Research Article

Asymptotical Convergence of the Solutions of a Linear Differential Equation with Delays

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The asymptotic behavior of the solutions of the first-order differential equation $\dot{y}(t) = \sum_{i=1}^{n} \beta_i(t)[y(t - \delta_i) - y(t - \tau_i)]$ containing delays is studied with $\beta_i : [t_0 - \tau, \infty) \to [0, \infty)$, $\tau = \max\{\tau_1, \ldots, \tau_n\}$, $\sum_{i=1}^{n} \beta_i(t) > 0$, $\tau_i > \delta_i > 0$. The attention is focused on an analysis of the asymptotical convergence of solutions. A criterion for the asymptotical convergence of all solutions, characterized by the existence of a strictly increasing bounded solution, is proved. Relationships with the previous results are discussed, too.

1. Introduction

We investigate the asymptotic behavior of the solutions of a linear homogeneous differential equation with delayed terms

$$\dot{y}(t) = \sum_{i=1}^{n} \beta_i(t) \left[y(t - \delta_i) - y(t - \tau_i) \right] \tag{1.1}$$

as $t \to \infty$. In (1.1) we assume $\delta_i, \tau_i \in \mathbb{R}^+ = (0, +\infty)$, $\tau_i > \delta_i$, functions $\beta_i : I_{-1} \to \mathbb{R}_+ := [0, +\infty)$ where $I_{-1} := [t_0 - \tau, \infty)$, $t_0 \in \mathbb{R}$, $\tau = \max\{\tau_1, \ldots, \tau_n\}$ are continuous and such that $\sum_{i=1}^{n} \beta_i(t) > 0$ on I_{-1}. Set $I := [t_0, \infty)$. Throughout the paper the symbol "·" denotes the *right-hand* derivative. Similarly, if necessary, the value of a function at a point of I_{-1} is understood as the value of the corresponding limit *from the right*.

We call a solution $y = y(t)$ of (1.1) asymptotically convergent if it has a finite limit $\lim_{t \to \infty} y(t)$. The main results concern the asymptotical convergence of all solutions of (1.1). Besides, the proof of the results is based on the comparison of solutions of (1.1) with solutions

of an auxiliary inequality which formally copies (1.1). At first, we prove that, under certain conditions, (1.1) has a strictly increasing *asymptotically convergent* solution. Then we extend this statement to all the solutions of (1.1). Moreover, in the general case, the asymptotical convergence of all solutions is characterized by the existence of a strictly increasing bounded solution.

The problem concerning the asymptotical convergence of solutions of delayed differential equations (or delayed difference equation, etc.) is a classical one. But the problem of the asymptotic convergence or divergence of solutions of delayed equations receives permanent attention. Let us mention at least investigations [1–18]. Comparing the known investigations with the results presented we conclude that our results give more sharp sufficient conditions.

The paper is organized as follows. In Section 2 an auxiliary inequality is studied and the relationship of its solutions with solutions of (1.1) is derived. The existence of a strictly increasing and convergent solution of (1.1) is established in Section 3. Section 4 contains results concerning the asymptotical convergence of all the solutions of (1.1). The related previous results are discussed in Section 5.

Let $\mathcal{C} := C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

Let $v \in I$ be given. The function $y : [v - \tau, \infty) \to \mathbb{R}$ is said to be a *solution of* (1.1) *on* $[v - \tau, \infty)$ if y is continuous on $[v - \tau, \infty)$, continuously differentiable on $[v, \infty)$ and satisfies (1.1) for $t \in [v, \infty)$.

For $v \in I$, $\varphi \in \mathcal{C}$, we say that $y(v, \varphi)$ is a *solution* of (1.1) *through* (v, φ) (or that $y(v, \varphi)$ *corresponds to the initial point v*) if $y(v, \varphi)$ is a solution of (1.1) on $[v - \tau, \infty)$ and $y(v, \varphi)(v + \theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.

2. Auxiliary Inequality

The inequality

$$\dot{\omega}(t) \geq \sum_{i=1}^{n} \beta_i(t)[\omega(t - \delta_i) - \omega(t - \tau_i)] \tag{2.1}$$

plays an important role in the analysis of (1.1). Let $v \in I$ and $A > 0$ be given. The function $\omega : [v - \tau, v + A] \to \mathbb{R}$ is said to be a *solution of* (2.1) *on* $[v - \tau, v + A]$ if ω is continuous on $[v - \tau, v + A]$, continuously differentiable on $[v, v + A]$, and satisfies inequality (2.1) for $t \in [v, v + A]$. If $A = \infty$, we call the solution ω of (2.1) asymptotically convergent if it has a finite limit $\lim_{t \to \infty} \omega(t)$.

