))^\delta \Delta v(n-1)\Big) + q(n)(u(n))^\delta v(n) - f(n, u(n))v(n)\Big] \quad \forall u, v \in E.\end{aligned} \tag{2.7}$$

Furthermore, the critical points of I in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by using the Mountain Pass Theorem. We recall it and a minimization theorem as follows.

Lemma 2.1 (see [15, 17]). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(iii) *there exists $e \in E \setminus \overline{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)), \tag{2.8}$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0 and $\Gamma = \{g \in C([0,1], E) : g(0) = 0,\ g(1) = e\}$.

Lemma 2.2. *For $u \in E$*

$$q\|u\|_{\infty}^{\delta+1} \leq \|u\|^{\delta+1}, \tag{2.9}$$

where $q = \inf_{n\in\mathbb{Z}} q(n)$.

Proof. Since $u \in E$, it follows that $\lim_{|n|\to\infty}|u(n)| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$|u(n^*)| = \max_{n\in\mathbb{Z}}|u(n)|. \tag{2.10}$$

So, we have

$$\|u\|_E^{\delta+1} \geq \sum_{n\in\mathbb{Z}} q(n)(u(n))^{\delta+1} \geq q\sum_{n\in\mathbb{Z}}|u(n)|^{\delta+1} \geq q\|u\|_{\infty}^{\delta+1}. \tag{2.11}$$

The proof is completed. □

Lemma 2.3. *Assume that* (F2) *and* (F3) *hold. Then for every* $(n, x) \in \mathbb{Z} \times \mathbb{R}$,

(i) $s^{-\mu}F_1(n, sx)$ *is nondecreasing on* $(0, +\infty)$;

(ii) $s^{-\varrho}F_2(n, sx)$ *is nonincreasing on* $(0, +\infty)$.

The proof of Lemma 2.3 is routine and so we omit it.

Lemma 2.4 (see [18]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

3. Proofs of Theorems

Proof of Theorem 1.1. In our case, it is clear that $I(0) = 0$. We show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_k)| \leq c, \quad \|I'(u_k)\|_{E^*} \leq \varrho c \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

From (2.6), (2.7), (3.1), (F2), and (F3), we obtain

$$\begin{aligned}
&(\delta+1)c + (\delta+1)c\|u_k\| \\
&\quad\geq (\delta+1)I(u_k) - \frac{\delta+1}{\varrho}\langle I'(u_k), u_k\rangle \\
&\quad= \frac{\varrho-(\delta+1)}{\varrho}\|u_k\|^{\delta+1} + (\delta+1)\sum_{n\in\mathbb{Z}}\left[F_2(n,u_k(n)) - \frac{1}{\varrho}u_k(n)f_2(n,u_k(n))\right] \\
&\qquad - (\delta+1)\sum_{n\in\mathbb{Z}}\left[F_1(n,u_k(n)) - \frac{1}{\varrho}u_k(n)f_1(n,u_k(n))\right] \\
&\quad\geq \frac{\varrho-(\delta+1)}{\varrho}\|u_k\|^{\delta+1}, \quad k \in \mathbb{N}.
\end{aligned} \tag{3.2}$$

It follows that there exists a constant $A > 0$ such that

$$\|u_k\| \leq A \quad \text{for } k \in \mathbb{N}. \tag{3.3}$$

Then, u_k is bounded in E. Going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (F1), we can choose $\zeta > 0$ such that

$$|f(n,x)| \leq \varepsilon q(n)|x|^{\delta} \quad \text{for } n \in \mathbb{Z} \setminus J,\ x \in \mathbb{R},\ |x| \leq \zeta. \tag{3.4}$$

Since $q(n) \to \infty$, we can also choose an integer $\Pi > \max\{|k| : k \in J\}$ such that

$$q(n) \geq \frac{A^{\delta+1}}{\zeta^{\delta+1}}, \quad |n| \geq \Pi. \tag{3.5}$$

