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Research Article

Solutions of Linear Impulsive Differential Systems Bounded on the Entire Real Axis

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We consider the problem of existence and structure of solutions bounded on the entire real axis of nonhomogeneous linear impulsive differential systems. Under assumption that the corresponding homogeneous system is exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$ and by using the theory of pseudoinverse matrices, we establish necessary and sufficient conditions for the indicated problem.

The research in the theory of differential systems with impulsive action was originated by Myshkis and Samoilenko [1], Samoilenko and Perestyuk [2], Halanay and Wexler [3], and Schwabik et al. [4]. The ideas proposed in these works were developed and generalized in numerous other publications [5]. The aim of this contribution is, using the theory of impulsive differential equations, using the well-known results on the splitting index by Sacker [6] and by Palmer [7] on the Fredholm property of the problem of bounded solutions and using the theory of pseudoinverse matrices [5, 8], to investigate, in a relevant space, the existence of solutions bounded on the entire real axis of linear differential systems with impulsive action.

We consider the problem of existence and construction of solutions bounded on the entire real axis of linear systems of ordinary differential equations with impulsive action at fixed points of time

$$\begin{aligned} \dot{x} &= A(t)x + f(t), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= \gamma_i, \quad i \in \mathbb{Z},\ t, \tau_i \in \mathbb{R},\ \gamma_i \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $A(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ is an $n \times n$ matrix of functions; $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ is an $n \times 1$ vector function; $\mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ is the Banach space of real vector functions continuous for $t \in \mathbb{R}$

with discontinuities of the first kind at $t = \tau_i$; γ_i are n-dimensional column constant vectors; $\cdots < \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 < \cdots$.

The solution $x(t)$ of the problem (1) is sought in the Banach space of n-dimensional piecewise continuously differentiable vector functions with discontinuities of the first kind at $t = \tau_i$: $x(t) \in \mathrm{BC}^1(\mathbb{R} \setminus \{\tau_i\}_I)$.

Parallel with the nonhomogeneous impulsive system (1) we consider the homogeneous system

$$\dot{x} = A(t)x, \quad t \in \mathbb{R}, \tag{2}$$

which is the homogeneous system without impulses.

Assume that the homogeneous system (2) is exponentially dichotomous (e-dichotomous) on semiaxes $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$; i.e. there exist projectors P and Q ($P^2 = P$, $Q^2 = Q$) and constants $K_i \geq 1$, $\alpha_i > 0$ ($i = 1, 2$) such that the following inequalities are satisfied:

$$\begin{gathered}
\left\| X(t)PX^{-1}(s) \right\| \leq K_1 e^{-\alpha_1(t-s)}, \quad t \geq s, \\
\left\| X(t)(I-P)X^{-1}(s) \right\| \leq K_1 e^{-\alpha_1(s-t)}, \quad s \geq t, \ t, s \in \mathbb{R}_+, \\
\left\| X(t)QX^{-1}(s) \right\| \leq K_2 e^{-\alpha_2(t-s)}, \quad t \geq s, \\
\left\| X(t)(I-Q)X^{-1}(s) \right\| \leq K_2 e^{-\alpha_2(s-t)}, \quad s \geq t, \ t, s \in \mathbb{R}_-,
\end{gathered} \tag{3}$$

where $X(t)$ is the normal fundamental matrix of system (2).

By using the results developed in [5] for problems without impulses, the general solution of the problem (1) bounded on the semiaxes has the form

$$x(t,\xi) = X(t) \begin{cases} P\xi + \displaystyle\int_0^t PX^{-1}(s)f(s)ds - \int_t^\infty (I-P)X^{-1}(s)f(s)ds \\ \quad + \displaystyle\sum_{i=1}^{j} PX^{-1}(\tau_i)\gamma_i - \sum_{i=j+1}^{\infty} (I-P)X^{-1}(\tau_i)\gamma_i, & t \geq 0; \\ (I-Q)\xi + \displaystyle\int_{-\infty}^t QX^{-1}(s)f(s)ds - \int_t^0 (I-Q)X^{-1}(s)f(s)ds \\ \quad + \displaystyle\sum_{i=-\infty}^{-(j+1)} QX^{-1}(\tau_i)\gamma_i - \sum_{i=-j}^{-1} (I-Q)X^{-1}(\tau_i)\gamma_i, & t \leq 0. \end{cases} \tag{4}$$

