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Research Article

Oscillation Criteria for Second-Order Nonlinear Neutral Delay Differential Equations

Zhenlai Han,[1,2] Tongxing Li,[1,2] Shurong Sun,[1,3] and Weisong Chen[1]

[1] *School of Science, University of Jinan, Jinan, Shandong 250022, China*
[2] *School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China*
[3] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA*

Correspondence should be addressed to Zhenlai Han, hanzhenlai@163.com

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Some sufficient conditions are established for the oscillation of second-order neutral differential equation $(x(t)+p(t)x(\tau(t)))''+q(t)f(x(\sigma(t)))=0$, $t \ge t_0$, where $0 \le p(t) \le p_0 < +\infty$. The results complement and improve those of Grammatikopoulos et al. Ladas, A. Meimaridou, Oscillation of second-order neutral delay differential equations, Rat. Mat. 1 (1985), Grace and Lalli (1987), Ruan (1993), H. J. Li (1996), H. J. Li (1997), Xu and Xia (2008).

1. Introduction

In recent years, the oscillatory behavior of differential equations has been the subject of intensive study; we refer to the articles [1–13]; Especially, the study of the oscillation of neutral delay differential equations is of great interest in the last three decades; see for example [14–38] and references cited therein. Second-order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems (see [39]).

This paper is concerned with the oscillatory behavior of the second-order neutral delay differential equation

$$\left(x(t)+p(t)x(\tau(t))\right)''+q(t)f(x(\sigma(t)))=0, \quad t \ge t_0, \tag{1.1}$$

where $p,\ q \in C([t_0,\infty),\mathbb{R})$, $f \in C(\mathbb{R},\mathbb{R})$. Throughout this paper, we assume that

(a) $0 \leq p(t) \leq p_0 < +\infty$, $q(t) \geq 0$, and $q(t)$ is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \geq t_0$, where p_0 is a constant;

(b) $f(u)/u \geq k > 0$, for $u \neq 0, k$ is a constant;

(c) $\tau, \sigma \in C^1([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\tau'(t) \equiv \tau_0 > 0$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\sigma(t) = \infty$, $\tau \circ \sigma = \sigma \circ \tau$, where τ_0 is a constant.

In the study of oscillation of differential equations, there are two techniques which are used to reduce the higher-order equations to the first-order Riccati equation (or inequality). One of them is the Riccati transformation technique. The other one is called the generalized Riccati technique. This technique can introduce some new sufficient conditions for oscillation and can be applied to different equations which cannot be covered by the results established by the Riccati technique.

Philos [7] examined the oscillation of the second-order linear ordinary differential equation

$$x''(t) + p(t)x(t) = 0 \tag{1.2}$$

and used the class of functions as follows. Suppose there exist continuous functions H, $h : \mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ such that $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$, $t > s \geq t_0$, and H has a continuous and nonpositive partial derivative on $\mathbb{D}$ with respect to the second variable. Moreover, let $h : \mathbb{D} \to \mathbb{R}$ be a continuous function with

$$\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}, \quad t,s \in \mathbb{D}. \tag{1.3}$$

The author obtained that if

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)p(s) - \frac{1}{4}h^2(t,s)\right] \mathrm{d}s = \infty, \tag{1.4}$$

then every solution of (1.2) oscillates. Li [4] studied the equation

$$\left(r(t)x'(t)\right)' + p(t)x(t) = 0, \tag{1.5}$$

used the generalized Riccati substitution, and established some new sufficient conditions for oscillation. Li utilized the class of functions as in [7] and proved that if there exists a positive function $g \in C^1([t_0,\infty), \mathbb{R}^+)$ such that

$$\begin{gathered}\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)r(s)h(t,s)\mathrm{d}s < \infty, \\ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)\left[H(t,s)\psi(s) - \frac{1}{4}r(s)h^2(t,s)\right] \mathrm{d}s = \infty,\end{gathered} \tag{1.6}$$

where $a(s) = \exp\{-\int_0^s g(u)\mathrm{d}u\}$ and $\psi(s) = \{p(s) + r(s)g^2(s) - (r(s)g(s))'\}$, then every solution of (1.5) oscillates. Yan [13] used Riccati technique to obtain necessary and sufficient conditions for nonoscillation of (1.5). Applying the results given in [4, 13], every solution of the equation

$$\left(\frac{1}{t}x'(t)\right)' + \frac{1}{t^3}x(t) = 0 \tag{1.7}$$

is oscillatory.

An important tool in the study of oscillation is the integral averaging technique. Just as we can see, most oscillation results in [1, 3, 5, 7, 11, 12] involved the function class $\mathscr{H}$. Say a function $H = H(t,s)$ belongs to a function class $\mathscr{H}$, denoted by $H \in \mathscr{H}$, if $H \in C(D, \mathbb{R}_+ \cup \{0\})$, where $D = \{(t,s) : t_0 \le s \le t < \infty\}$ and $\mathbb{R}_+ = (0,\infty)$, which satisfies

$$H(t,t) = 0, \quad H(t,s) > 0, \quad \text{for } t > s, \tag{1.8}$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}. \tag{1.9}$$

In [10], Sun defined another type of function class X and considered the oscillation of the second-order nonlinear damped differential equation

$$\left(r(t)y'(t)\right)' + p(t)y'(t) + q(t)f\left(y(t)\right) = 0. \tag{1.10}$$

Say a function $\Phi = \Phi(t,s,l)$ is said to belong to X, denoted by $\Phi \in X$, if $\Phi \in C(E,\mathbb{R})$, where $E = \{(t,s,l) : t_0 \le l \le s \le t < \infty\}$, which satisfies $\Phi(t,t,l) = 0$, $\Phi(t,l,l) = 0$, $\Phi(t,s,l) > 0$, for $l < s < t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E.