By (3.3) and (3.5), we have

$$|u_k(n)|^{\delta+1} = \frac{1}{q(n)} q(n)|u_k(n)|^{\delta+1} \leq \frac{\zeta^{\delta+1}}{A^{\delta+1}} \|u_k\|^{\delta+1} \leq \zeta^{\delta+1}, \quad \text{for } |n| \geq \Pi,\ k \in \mathbb{N}. \tag{3.6}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(n)$ converges to $u_0(n)$ pointwise for all $n \in \mathbb{Z}$, that is,

$$\lim_{k \to \infty} u_k(n) = u_0(n), \quad \forall n \in \mathbb{Z}. \tag{3.7}$$

Hence, we have by (3.6) and (3.7)

$$|u_0(n)| \leq \zeta, \quad \text{for } |n| \geq \Pi. \tag{3.8}$$

It follows from (3.7) and the continuity of $f(n,x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} |f(n,u_k(n)) - f(n,u_0(n))||u_k(n) - u_0(n)| < \varepsilon, \quad \text{for } k \geq k_0. \tag{3.9}$$

On the other hand, it follows from (3.3), (3.4), (3.6), and (3.8) that

$$\begin{aligned}
&\sum_{|n|>\Pi} \left| f(n,u_k(n)) - f(n,u_0(n)) \right| \, |u_k(n) - u_0(n)| \\
&\quad \le \sum_{|n|>\Pi} \left(\left| f(n,u_k(n)) \right| + \left| f(n,u_0(n)) \right| \right) (|u_k(n)| + |u_0(n)|) \\
&\quad \le \varepsilon \sum_{|n|>\Pi} q(n) \left(|u_k(n)|^{\delta} + |u_0(n)|^{\delta} \right) (|u_k(n)| + |u_0(n)|) \\
&\quad \le 2\varepsilon \sum_{|n|>\Pi} q(n) \left(|u_k(n)|^{\delta+1} + |u_0(n)|^{\delta+1} \right) \\
&\quad \le 2\varepsilon \left(\|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} \right) \\
&\quad \le 2\varepsilon \left(A^{\delta+1} + \|u_0\|^{\delta+1} \right), \quad k \in \mathbb{N}.
\end{aligned} \tag{3.10}$$

Since ε is arbitrary, combining (3.9) with (3.10), we get

$$\sum_{n\in\mathbb{Z}} \left| f(n,u_k(n)) - f(n,u_0(n)) \right| |u_k(n) - u_0(n)| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.11}$$

It follows from (2.7) and the Hölder's inequality that

$$\begin{aligned}
&\left\langle I'(u_k) - I'(u_0), u_k - u_0 \right\rangle \\
&= \sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta} (\Delta u_k(n-1) - \Delta u_0(n-1)) \\
&\quad + \sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta} (u_k(n) - u_0(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta} (\Delta u_k(n-1) - \Delta u_0(n-1)) \\
&\quad - \sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta} (u_k(n) - u_0(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} \left(f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n) \right) \\
&= \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta} \Delta u_0(n-1) \\
&\quad - \sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta} u_0(n) \\
&\quad - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta} \Delta u_k(n-1) - \sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta} u_k(n) \\
&\quad - \sum_{n\in\mathbb{Z}} \left(f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n) \right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} - \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta+1}\right)^{1/\delta+1} \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta+1}\right)^{\delta/\delta+1} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta+1}\right)^{1/\delta+1} \left(\sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta+1}\right)^{\delta/\delta+1} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta+1}\right)^{1/\delta+1} \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta+1}\right)^{\delta/\delta+1} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta+1}\right)^{1/\delta+1} \left(\sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta+1}\right)^{\delta/\delta+1} \\
&\quad - \sum_{n\in\mathbb{Z}} (f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n)) \\
&\geq \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u_0(n-1))^{\delta+1} + q(n)(u_0(n))^{\delta+1}\right]\right)^{1/\delta+1} \\
&\quad \times \left(\sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u_k(n-1))^{\delta+1} + q(n)(u_k(n))^{\delta+1}\right]\right)^{\delta/\delta+1} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u_k(n-1))^{\delta+1} + q(n)(u_k(n))^{\delta+1}\right]\right)^{1/\delta+1} \\
&\quad \times \left(\sum_{n\in\mathbb{Z}} \left[p(n)(\Delta u_0(n-1))^{\delta+1} + q(n)(u_0(n))^{\delta+1}\right]\right)^{\delta/\delta+1} \\
&\quad - \sum_{n\in\mathbb{Z}} (f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n)) \\
&= \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} - \|u_0\|\|u_k\|^{\delta} - \|u_k\|\|u_0\|^{\delta} \\
&\quad - \sum_{n\in\mathbb{Z}} (f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n)) \\
&= \left(\|u_k\|^{\delta} - \|u_0\|^{\delta}\right)(\|u_k\| - \|u_0\|) \\
&\quad - \sum_{n\in\mathbb{Z}} (f(n,u_k(n)) - f(n,u_0(n)), u_k(n) - u_0(n)).
\end{aligned}
\tag{3.12}
$$