For getting the solution $x(t) \in \mathrm{BC}^1(\mathbb{R} \setminus \{\tau_i\}_I)$ bounded on the entire axis, we assume that it has continuity in $t = 0$:

$$x(0+,\xi) - x(0-,\xi) = \gamma_0 = 0 \tag{5}$$

or

$$
\begin{aligned}
P\xi - \int_0^{\infty} (I-P)X^{-1}(s)f(s)ds - \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \\
= (I-Q)\xi + \int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i.
\end{aligned}
\tag{6}
$$

Thus, the solution (4) will be bounded on $\mathbb{R}$ if and only if the constant vector $\xi \in \mathbb{R}^n$ is the solution of the algebraic system:

$$
D\xi = \int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i,
\tag{7}
$$

where D is an $n \times n$ matrix, $D := P - (I-Q)$. The algebraic system (7) is solvable if and only if the condition

$$
\begin{aligned}
P_{D^*}\Bigg\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \\
+ \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i\Bigg\} = 0
\end{aligned}
\tag{8}
$$

is satisfied, where P_{D^*} is the $n \times n$ matrix-orthoprojector; $P_{D^*} : \mathbb{R}^n \to N(D^*)$.

Therefore, the constant $\xi \in \mathbb{R}^n$ in the expression (4) has the form

$$
\begin{aligned}
\xi = D^+\Bigg\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \\
+ \sum_{i=-\infty}^{-1} X(t)QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}X(t)(I-P)X^{-1}(\tau_i)\gamma_i\Bigg\} + P_D c, \quad \forall c \in \mathbb{R}^n,
\end{aligned}
\tag{9}
$$

where P_D is the $n \times n$ matrix-orthoprojector; $P_D : \mathbb{R}^n \to N(D)$; D^+ is a Moore-Penrose pseudoinverse matrix to D. Since $P_{D^*}D = 0$, we have $P_{D^*}Q = P_{D^*}(I-P)$. Let

$$
d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I-P)] \le n.
\tag{10}
$$

Then we denote by $[P_{D^*}Q]_d$ a $d \times n$ matrix composed of a complete system of d linearly independent rows of the matrix $[P_{D^*}Q]$ and by $H_d(t) = [P_{D^*}Q]_d X^{-1}(t)$ a $d \times n$ matrix.

Thus, the necessary and sufficient condition for the existence of the solution of problem (1) has the form

$$\int_{-\infty}^{\infty} H_d(t) f(t) dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)\gamma_i = 0 \tag{11}$$

and consists of d linearly independent conditions.

If we substitute the constant $\xi \in \mathbb{R}^n$ given by relation (9) into (4), we get the general solution of problem (1) in the form

$$x(t,c) = X(t)\begin{cases} PP_Dc + \int_0^t PX^{-1}(s)f(s)ds - \int_t^{\infty}(I-P)X^{-1}(s)f(s)ds \\ \quad + \sum_{i=1}^{j} PX^{-1}(\tau_i)\gamma i - \sum_{i=j+1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \\ \quad + PD^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \qquad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \geq 0; \\ (I-Q)P_Dc + \int_{-\infty}^{t} QX^{-1}(s)f(s)ds - \int_t^{0}(I-Q)X^{-1}(s)f(s)ds \\ \quad + \sum_{i=-\infty}^{-(j+1)} QX^{-1}(\tau_i)\gamma_i - \sum_{i=-j}^{-1}(I-Q)X^{-1}(\tau_i)\gamma_i \\ \quad + (I-Q)D^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \qquad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \leq 0. \end{cases} \tag{12}$$

Since $DP_D = 0$, we have $PP_D = (I-Q)P_D$. Let

$$r = \operatorname{rank}[PP_D] = \operatorname{rank}[(I-Q)P_D] \leq n. \tag{13}$$

Then we denote by $[PP_D]_r$ an $n \times r$ matrix composed of a complete system of r linearly independent columns of the matrix $[PP_D]$.

Thus, we have proved the following statement.