2.1. Relationship between the Solutions of Inequality (2.1) and Equation (1.1)

In this part, we will derive some properties of the solutions of type (2.1) inequalities and compare the solutions of (1.1) with those of inequality (2.1).

Lemma 2.1. *Let $\varphi \in \mathcal{C}$ be strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $[-\tau, 0]$. Then the corresponding solution $y(t^*, \varphi)(t)$ of (1.1) with $t^* \in I$ is strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $[t^* - \tau, \infty)$, respectively. If φ is strictly*

increasing (nondecreasing) and $\omega : I_{-1} \rightarrow R$ is a solution of (2.1) with $\omega(t_0 + \theta) = \varphi(\theta)$, $\theta \in [-1, 0]$, then ω is strictly increasing (nondecreasing) on I_{-1}.

Proof. This is clear from (1.1) and (2.1) and from $\beta_i(t) \geq 0$, $\sum_{i=1}^{n} \beta_i(t) > 0$, $t \in I$, $\tau_i > \delta_i > 0$, $i = 1, 2, \ldots, n$. □

Theorem 2.2. *Let $\omega(t)$ be a solution of inequality (2.1) on I_{-1}. Then there exists a solution $y(t)$ of (1.1) on I_{-1} such that the inequality*

$$y(t) \leq \omega(t) \tag{2.2}$$

holds on I_{-1}. In particular, a solution $y(t_0, \phi)$ of (1.1) with $\phi \in C$ defined by the relation

$$\phi(\theta) := \omega(t_0 + \theta), \quad \theta \in [-\tau, 0] \tag{2.3}$$

is such a solution.

Proof. Let $\omega(t)$ be a solution of inequality (2.1) on I_{-1}. We will show that the solution $y(t) := y(t_0, \phi)(t)$ of (1.1) satisfies inequality (2.2), that is,

$$y(t_0, \phi)(t) \leq \omega(t) \tag{2.4}$$

on I_{-1}. Define on I_{-1} the continuous function $W(t) = \omega(t) - y(t)$. Then $W = 0$ on $[t_0 - \tau, t_0]$, and W is a solution of (2.1) on I_{-1}. **Lemma 2.1** implies that W is nondecreasing. Consequently, $\omega(t) - y(t) \geq \omega(t_0) - y(t_0) = 0$ for all $t \geq t_0$. □

Remark 2.3. Let us note that the assertion, opposite in a sense with to that statement of Theorem 2.2, is obvious. Namely, if a solution $y(t)$ of (1.1) on I_{-1} is given, then there exists a solution $\omega(t)$ of inequality (2.1) on I_{-1} such that the inequality

$$\omega(t) \leq y(t) \tag{2.5}$$

holds on I_{-1} since it can be put $\omega(t) \equiv y(t)$. Moreover, if we put, for example, $\omega_1(t) \equiv y(t) - 1$, then

$$\omega_1(t) < y(t) \tag{2.6}$$

on I_{-1}.

2.2. A Solution of Inequality (2.1)

It is easy to get a solution of inequality (2.1) in an exponential form. We will indicate this form in the following lemma. This auxiliary result will help us derive concrete sufficient conditions for the existence of strictly increasing and convergent solution of (1.1).

Lemma 2.4. *Let there exist a function $\varepsilon : I_{-1} \to \mathbb{R}$, continuous on $I_{-1} \setminus \{t_0\}$ with at most first-order discontinuity at the point $t = t_0$ and satisfying the inequality*

$$\varepsilon(t) + \sum_{i=1}^{n} \beta_i(t) \exp\left[-\int_{t-\tau_i}^{t} \varepsilon(s)\mathrm{d}s \right] \geq \sum_{i=1}^{n} \beta_i(t) \exp\left[-\int_{t-\delta_i}^{t} \varepsilon(s)\mathrm{d}s \right] \tag{2.7}$$

on I. Then there exists a solution $\omega(t) = \omega_{\mathrm{e}}(t)$ of inequality (2.1), defined on I_{-1}, and having the form

$$\omega_{\mathrm{e}}(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\mathrm{d}s \right]. \tag{2.8}$$

Proof. Inequality (2.7) follows immediately from inequality (2.1) for $\omega = \omega_{\mathrm{e}}$. $\square$

3. Existence of an Asymptotically Convergent Solution of (1.1)

In this part we indicate sufficient conditions for the existence of a convergent solution of (1.1). First, let us introduce two obvious statements concerning asymptotical convergence. From Theorem 2.2 and Lemma 2.1, we immediately get the following.