Since $\langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \to 0$, it follows from (3.11) and (3.12) that $u_k \to u_0$ in E. Hence, I satisfies the (PS)-condition.

We now show that there exist constants $\rho, \alpha > 0$ such that I satisfies assumption (ii) of Lemma 2.1. By (F1), there exists $\eta \in (0,1)$ such that

$$\left|f(n,x)\right| \le \frac{1}{2} q(n) |x|^{\delta} \quad \text{for } n \in \mathbb{Z} \setminus J,\ x \in \mathbb{R},\ |x| \le \eta. \tag{3.13}$$

It follows from $F(n,0) \equiv 0$ that

$$|F(n,x)| \le \frac{1}{2(\delta+1)} q(n) |x|^{\delta+1} \quad \text{for } n \in \mathbb{Z} \setminus J,\ x \in \mathbb{R},\ |x| \le \eta. \tag{3.14}$$

Set

$$M = \sup\left\{ \frac{F_1(n,x)}{q(n)} \mid n \in J,\ x \in \mathbb{R},\ |x| = 1 \right\}, \tag{3.15}$$

$$v = \min\left\{ \left(\frac{1}{2(\delta+1)M + 1} \right)^{\delta+1-\mu}, \eta \right\}. \tag{3.16}$$

If $\|u\| = q^{1/(\delta+1)}\, v := \rho$, then by Lemma 2.2, $|u(n)| \le v \le \eta < 1$ for $n \in \mathbb{Z}$, we have by (q), (3.15), and Lemma 2.3(i) that

$$\begin{aligned} \sum_{n \in J} F_1(n, u(n)) &\le \sum_{n \in J, u(n) \ne 0} F_1\left(n, \frac{u(n)}{|u(n)|}\right) |u(n)|^{\mu} \\ &\le M \sum_{n \in J} q(n) |u(n)|^{\mu} \\ &\le M v^{\mu-\delta-1} \sum_{n \in J} q(n) |u(n)|^{\delta+1} \\ &\le \frac{1}{2(\delta+1)} \sum_{n \in J} q(n) |u(n)|^{\delta+1}. \end{aligned} \tag{3.17}$$

Set $\alpha = (1/2(\delta+1)) q v^{\delta+1}$. Hence, from (2.6), (3.14), (3.17), (q), and (F1), we have