Theorem 1. *Assume that the linear nonhomogeneous impulsive differential system* (1) *has the corresponding homogeneous system* (2) *e-dichotomous on the semiaxes* $\mathbb{R}_- = (-\infty, 0]$ *and* $\mathbb{R}_+ = [0,\infty)$ *with projectors* P *and* Q*, respectively. Then the homogeneous system* (2) *has exactly* r ($r = \operatorname{rank} PP_D = \operatorname{rank}(I-Q)P_D,\ D = P-(I-Q)$) *linearly independent solutions bounded on the entire real axis. If nonhomogenities* $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ *and* $\gamma_i \in \mathbb{R}^n$ *satisfy* d ($d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I-P)]$) *linearly independent conditions* (11)*, then the nonhomogeneous system* (1) *possesses an r-parameter family of linearly independent solutions bounded on the entire real axis $\mathbb{R}$ in the form*

$$x(t,c_r) = X_r(t)c_r + \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t), \quad \forall c_r \in \mathbb{R}^r, \tag{14}$$

where

$$X_r(t) := X(t)[PP_D]_r = X(t)[(I-Q)P_D]_r \tag{15}$$

is an $n \times r$ matrix formed by a complete system of r linearly independent solutions of homogeneous problem (2) *and* $\left(G\left[\begin{smallmatrix} f \\ \gamma_i \end{smallmatrix}\right]\right)(t)$ *is the generalized Green operator of the problem of finding solutions of the impulsive problem* (1) *bounded on $\mathbb{R}$, acting upon $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ and $\gamma_i \in \mathbb{R}^n$, defined by the formula*

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t) = X(t)\begin{cases} \displaystyle\int_0^t PX^{-1}(s)f(s)ds - \int_t^\infty (I-P)X^{-1}(s)f(s)ds \\ \displaystyle\quad +\sum_{i=1}^{j} PX^{-1}(\tau_i)\gamma i - \sum_{i=j+1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad +PD^+\left\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^\infty (I-P)X^{-1}(s)f(s)ds\right. \\ \displaystyle\qquad\qquad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i\right\}, & t \geq 0; \\ \displaystyle\int_{-\infty}^t QX^{-1}(s)f(s)ds - \int_t^0 (I-Q)X^{-1}(s)f(s)ds \\ \displaystyle\quad + \sum_{i=-\infty}^{-(j+1)} QX^{-1}(\tau_i)\gamma_i - \sum_{i=-j}^{-1}(I-Q)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad +(I-Q)D^+\left\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^\infty (I-P)X^{-1}(s)f(s)ds\right. \\ \displaystyle\qquad\qquad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i\right\}, & t \leq 0. \end{cases} \tag{16}$$

The generalized Green operator (16) *has the following property:*

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0-0) - \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0+0) = \int_{-\infty}^{\infty} H(t)f(t)dt + \sum_{i=-\infty}^{\infty} H(\tau_i)\gamma_i, \tag{17}$$

where $H(t) = [P_{D^*}Q]X^{-1}(t)$.

We can also formulate the following corollaries.

Corollary 2. *Assume that the homogeneous system* (2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively, and such that* $PQ = QP = Q$*. In this case, the system* (2) *has r-parameter set of solutions bounded on* $\mathbb{R}$ *in the form* (14)*. The nonhomogeneous impulsive system* (1) *has for arbitrary* $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ *and* $\gamma_i \in \mathbb{R}^n$ *an r-parameter set of solutions bounded on* $\mathbb{R}$ *in the form*

$$x(t, c_r) = X_r(t)c_r + \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t), \quad \forall c_r \in \mathbb{R}^r, \tag{18}$$

where $\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t)$ *is the generalized Green operator* (16) *of the problem of finding bounded solutions of the impulsive system* (1) *with the property*

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0-0) - \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0+0) = 0. \tag{19}$$

Proof. Since $DP = (P - (I - Q))P = QP = Q$ and $P_{D^*}D = 0$, we have $P_{D^*}Q = P_{D^*}DP = 0$. Thus condition (11) for the existence of bounded solution of system (1) is satisfied for all $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ and $\gamma_i \in \mathbb{R}^n$. □

Corollary 3. *Assume that the homogenous system* (2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively, and such that* $PQ = QP = P$*. In this case, the system* (2) *has only trivial solution bounded on* $\mathbb{R}$*. If condition* (11) *is satisfied, then the nonhomogeneous impulsive system* (1) *possesses a unique solution bounded on* $\mathbb{R}$ *in the form*

$$x(t) = \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t), \tag{20}$$

where $\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t)$ *is the generalized Green operator* (16) *of the problem of finding bounded solutions of the impulsive system* (1).