Theorem 3.1. *If ω is a strictly increasing asymptotically convergent solution of (2.1) on I_{-1}, then there exists a strictly increasing asymptotically convergent solution $y(t)$ of (1.1) on I_{-1}.*

From Lemma 2.1, Theorem 2.2, and Lemma 2.4, we get the following.

Theorem 3.2. *If there exists a function $\varepsilon : I_{-1} \to \mathbb{R}^+$, continuous on $I_{-1} \setminus \{t_0\}$ with at most the first order discontinuity at the point $t = t_0$ satisfying $\int^{\infty} \varepsilon(s)\mathrm{d}s < \infty$, and the inequality (2.7) on I, the initial function*

$$\varphi(\theta) = \exp\left[\int_{t_0-\tau}^{t_0+\theta} \varepsilon(s)\mathrm{d}s \right], \quad \theta \in [-\tau, 0] \tag{3.1}$$

defines a strictly increasing and asymptotically convergent solution $y(t_0, \varphi)(t)$ of (1.1) on I_{-1} satisfying the inequality

$$y(t) \leq \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\mathrm{d}s \right] \tag{3.2}$$

on I.

Theorem 3.3. *If there exists a constant $\alpha > 1$ such that*

$$\liminf_{t \to \infty} t \cdot \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2 \right) \right] \right] > 0, \tag{3.3}$$

there exists a strictly increasing and asymptotically convergent solution $y(t)$ of (1.1) as $t \to \infty$.

Proof. Let us verify that the integral inequality (2.7) has (for every sufficiently large t) a solution ε such that

$$\int^{\infty} \varepsilon(s)ds < \infty. \tag{3.4}$$

In inequality (2.7), we put

$$\varepsilon(t) := t^{-\alpha}, \tag{3.5}$$

where $\alpha > 1$. Then (3.4) holds. Now we perform an auxiliary asymptotical analysis for $t \to \infty$. The symbol O used below is the Landau order symbol. All asymptotical decompositions are developed with sufficient accuracy. Let γ be a nonzero constant. Then

$$
\begin{aligned}
\int_{t-\gamma}^{t} \varepsilon(s)ds &= \int_{t-\gamma}^{t} \frac{1}{s^{\alpha}}ds \\
&= \frac{1}{1-\alpha}\left[\frac{1}{t^{\alpha-1}} - \frac{1}{(t-\gamma)^{\alpha-1}}\right] \\
&= \frac{1}{1-\alpha}\left[\frac{1}{t^{\alpha-1}} - \frac{1}{t^{\alpha-1}}\left(1 - \frac{\gamma}{t}\right)^{1-\alpha}\right] \\
&= \frac{1}{1-\alpha}\left[\frac{1}{t^{\alpha-1}} - \frac{1}{t^{\alpha-1}}\left(1 - \frac{(1-\alpha)\gamma}{t} - \frac{(1-\alpha)\alpha\gamma^2}{2t^2} + O\left(\frac{1}{t^3}\right)\right)\right] \\
&= \frac{1}{t^{\alpha}}\left[\gamma + \frac{\alpha\gamma^2}{2t} + O\left(\frac{1}{t^2}\right)\right].
\end{aligned}
\tag{3.6}
$$

Moreover

$$
\begin{aligned}
\exp\left[-\int_{t-\gamma}^{t}\varepsilon(s)ds\right] &= \exp\left[-\frac{1}{t^{\alpha}}\left[\gamma + \frac{\alpha\gamma^2}{2t} + O\left(\frac{1}{t^2}\right)\right]\right] \\
&= 1 - \frac{1}{t^{\alpha}}\left[\gamma + \frac{\alpha\gamma^2}{2t} + O\left(\frac{1}{t^2}\right)\right] + \frac{1}{2t^{2\alpha}}\left[\gamma + \frac{\alpha\gamma^2}{2t} + O\left(\frac{1}{t^2}\right)\right]^2 + O\left(\frac{1}{t^{3\alpha}}\right) \\
&= 1 - \frac{\gamma}{t^{\alpha}} - \frac{\alpha\gamma^2}{2t^{\alpha+1}} + O\left(\frac{1}{t^{\alpha+2}}\right) + O\left(\frac{1}{t^{2\alpha}}\right).
\end{aligned}
\tag{3.7}
$$