$$\begin{aligned} I(u) &= \frac{1}{\delta+1} \|u\|^{\delta+1} - \sum_{n \in \mathbb{Z}} F(n, u(n)) \\ &= \frac{1}{\delta+1} \|u\|^{\delta+1} - \sum_{n \in \mathbb{Z} \setminus J} F(n, u(n)) - \sum_{n \in J} F(n, u(n)) \\ &\ge \frac{1}{\delta+1} \|u\|^{\delta+1} - \frac{1}{2(\delta+1)} \sum_{n \in \mathbb{Z} \setminus J} q(n) |u(n)|^{\delta+1} - \sum_{n \in J} F_1(n, u(n)) \\ &\ge \frac{1}{\delta+1} \|u\|^{\delta+1} - \frac{1}{2(\delta+1)} \sum_{n \in \mathbb{Z} \setminus J} q(n) |u(n)|^{\delta+1} - \frac{1}{2(\delta+1)} \sum_{n \in J} q(n) |u(n)|^{\delta+1} \\ &\ge \frac{1}{2(\delta+1)} \|u\|^{\delta+1} \\ &= \alpha. \end{aligned} \tag{3.18}$$

Equation (3.18) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, that is, I satisfies assumption (ii) of Lemma 2.1. Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. For any $u \in E$, it follows from (2.9) and Lemma 2.3(ii) that

$$\begin{aligned}
\sum_{n=-2}^{2} F_2(n, u(n)) &= \sum_{\{n\in[-2,2]:|u(n)|>1\}} F_2(n, u(n)) + \sum_{\{n\in[-2,2]:|u(n)|\leq 1\}} F_2(n, u(n)) \\
&\leq \sum_{\{n\in[-2,2]:|u(n)|>1\}} F_2\left(n, \frac{u(n)}{|u(n)|}\right)|u(n)|^{\varrho} + \sum_{n=-2}^{2} \max_{|x|\leq 1}|F_2(n,x)| \\
&\leq \|u\|_{\infty}^{\varrho} \sum_{n=-2}^{2} \max_{|x|=1}|F_2(n,x)| + \sum_{n=-2}^{2} \max_{|x|\leq 1}|F_2(n,x)| \\
&\leq q^{-\varrho/(\delta+1)}\|u\|^{\varrho} \sum_{n=-2}^{2} \max_{|x|=1}|F_2(n,x)| + \sum_{n=-2}^{2} \max_{|x|\leq 1}|F_2(n,x)| \\
&= M_1\|u\|^{\varrho} + M_2,
\end{aligned} \tag{3.19}$$

where

$$M_1 = q^{-\varrho/(\delta+1)} \sum_{n=-2}^{2} \max_{|x|=1}|F_2(n,x)|, \qquad M_2 = \sum_{n=-2}^{2} \max_{|x|\leq 1}|F_2(n,x)|. \tag{3.20}$$

Take $\omega \in E$ such that

$$|\omega(n)| = \begin{cases} 1, & \text{for } |n| \leq 1, \\ 0, & \text{for } |n| \geq 2, \end{cases} \tag{3.21}$$

and $|\omega(n)| \leq 1$ for $|n| \in (1,2)$. For $\sigma > 1$, by Lemma 2.3(i) and (3.21), we have

$$\sum_{n=-1}^{1} F_1(n, \sigma\omega(n)) \geq \sigma^{\mu} \sum_{n=-1}^{1} F_1(n, \omega(n)) = m\sigma^{\mu}, \tag{3.22}$$

where $m = \sum_{n=-1}^{1} F_1(n, \omega(n)) > 0$. By (2.6), (3.19), (3.21), and (3.22), we have for $\sigma > 1$

$$\begin{aligned}
I(\sigma\omega) &= \frac{1}{\delta+1}\|\sigma\omega\|^{\delta+1} + \sum_{n\in\mathbb{Z}}[F_2(n, \sigma\omega(n)) - F_1(n, \sigma\omega(n))] \\
&\leq \frac{\sigma^{\delta+1}}{\delta+1}\|\omega\|^{\delta+1} + \sum_{n=-2}^{2} F_2(n, \sigma\omega(n)) - \sum_{n=-1}^{1} F_1(n, \sigma\omega(n)) \\
&\leq \frac{\sigma^{\delta+1}}{\delta+1}\|\omega\|^{\delta+1} + M_1\sigma^{\varrho}\|\omega\|^{\varrho} + M_2 - m\sigma^{\mu}.
\end{aligned} \tag{3.23}$$