Proof. Since $PD = (PP - (I - Q)) = PQ = P$ and $DP_D = 0$, we have $PP_D = PDP_D = 0$. By virtue of Theorem 1, we have $r = 0$ and thus the homogenous system (2) has only trivial solution bounded on $\mathbb{R}$. Moreover, the nonhomogeneous impulsive system (1) possesses a unique solution bounded on $\mathbb{R}$ for $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ and $\gamma_i \in \mathbb{R}^n$ satisfying the condition (11). □

Corollary 4. *Assume that the homogenous system* (2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively, and such that* $PQ = QP = P = Q$*. Then the system* (2) *is e-dichotomous on* $\mathbb{R}$ *and has only trivial solution bounded on* $\mathbb{R}$*. The nonhomogeneous impulsive system* (1) *has for arbitrary* $f(t) \in \mathrm{BC}(\mathbb{R} \setminus \{\tau_i\}_I)$ *and* $\gamma_i \in \mathbb{R}^n$ *a unique solution bounded on* $\mathbb{R}$ *in the form*

$$x(t) = \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t), \tag{21}$$

where $\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t)$ *is the Green operator* (16) $(D^+ = D^{-1})$ *of the problem of finding bounded solutions of the impulsive system* (1).

Proof. Since $PQ = QP = Q = P$ and det $D \neq 0$, we have $P_{D^*} = P_D = 0$, $D^+ = D^{-1}$. By virtue of Theorem 1, we have $r = d = 0$ and thus the homogenous system (2) has only trivial solution bounded on $\mathbb{R}$. Moreover, the nonhomogeneous impulsive system (1) possesses a unique solution bounded on $\mathbb{R}$ for all $f(t) \in BC(\mathbb{R} \setminus \{\tau_i\}_I)$ and $\gamma_i \in \mathbb{R}^n$. □

Regularization of Linear Problem

The condition of solvability (11) of impulsive problem (1) for solutions bounded on $\mathbb{R}$ enables us to analyze the problem of regularization of linear problem that is not solvable everywhere by adding an impulsive action.

Consider the problem of finding solutions bounded on the entire real axis of the system

$$\dot{x} = A(t)x + f(t), \quad A(t) \in BC(\mathbb{R}),\ f(t) \in BC(\mathbb{R}), \tag{22}$$

the corresponding homogeneous problem of which is e-dichotomous on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$. Assume that this problem has no solution bounded on $\mathbb{R}$ for some $f_0(t) \in BC(\mathbb{R})$; i.e. the solvability condition of (22) is not satisfied. This means that

$$\int_{-\infty}^{\infty} H_d(t) f_0(t)dt \neq 0. \tag{23}$$

In this problem, we introduce an impulsive action for $t = \tau_1 \in \mathbb{R}$ as follows:

$$\Delta x|_{t=\tau_1} = \gamma_1, \quad \gamma_1 \in \mathbb{R}^n, \tag{24}$$

and we consider the existence of solution of the impulsive problem (22)-(24) from the space $BC^1(\mathbb{R} \setminus \{\tau_1\}_I)$ bounded on the entire real axis. The parameter γ_1 is chosen from a condition similar to (11) guaranteeing that the impulsive problem (22)-(24) is solvable for any $f_0(t) \in BC(\mathbb{R})$ and some $\gamma_1 \in \mathbb{R}^n$:

$$\int_{-\infty}^{\infty} H_d(t) f_0(t)dt + H_d(\tau_1)\gamma_1 = 0, \tag{25}$$

where $H_d(\tau_1)$ is a $d \times n$ matrix, $H_d^+(\tau_1)$ is an $n \times d$ matrix pseudoinverse to the matrix $H_d(\tau_1)$, $P_{N(H_d^*)}$ is a $d \times d$ matrix (othoprojector), $P_{N(H_d^*)} : R^d \to N(H_d^*)$, and $P_{N(H_d)}$ is an $n \times n$ matrix (othoprojector), $P_{N(H_d)} : R^n \to N(H_d)$. The algebraic system (25) is solvable if and only if the condition

$$P_{N(H_d^*)} \left\{ \int_{-\infty}^{\infty} H_d(t) f_0(t)dt \right\} = 0 \tag{26}$$

is satisfied. Thus, Theorem 1 yields the following statement.

Corollary 5. *By adding an impulsive action, the problem of finding solutions bounded on* $\mathbb{R}$ *of linear system* (22)*, that is solvable not everywhere, can be made solvable for any* $f_0(t) \in \mathrm{BC}(\mathbb{R})$ *if and only if*

$$P_{N(H_d^*)} = 0 \quad or \quad \operatorname{rank} H_d(\tau_1) = d. \tag{27}$$

The indicated additional (regularizing) impulse γ_1 *should be chosen as follows:*

$$\gamma_1 = -H_d^+(\tau_1)\left\{\int_{-\infty}^{\infty} H_d(t) f_0(t)dt\right\} + P_{N(H_d)}c, \quad \forall c \in R^n. \tag{28}$$

So the impulsive action can be regarded as a control parameter which guarantees the solvability of not everywhere solvable problems.