We use the asymptotical decomposition (3.7) and rewrite the integral inequality (2.7). We get

$$
\begin{aligned}
\frac{1}{t^{\alpha}} + \sum_{i=1}^{n}\beta_i(t)\left[1 - \frac{\tau_i}{t^{\alpha}} - \frac{\alpha\tau_i^2}{2t^{\alpha+1}} + O\left(\frac{1}{t^{\alpha+2}}\right) + O\left(\frac{1}{t^{2\alpha}}\right)\right] \\
\geq \sum_{i=1}^{n}\beta_i(t)\left[1 - \frac{\delta_i}{t^{\alpha}} - \frac{\alpha\delta_i^2}{2t^{\alpha+1}} + O\left(\frac{1}{t^{\alpha+2}}\right) + O\left(\frac{1}{t^{2\alpha}}\right)\right],
\end{aligned}
\tag{3.8}
$$

or

$$\frac{1}{t^{\alpha}} + \sum_{i=1}^{n} \beta_i(t) \left[-\frac{\tau_i - \delta_i}{t^{\alpha}} - \frac{\alpha(\tau_i^2 - \delta_i^2)}{2t^{\alpha+1}} + O\left(\frac{1}{t^{\alpha+2}}\right) + O\left(\frac{1}{t^{2\alpha}}\right) \right] \geq 0. \tag{3.9}$$

Multiplying this inequality by $t^{\alpha+1}$, we get

$$t \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right) + O\left(\frac{1}{t^2}\right) + O\left(\frac{1}{t^{\alpha}}\right) \right] \right] \geq 0. \tag{3.10}$$

Analysing inequality (3.10), we conclude that the inequality

$$1 + \sum_{i=1}^{n} \beta_i(t)(\delta_i - \tau_i) > 0 \tag{3.11}$$

or, equivalently,

$$\sum_{i=1}^{n} \beta_i(t)(\tau_i - \delta_i) < 1 \tag{3.12}$$

is a necessary condition for its validity as $t \to \infty$ (because $\alpha > 1$ and $\tau_i > \delta_i$, $i = 1,\dots,n$). Consequently,

$$\beta_i(t) < \frac{1}{\tau_i - \delta_i} \leq M := \max_{i=1,\dots,n} \left\{ \frac{1}{\tau_i - \delta_i} \right\}, \quad i = 1,\dots,n, \ t \in I. \tag{3.13}$$

Then, inequality (3.10) will be valid as $t \to \infty$ if there exist positive constants d and $\pi \in (1, \min\{2, \alpha\})$ such that

$$t \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right) + O\left(\frac{1}{t^2}\right) + O\left(\frac{1}{t^{\alpha}}\right) \right] \right]$$

$$> t \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right) \right] - \frac{dnM}{t^{\pi}} \right] \tag{3.14}$$

$$= t \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right) \right] \right] - \frac{dnM}{t^{\pi-1}} > 0.$$

Inequality (3.3) implies that there is a positive constant d^* such that

$$t \left[1 + \sum_{i=1}^{n} \beta_i(t) \left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right) \right] \right] > d^*. \tag{3.15}$$

Finally, since

$$t\left[1 + \sum_{i=1}^{n}\beta_i(t)\left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right)\right]\right] - \frac{dnM}{t^{\pi-1}} > d^* - \frac{dnM}{t^{\pi-1}} > 0 \tag{3.16}$$

as $t \to \infty$, we conclude that (3.14) holds and, consequently, the integral inequality (2.7) has a solution $\varepsilon(t) = t^{-\alpha}$ for every sufficiently large t. Lemma 2.4 holds. We finalize the proof by noticing that the statement of the theorem directly follows from Theorem 3.1. □

Assuming that functions β_i, $i = 1, 2, \ldots, n$, $t \in I$ can be estimated by suitable functions, we will prove that (1.1) has an asymptotically convergent solution. This yields two interesting corollaries directly following from inequality (3.3) in Theorem 3.3.