Since $\mu > \varrho \geq \delta + 1$ and $m > 0$, (3.23) implies that there exists $\sigma_0 > 1$ such that $\|\sigma_0 \omega\| > \rho$ and $I(\sigma_0\omega) < 0$. Set $e = \sigma_0\omega(n)$. Then $e \in E$, $\|e\| = \|\sigma_0\omega\| > \rho$, and $I(e) = I(\sigma_0\omega) < 0$. By Lemma 2.1, I possesses a critical value $d \geq \alpha$ given by

$$d = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\big(g(s)\big), \tag{3.24}$$

where

$$\Gamma = \big\{g \in C([0,1], E) : g(0) = 0,\ g(1) = e\big\}. \tag{3.25}$$

Hence, there exists $u^* \in E$ such that

$$I(u^*) = d, \qquad I'(u^*) = 0. \tag{3.26}$$

Then function u^* is a desired classical solution of (1.1). Since $d > 0$, u^* is a nontrivial homoclinic solution. The proof is complete. □

Proof of Theorem 1.2. In the proof of Theorem 1.1, the condition that $F_2(n, x) \geq 0$ for $(n, x) \in J\times\mathbb{R}$, $|x| \leq 1$ in (F1), is only used in the the proofs of assumption (ii) of Lemma 2.1. Therefore, we only proves assumption (ii) of Lemma 2.1 still hold that using (F1′) instead of (F1). By (F1′), there exists $\eta \in (0, 1)$ such that

$$\big|f(n,x)\big| \leq \frac{1}{2}q(n)|x|^{\delta} \quad \text{for } (n,x) \in \mathbb{Z}\times\mathbb{R},\ |x| \leq \eta. \tag{3.27}$$

Since $F(n, 0) \equiv 0$, it follows that

$$|F(n,x)| \leq \frac{1}{2(\delta+1)}q(n)|x|^{\delta+1} \quad \text{for } (n,x) \in \mathbb{Z}\times\mathbb{R},\ \ |x| \leq \eta. \tag{3.28}$$

If $\|u\| = q^{1/(\delta+1)}\eta := \rho$, then by Lemma 2.2, $|u(n)| \leq \eta$ for $n \in \mathbb{Z}$. Set $\alpha = q\eta^{\delta+1}/2(\delta + 1)$. Hence, from (2.6) and (3.28), we have

$$\begin{aligned}
I(u) &= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n,u(n)) \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}} q(n)(u(n))^{\delta+1} \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\|u\|^{\delta+1} \\
&= \frac{1}{2(\delta+1)}\|u\|^{\delta+1} \\
&= \alpha.
\end{aligned} \tag{3.29}$$

Equation (3.29) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, that is, assumption (ii) of Lemma 2.1 holds. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.4. In view of Lemma 2.4, $I \in C^1(E, \mathbb{R})$. We first show that I is bounded from below. By (F4), (2.6), and Hölder inequality, we have