Example 6. In this example we illustrate the assertions proved above.

Consider the impulsive system

$$\begin{gathered}\dot{x} = A(t)x + f(t), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} = \gamma_i = \begin{pmatrix}\gamma_i^{(1)} \\ \gamma_i^{(2)} \\ \gamma_i^{(3)}\end{pmatrix} \in \mathbb{R}^3, \quad t, \tau_i \in \mathbb{R}, \ i \in \mathbb{Z},\end{gathered} \tag{29}$$

where $A(t) = \operatorname{diag}\{-\tanh t, -\tanh t, \tanh t\}$, $f(t) = \operatorname{col}(f_1(t), f_2(t), f_3(t)) \in \mathrm{BC}(\mathbb{R})$. The normal fundamental matrix of the corresponding homogenous system

$$\dot{x} = A(t)x, \qquad t \neq \tau_i, \qquad \Delta x|_{t=\tau_i} = 0 \tag{30}$$

is

$$X(t) = \operatorname{diag}\left\{\frac{2}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}, \frac{e^t + e^{-t}}{2}\right\}, \tag{31}$$

and this system is e-dichotomous (as shown in [9]) on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors $P = \operatorname{diag}\{1, 1, 0\}$ and $Q = \operatorname{diag}\{0, 0, 1\}$, respectively. Thus, we have

$$\begin{gathered}D = 0, \qquad D^+ = 0, \quad P_{N(D)} = P_{N(D^*)} = I_3, \\ r = \operatorname{rank} PP_{N(D)} = 2, \qquad d = \operatorname{rank} P_{N(D^*)}Q = 1, \\ X_r(t) = \begin{pmatrix}\dfrac{2}{e^t + e^{-t}} & 0 \\ 0 & \dfrac{2}{e^t + e^{-t}} \\ 0 & 0\end{pmatrix}, \\ H_d(t) = \left(0, \ 0, \ \frac{2}{e^t + e^{-t}}\right).\end{gathered} \tag{32}$$

In order that the impulsive system (29) with the matrix $A(t)$ specified above has solutions bounded on the entire real axis, the nonhomogenities $f(t) = (\mathrm{col}\, f_1(t), f_2(t), f_3(t)) \in BC(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^3$ must satisfy condition (11). In the analyzed impulsive problem, this condition takes the following form:

$$\int_{-\infty}^{\infty} \frac{2f_3(t)}{e^t + e^{-t}} dt + \sum_{i=-\infty}^{\infty} \frac{2}{e^{\tau_i} + e^{-\tau_i}} \gamma_i^{(3)} = 0, \quad \forall f_1(t), f_2(t) \in BC(\mathbb{R}), \ \forall \gamma_i^{(1)}, \gamma_i^{(2)} \in \mathbb{R}. \tag{33}$$

If we consider the system (29) only with one point of discontinuity of the first kind $t = \tau_1 \in \mathbb{R}$ with impulse

$$\Delta x|_{t=\tau_1} = \gamma_1 \in \mathbb{R}^3, \tag{34}$$

then we rewrite the condition (33) in the form

$$\int_{-\infty}^{\infty} \frac{2f_3(t)}{e^t + e^{-t}} dt + \frac{2}{e^{\tau_1} + e^{-\tau_1}} \gamma_1^{(3)} = 0. \tag{35}$$

It is easy to see that (35) is always solvable and, according to Corollary 5, the analyzed impulsive problem has bounded solution for arbitrary $f_0(t) \in BC(\mathbb{R})$ if the pulse parameter γ_1 should be chosen as follows:

$$\gamma_1^{(3)} = -\left(e^{\tau_1} + e^{-\tau_1}\right) \int_{-\infty}^{\infty} \frac{f_3(t)}{e^t + e^{-t}} dt, \quad \forall \gamma_1^{(1)}, \gamma_1^{(2)} \in \mathbb{R}. \tag{36}$$

Remark 7. It seems that a possible generalization to systems with delay will be possible. In a particular case when the matrix of linear terms is constant, a representation of the fundamental matrix given by a special matrix function (so-called delayed matrix exponential, etc.), for example, in [10, 11] (for a continuous case) and in [12, 13] (for a discrete case), can give concrete formulas expressing solution of the considered problem in analytical form.

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