Corollary 3.4. *Let*

$$0 \le \beta_i(t) \le b_i - \frac{c_i}{t}, \tag{3.17}$$

where b_i and c_i are nonnegative constants and $\sum_{i=1}^{n}\beta_i(t) > 0$ on I. If, moreover,

$$\sum_{i=1}^{n}b_i(\tau_i - \delta_i) = q < 1, \tag{3.18}$$

then there exists a strictly increasing and convergent solution $y(t)$ of (1.1) as $t \to \infty$.

Proof. We show that the inequality (3.3) in Theorem 3.3 holds for any $\alpha > 1$. Estimating the left-hand side of (3.3), we get

$$\liminf_{t \to \infty} t \cdot \left[1 + \sum_{i=1}^{n}\beta_i(t)\left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right)\right]\right]$$

$$\ge \liminf_{t \to \infty} t \cdot \left[1 - \sum_{i=1}^{n}\left(b_i - \frac{c_i}{t}\right)\left[\tau_i - \delta_i + \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right)\right]\right] \tag{3.19}$$

$$= \liminf_{t \to \infty} t \cdot \left[(1 - q) + \sum_{i=1}^{n}\frac{c_i}{t}(\tau_i - \delta_i) - \frac{\alpha}{2t}\sum_{i=1}^{n}\left(b_i - \frac{c_i}{t}\right)\left(\tau_i^2 - \delta_i^2\right)\right] > 0$$

since $q \in (0, 1)$. □

Corollary 3.5. *Let*

$$0 \le \beta_i(t) \le b_i - \frac{c_i}{t}, \tag{3.20}$$

where b_i and c_i are nonnegative constants and $\sum_{i=1}^{n} \beta_i(t) > 0$ on I. If, moreover,

$$\sum_{i=1}^{n} b_i(\tau_i - \delta_i) = 1 \tag{3.21}$$

and there exists a constant $\alpha > 1$ such that

$$\sum_{i=1}^{n} c_i(\tau_i - \delta_i) > \frac{\alpha}{2} \sum_{i=1}^{n} b_i\left(\tau_i^2 - \delta_i^2\right), \tag{3.22}$$

then there exists a strictly increasing and convergent solution $y = y(t)$ of (1.1) as $t \to \infty$.

Proof. Employing a part of the proof of Corollary 3.4 with $q = 1$, we use inequality (3.22). Then

$$
\begin{aligned}
\liminf_{t \to \infty} t \cdot &\left[1 + \sum_{i=1}^{n} \beta_i(t)\left[\delta_i - \tau_i - \frac{\alpha}{2t}\left(\tau_i^2 - \delta_i^2\right)\right] \right] \\
&\geq \liminf_{t \to \infty} \left[\sum_{i=1}^{n} c_i(\tau_i - \delta_i) - \frac{\alpha}{2} \sum_{i=1}^{n} \left(b_i - \frac{c_i}{t}\right)\left(\tau_i^2 - \delta_i^2\right) \right] \\
&= \sum_{i=1}^{n} c_i(\tau_i - \delta_i) - \frac{\alpha}{2} \sum_{i=1}^{n} b_i\left(\tau_i^2 - \delta_i^2\right) > 0.
\end{aligned}
\tag{3.23}
$$

Thus, the inequality (3.3) in Theorem 3.3 holds. $\qquad\square$

4. Asymptotical Convergence of All Solutions

In this part we prove results concerning the asymptotical convergence of all the solutions of (1.1). First, we use inequality (3.3) to establish conditions for the asymptotical convergence of all the solutions.

Theorem 4.1. *Let there exist $\alpha > 1$ such that inequality (3.3) holds. Then all the solutions of (1.1) are asymptotically convergent for $t \to \infty$.*

Proof. First we prove that every solution defined by a *monotone* initial function is asymptotically convergent. We will assume that a monotone initial function $\varphi \in C$ is given. For the definiteness, let φ be strictly increasing or nondecreasing (the strictly decreasing or nonincreasing case can be dealt with in much the same way). By Lemma 2.1, the solution $y(t_0, \varphi)(t)$ is monotone (either strictly increasing or nondecreasing) on I_{-1}. In what follows, we will prove that $y(t_0, \varphi)$ is asymptotically convergent.