$$
\begin{aligned}
I(u) &= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n,u(n)) \\
&= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{\mathbb{Z}(|u(n)|\le 1)} F(n,u(n)) - \sum_{\mathbb{Z}(|u(n)|>1)} F(n,u(n)) \\
&\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{\mathbb{Z}(|u(n)|\le 1)} a_1(n)|u(n)|^{\gamma_1} - \sum_{\mathbb{Z}(|u(n)|>1)} a_2(n)|u(n)|^{\gamma_2} \\
&\ge \frac{1}{\delta+1}\|u\|^{\delta+1} \\
&\quad - q^{-\gamma_1/(\delta+1)}\left(\sum_{\mathbb{Z}(|u(n)|\le 1)} |a_1(n)|^{(\delta+1)/(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \\
&\quad \times \left(\sum_{\mathbb{Z}(|u(n)|\le 1)} q(n)(u(n))^{\delta+1}\right)^{\gamma_1/(\delta+1)} \\
&\quad - q^{-\gamma_1/(\delta+1)}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{(\delta+1)/(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \\
&\quad \times \left(\sum_{\mathbb{Z}(|u(n)|>1)} |u(n)|^{(\delta+1)(\gamma_2-\gamma_1)/\gamma_1} q(n)(u(n))^{\delta+1}\right)^{\gamma_1/(\delta+1)} \\
&\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - q^{-\gamma_1/(\delta+1)}\left(\sum_{\mathbb{Z}(|u(n)|\le 1)} |a_1(n)|^{(\delta+1)(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \|u\|^{\gamma_1} \\
&\quad - q^{-\gamma_1/(\delta+1)}\|u\|_\infty^{\gamma_2-\gamma_1}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{(\delta+1)(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \|u\|^{\gamma_1} \\
&\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - q^{-\gamma_1/(\delta+1)}\left(\sum_{\mathbb{Z}(|u(n)|\le 1)} |a_1(n)|^{(\delta+1)/(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \|u\|^{\gamma_1} \\
&\quad - q^{-\gamma_1/(\delta+1)} q^{(\gamma_1-\gamma_2)/(\delta+1)}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{(\delta+1)/(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} \|u\|^{\gamma_2} \\
&\ge \frac{1}{\delta+1}\|u\|^{\delta+1} - q^{-\gamma_1/(\delta+1)}\|a_1\|_{(\delta+1)/(\delta+1-\gamma_1)}\|u\|^{\gamma_1} \\
&\quad - q^{-\gamma_2/(\delta+1)}\|a_2\|_{(\delta+1)/(\delta+1-\gamma_1)}\|u\|^{\gamma_2}.
\end{aligned}
\tag{3.30}
$$

Since $1 < \gamma_1 < \gamma_2 < \delta + 1$, (3.30) implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then by (2.6), (2.9), and (3.30), there exists a constant $A > 0$ such that

$$\|u_k\|_\infty \le q^{-1/(\delta+1)}\|u_k\| \le A, \quad k \in \mathbb{N}. \tag{3.31}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. It is easy to verify that $u_k(n)$ converges to $u_0(n)$ pointwise for all $n \in \mathbb{Z}$, that is,

$$\lim_{k\to\infty} u_k(n) = u_0(n), \quad \forall n \in \mathbb{Z}. \tag{3.32}$$

Hence, we have, by (3.31) and (3.32),

$$\|u_0\|_\infty \le A. \tag{3.33}$$

By (F5), there exists $M_2 > 0$ such that

$$\varphi(|x|) \le M_2|x|^{\gamma_1-1}, \quad \forall x \in \mathbb{R}, \ |x| \le A. \tag{3.34}$$

For any given number $\varepsilon > 0$, by (F5), we can choose an integer $\Pi > 0$ such that

$$\left(\sum_{|n|>\Pi} |b(n)|^{(\delta+1)/(\delta+1-\gamma_1)}\right)^{(\delta+1-\gamma_1)/(\delta+1)} < \varepsilon. \tag{3.35}$$

It follows from (3.32) and the continuity of $f(n,x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} \left|f(n,u_k(n)) - f(n,u_0(n))\right||u_k(n) - u_0(n)| < \varepsilon, \quad \text{for } k \ge k_0. \tag{3.36}$$

On the other hand, it follows from (3.31), (3.33), (3.34), (3.35), and (F5) that

$$\begin{aligned}
&\sum_{|n|>\Pi} \left| f(n, u_k(n)) - f(n, u_0(n)) \right| |u_k(n) - u_0(n)| \\
&\quad \leq \sum_{|n|>\Pi} |b(n)| \left[\varphi(|u_k(n)|) + \varphi(|u_0(n)|) \right] (|u_k(n)| + |u_0(n)|) \\
&\quad \leq M_2 \sum_{|n|>\Pi} |b(n)| \left(|u_k(n)|^{\gamma_1 - 1} + |u_0(n)|^{\gamma_1 - 1} \right) (|u_k(n)| + |u_0(n)|) \\
&\quad \leq 2M_2 \sum_{|n|>\Pi} |b(n)| \left(|u_k(n)|^{\gamma_1} + |u_0(n)|^{\gamma_1} \right) \\
&\quad \leq 2M_2 q^{-\gamma_1/(\delta+1)} \left(\sum_{|n|>\Pi} |b(n)|^{(\delta+1)/(\delta+1-\gamma_1)} \right)^{(\delta+1-\gamma_1)/(\delta+1)} \left(\|u_k\|^{\gamma_1} + \|u_0\|^{\gamma_1} \right) \\
&\quad \leq 2M_2 q^{-\gamma_1/(\delta+1)} \left(\sum_{|n|>\Pi} |b(n)|^{(\delta+1)/(\delta+1-\gamma_1)} \right)^{(\delta+1-\gamma_1)/\delta+1} \left[q^{\gamma_1/(\delta+1)} A^{\gamma_1} + \|u_0\|^{\gamma_1} \right] \\
&\quad \leq 2M_2 q^{-\gamma_1/(\delta+1)} \left[q^{\gamma_1/(\delta+1)} A^{\gamma_1} + \|u_0\|^{\gamma_1} \right] \varepsilon, \quad k \in \mathbb{N}.
\end{aligned} \tag{3.37}$$

Since ε is arbitrary, combining (3.36) with (3.37), we get

$$\sum_{n \in \mathbb{Z}} \left(f(n, u_k(n)) - f(n, u_0(n)), u_k(n) - u_0(n) \right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.38}$$

Similar to the proof of Theorem 1.1, it follows from (3.12) that

$$\begin{aligned}
\left\langle I'(u_k) - I'(u_0), u_k - u_0 \right\rangle \geq{} & \left(\|u_k\|^{\delta} - \|u_0\|^{\delta} \right) (\|u_k\| - \|u_0\|) \\
& - \sum_{n \in \mathbb{Z}} \left(f(n, u_k(n)) - f(n, u_0(n)), u_k(n) - u_0(n) \right).
\end{aligned} \tag{3.39}$$

Since $\langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \to 0$, it follows from (3.38) and (3.39) that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition.

By Lemma 2.4, $c = \inf_E I(u)$ is a critical value of I, that is, there exists a critical point $u^* \in E$ such that $I(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_0(n_0) = 1$ and $u_0(n) = 0$ for $n \neq n_0$. Then by (F4) and (F6), we have

$$\begin{aligned} I(su_0) &= \frac{s^{\delta+1}}{\delta+1}\|u_0\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n, su_0) \\ &= \frac{s^{\delta+1}}{\delta+1}\|u_0\|^{\delta+1} - F(n_0, su_0(n_0)) \\ &\leq \frac{s^{\delta+1}}{\delta+1}\|u_0\|^{\delta+1} - \eta s^{\gamma_3}|u_0(n_0)|^{\gamma_3}, \quad 0 < s < 1. \end{aligned} \tag{3.40}$$

Since $1 < \gamma_3 < \delta + 1$, it follows from (3.40) that $I(su_0) < 0$ for $s > 0$ small enough. Hence $I(u^*) = c < 0$, therefore u^* is nontrivial critical point of I, and so $u^* = u^*(n)$ is a nontrivial homoclinic solution of (1.1). The proof is complete. □

Proof of Corollary 1.5. Obviously, (F7) and (F8) imply that (F4) holds, and (F7) and (F9) imply that (F5) holds with $a_1(n) = a_2(n) = b(n) = |a(n)|$. In addition, by (F7) and (F8), we have

$$F(n_0, x) = a(n_0)V(x) \geq M'a(n_0)|x|^{\gamma_3}, \quad \forall x \in \mathbb{R}, \ |x| \leq 1. \tag{3.41}$$

This shows that (F6) holds also. Hence, by Theorem 1.4, the conclusion of Corollary 1.5 is true. The proof is complete. □