By Theorem 3.3, there exists a strictly increasing and asymptotically convergent solution Y of (1.1) on I_{-1}. Without loss of generality, we can assume that $y(t_0, \varphi)(t) \not\equiv Y(t)$ on I_{-1} since, in the opposite case, we can choose another initial function. Moreover, we can assume that both solutions $y(t_0, \varphi)(t)$ and $Y(t)$ are continuously differentiable on I_{-1}. In the opposite case, we can start our reasoning with the interval I instead of I_{-1}, that is, we can

replace $t_0 - \tau$ by t_0 and t_0 by $t_0 + \tau$. Similarly, without loss of generality we can assume that $\dot{Y}(t) > 0, t \in [t_0 - \tau, t_0]$. Hence, there is a $k > 0$ such that

$$\left(\dot{Y}(t) - k\dot{y}(t_0, \varphi)(t)\right)\big|_{t \in [t_0 - \tau, t_0]} = \left(\dot{Y}(t) - k\dot{\varphi}(t)\right)\big|_{t \in [t_0 - \tau, t_0]} > 0. \tag{4.1}$$

Then, by Lemma 2.1, $Y(t) - ky(t_0, \varphi)(t)$ is strictly increasing in I_{-1}. Thus

$$Y(t) - ky(t_0, \varphi)(t) > Y(t_0 - \tau) - ky(t_0, \varphi)(t_0 - \tau), \quad t > t_0 - \tau,$$

$$y(t_0, \varphi)(t) < \frac{1}{k}(Y(t) - Y(t_0 - \tau)) + y(t_0, \varphi)(t_0 - \tau) \tag{4.2}$$

is a bounded function for all $t > t_0 - \tau$.

Summarizing the previous part, we state that every monotone solution is asymptotically convergent. It remains to consider a class of all nonmonotone initial functions. For the behavior of a solution $y(t_0, \varphi)$, generated by a nonmonotone initial function $\varphi \in C$ there are two possibilities: either $y(t_0, \varphi)$ is eventually monotone and, consequently, asymptotically convergent, or $y(t_0, \varphi)$ is eventually nonmonotone.

We will also use the known fact that every absolutely continuous function can be decomposed into the difference of two strictly increasing absolutely continuous functions [19, page 315]. Assuming that an initial (nonmonotone) function $\varphi \in C$ is absolutely continuous on interval $[-\tau, 0]$, we can decompose it on the interval $[-\tau, 0]$ into the difference $\varphi = \varphi_1 - \varphi_2$ of two strictly increasing absolutely continuous functions $\varphi_1 \in C, \varphi_2 \in C$. In accordance with the previous part of the proof, every function φ_i, $i = 1, 2$ defines a strictly increasing convergent solution $y(t_0, \varphi_i)$. Now it becomes clear that the solution $y(t_0, \varphi)$ is asymptotically convergent. To complete the proof, it remains to prove that, without loss of generality, we can restrict the set of all initial functions to the set of absolutely continuous initial functions. To this end, we again consider the solution $y(t_0, \varphi)$ defined by $\varphi \in C$ and, if necessary, always without loss of generality, we can replace $t_0 - \tau$ by t_0 and t_0 by $t_0 + \tau$ since the solution has a finite derivative the interval $[t_0, t_0 + \tau]$. Finally, we remark that any function satisfying the Lipschitz condition on an interval $[a, b]$ is absolutely continuous in it [19, page 313]. $\qquad\square$

Tracing the proof of Theorem 4.1, we can see that the inequality (3.3) was used only as "input" information stating that, in accordance with Theorem 3.3, there exists a strictly increasing and asymptotically convergent solution Y of (1.1) on I_{-1}. If the existence of a strictly monotone and asymptotically convergent solution is assumed instead of (3.3), we obtain the following

Theorem 4.2. *If* (1.1) *has a strictly monotone and convergent solution on* I_{-1}, *then all the solutions of* (1.1) *defined on* I_{-1} *are asymptotically convergent.*

Moreover, combining the results formulated in Theorems 2.2, 3.1 and 4.2, we obtain the following

Theorem 4.3. *The following three statements are equivalent.*

(a) *Equation* (1.1) *has a strictly monotone and asymptotically convergent solution on* I_{-1}.

(b) *All solutions of* (1.1) *defined on* I_{-1} *are asymptotically convergent.*

(c) *Inequality* (2.1) *has a strictly monotone and asymptotically convergent solution on* I_{-1}.