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. In (1.1), let $p(n) > 0$ and

$$F(n, x) = q(n)\left[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - (2-|n|)|x|^{\varrho_1} - (2-|n|)|x|^{\varrho_2}\right], \tag{4.1}$$

where $q : \mathbb{Z} \to (0, \infty)$ such that $q(n) \to +\infty$ as $|n| \to +\infty$, $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > \delta + 1$, $a_1, a_2 > 0$. Let $\mu = \mu_2$, $\varrho = \varrho_1$, $J = \{-2, -1, 0, 1, 2\}$, and

$$F_1(n, x) = q(n)\left(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\right), \qquad F_2(n, x) = q(n)\left[(2-|n|)|x|^{\varrho_1} + (2-|n|)|x|^{\varrho_2}\right]. \tag{4.2}$$

Then it is easy to verify that all conditions of Theorem 1.1 are satisfied. By Theorem 1.1, (1.1) has at least a nontrivial homoclinic solution.

Example 4.2. In (1.1), let $p(n) > 0$, $q(n) > 0$ for all $n \in \mathbb{Z}$ and $\lim_{|n|\to+\infty} q(n) = +\infty$, and let

$$F(n, x) = q(n)\left(\sum_{i=1}^{m_1} a_i|x|^{\mu_i} - \sum_{j=1}^{m_2} b_j|x|^{\varrho_j}\right), \tag{4.3}$$

where $\mu_1 > \mu_2 > \cdots > \mu_{m_1} > \varrho_1 > \varrho_2 > \cdots > \varrho_{m_2} > \delta + 1$, $a_i, b_j > 0$, $i = 1, 2, \ldots, m_1$, and $j = 1, 2, \ldots, m_2$. Let $\mu = \mu_{m_1}$, $\varrho = \varrho_1$, and

$$F_1(n,x) = q(n) \sum_{i=1}^{m_1} a_i |x|^{\mu_i}, \qquad F_2(n,x) = q(n) \sum_{j=1}^{m_2} b_j |x|^{\varrho_j}. \tag{4.4}$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, (1.1) has at least a nontrivial homoclinic solution.

Example 4.3. In (1.1), let $q : \mathbb{Z} \to (0, \infty)$ such that $q(n) \to +\infty$ as $|n| \to +\infty$ and

$$F(n,x) = \frac{\cos n}{1 + |n|} |x|^{4/3} + \frac{\sin n}{1 + |n|} |x|^{3/2}. \tag{4.5}$$

Then

$$\begin{gathered} f(n,x) = \frac{4 \cos n}{3(1 + |n|)} |x|^{-2/3} x + \frac{3 \sin n}{2(1 + |n|)} |x|^{-1/2} x, \\ |F(n,x)| \le \frac{2|x|^{4/3}}{1 + |n|}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R},\ |x| \le 1, \\ |F(n,x)| \le \frac{2|x|^{3/2}}{1 + |n|}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R},\ |x| \ge 1, \\ |f(n,x)| \le \frac{8|x|^{1/3} + 9|x|^{1/2}}{6(1 + |n|)}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}. \end{gathered} \tag{4.6}$$

We can choose n_0 such that

$$\cos n_0 > 0, \qquad \sin n_0 > 0. \tag{4.7}$$

Let

$$\eta = \frac{\cos n_0 + \sin n_0}{1 + |n_0|}. \tag{4.8}$$

Then

$$F(n_0, x) \ge \eta |x|^{3/2}, \quad \forall x \in \mathbb{R},\ |x| \le 1. \tag{4.9}$$

These show that all conditions of Theorem 1.4 are satisfied, where

$$1 < \frac{4}{3} = \gamma_1 < \gamma_2 = \gamma_3 = \frac{3}{2} < \delta + 1, \qquad a_1(n) = a_2(n) = b(n) = \frac{2}{1 + |n|}, \qquad \varphi(s) = \frac{8s^{1/3} + 9s^{1/2}}{12}. \tag{4.10}$$

By Theorem 1.4, (1.1) has at least a nontrivial homoclinic solution.

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