5. Comparison with Previous Results

In [8], conditions for the asymptotical convergence of all the solutions of (1.1) with $n = 1$ and $\delta = 0$, that is,

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau)] \tag{5.1}$$

are given. A particular case of (1.1) with $n = 1$, that is, the case

$$\dot{y}(t) = \beta(t)[y(t - \delta) - y(t - \tau)] \tag{5.2}$$

is treated, for example, in [2, 4, 10, 11]. The following theorem (see [11, Theorems 4, 5, and 9]) gives corresponding results related to (5.2). Its first part gives sufficient conditions for the existence of a strictly increasing and unbounded solution of (5.2) whereas the second part provides sufficient conditions for the asymptotical convergence of all its solutions.

Theorem 5.1. *(a) Let there exist a constant $p \in (0,1)$ such that the inequality*

$$\beta(t) \geq \frac{1}{\tau - \delta} - \frac{p}{2t} \tag{5.3}$$

holds for all $t \in I_{-1}$. Then there exists a strictly increasing and unbounded solution of (5.2) as $t \to \infty$.
(b) Let there exist a constant $p > 1$ such that the inequality

$$\beta(t) \leq \frac{1}{\tau - \delta} - \frac{p(\tau + \delta)}{2(\tau - \delta)t} \tag{5.4}$$

holds for all $t \in I_{-1}$. Then all solutions of (5.2) defined on I_{-1} are convergent.

Comparing (5.3) with (5.4), we see that the maximal allowed values for β given in (5.4) are not strictly opposite with respect to the minimal allowed values for β given in (5.3). This is a gap since

$$\frac{\tau + \sigma}{\tau - \sigma} > 1 \tag{5.5}$$

and one would expect a pair of opposite inequalities stronger than (5.3) and (5.4). Can one of the inequalities (5.3) or (5.4) be improved? The following example shows that probably (5.3) can be improved because (5.4) provides a best possible criterion.

Example 5.2. Consider (5.2) with $\beta(t) = [t \ln((t - \delta)/(t - \tau))]^{-1}$, that is,

$$\dot{y}(t) = \frac{1}{t \ln((t - \delta)/(t - \tau))} \cdot (y(t - \delta) - y(t - \tau)). \tag{5.6}$$

It is easy to verify that $y(t) = \ln t$ is a strictly increasing and unbounded solution of (5.6) as $t \to \infty$. Decomposing asymptotically $\beta(t)$ as $t \to \infty$ we get

$$
\begin{aligned}
\beta(t) &= \frac{1}{t \ln((t-\delta)/(t-\tau))} = \frac{1}{t[\ln(1-\delta/t) - \ln(1-\tau/t)]} \\
&= \frac{1}{t} \cdot \frac{1}{-\delta/t - \delta^2/2t^2 - \delta^3/3t^3 + \tau/t + \tau^2/2t^2 + \tau^3/3t^3 + O(1/t^4)} \\
&= \frac{1}{\tau-\delta} \cdot \left[1 + \frac{1}{2t}(\tau+\delta) + \frac{1}{3t^2}\left(\tau^2 + \tau\delta + \delta^2\right) + O\left(\frac{1}{t^3}\right)\right]^{-1} \\
&= \frac{1}{\tau-\delta} \cdot \left[1 - \frac{1}{2t}(\tau+\delta) - \frac{1}{3t^2}\left(\tau^2 + \tau\delta + \delta^2\right) + \frac{1}{4t^2}(\tau+\delta)^2 + O\left(\frac{1}{t^3}\right)\right] \\
&= \frac{1}{\tau-\delta} - \frac{1}{2t}\frac{\tau+\delta}{\tau-\delta} - \frac{\tau-\delta}{12t^2} + O\left(\frac{1}{t^3}\right).
\end{aligned}
\tag{5.7}
$$

We conclude that this function can satisfy the inequality (5.4) as $t \to \infty$, that is, the inequality

$$
\begin{aligned}
\beta(t) &= \frac{1}{\tau-\delta} - \frac{1}{2t}\frac{\tau+\delta}{\tau-\delta} - \frac{\tau-\delta}{12t^2} + O\left(\frac{1}{t^3}\right) \\
&\leq \frac{1}{\tau-\delta} - \frac{p(\tau+\delta)}{2(\tau-\delta)t}
\end{aligned}
\tag{5.8}
$$

only if $p \leq 1$.

Acknowledgments

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