In [8], by employing a class of function $H \in \mathbb{H}$ and a generalized Riccati transformation technique, Rogovchenko and Tuncay studied the oscillation of (1.10). Let $\mathbb{D} = \{(t,s) : -\infty < s \le t < +\infty\}$. Say a continuous function $H(t,s)$, $H : \mathbb{D} \to [0,+\infty)$ belongs to the class $\mathbb{H}$ if:

(i) $H(t,t) = 0$ and $H(t,s) > 0$ for $-\infty < s < t < +\infty$;

(ii) H has a continuous and nonpositive partial derivative $\partial H/\partial s$ satisfying, for some $h \in L_{loc}(\mathbb{D},\mathbb{R})$, $\partial H/\partial s = -h(t,s)\sqrt{H(t,s)}$, where h is nonnegative.

Meng and Xu [22] considered the even-order neutral differential equations with deviating arguments

$$\left[x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t))\right]^{(n)} + \sum_{j=1}^{l} q_j(t)f_j\left(x\left(\sigma_j(t)\right)\right) = 0, \tag{1.11}$$

where $\sum_{i=1}^{m} p_i(t) \le p$, $p \in (0,1)$. The authors introduced a class of functions F^*. Let $D_0 = \{(t,s) \in \mathbb{R}^2; t > s \ge t_0\}$ and $D = \{(t,s) \in \mathbb{R}^2; t \ge s \ge t_0\}$. The function $H \in C(D,\mathbb{R})$ is said to belong to the class F^* (defined by $H \in F^*$, for short) if

(J_1) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D_0$;

(J_2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;

(J_3) there exists a nondecreasing function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that

$$h(t,s) = \frac{\partial H(t,s)}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}. \tag{1.12}$$

Xu and Meng [31] studied the oscillation of the second-order neutral delay differential equation

$$\left[r(t)\left(y(t)+p(t)y(\sigma(t))\right)'\right]' + \sum_{i=1}^{n} q_i(t) f_i\left(y(\tau_i(t))\right) = 0, \tag{1.13}$$

where $p \in C([t_0,\infty),[0,1))$; by using the function class X an operator $T[\cdot;l,t]$, and a Riccati transformation of the form

$$\omega(t) = r(t)\frac{z'(t)}{z(t)}, \qquad z(t) = y(t) + p(t)y(\sigma(t)), \tag{1.14}$$

the authors established some oscillation criteria for (1.13). In [31], the operator $T[\cdot;l,t]$ is defined by

$$T[g;l,t] = \int_l^t \Phi^2(t,s,l)g(s)\mathrm{d}s, \tag{1.15}$$

for $t \geq s \geq l \geq t_0$ and $g \in C([t_0,\infty),\mathbb{R})$. The function $\varphi = \varphi(t,s,l)$ is defined by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \varphi(t,s,l)\Phi(t,s,l). \tag{1.16}$$

It is easy to verify that $T[\cdot;l,t]$ is a linear operator and that it satisfies

$$T[g';l,t] = -2T[g\varphi;l,t], \quad \text{for } g(s) \in C^1([t_0,\infty),\mathbb{R}). \tag{1.17}$$

In 2009, by using the function class X and defining a new operator $T[\cdot;l,t]$, Liu and Bai [21] considered the oscillation of the second-order neutral delay differential equation

$$\left(r(t)\left|(x(t)+p(t)x(\tau(t)))'\right|^{m-1}(x(t)+p(t)x(\tau(t)))'\right)+q(t)f(x(\sigma(t)))=0, \tag{1.18}$$

where $0 \le p(t) \le 1$. The authors defined the operator $T[\cdot;l,t]$ by

$$T[g;l,t]=\int_l^t \Phi(t,s,l)g(s)\mathrm{d}s, \tag{1.19}$$

for $t \ge s \ge l \ge t_0$ and $g \in C([t_0,\infty),\mathbb{R})$. The function $\varphi=\varphi(t,s,l)$ is defined by

$$\frac{\partial\Phi(t,s,l)}{\partial s}=\varphi(t,s,l)\Phi(t,s,l). \tag{1.20}$$

It is easy to see that $T[\cdot;l,t]$ is a linear operator and that it satisfies

$$T[g';l,t]=-T[g\varphi;l,t], \quad \text{for } g \in C^1([t_0,\infty),\mathbb{R}). \tag{1.21}$$

Wang [11] established some results for the oscillation of the second-order differential equation

$$(r(t)y'(t))'+f(t,y(t),y'(t))=0 \tag{1.22}$$

by using the function class $\mathscr{H}$ and a generalized Riccati transformation of the form

$$\omega(t)=\delta(t)r(t)\left[\frac{y'(t)}{g(y(t))}+\rho(t)\right]. \tag{1.23}$$

Long and Wang [6] considered (1.22); by using the function class X and the operator $T[\cdot;l,t]$ which is defined in [31], the authors established some oscillation results for (1.22).

In 1985, Grammatikopoulos et al. [16] obtained that if $0 \le p(t) \le 1$, $q(t) \ge 0$, and $\int_{t_0}^{\infty} q(s)[1-p(s-\sigma)]\mathrm{d}s=\infty$, then equation

$$[x(t)+p(t)x(t-\tau)]''+q(t)x(t-\sigma)=0 \tag{1.24}$$

is oscillatory. Li [18] studied (1.1) when $0 \le p(t) \le 1$, $\tau(t)=t-\tau$, $\sigma(t)=t-\sigma$ and established some oscillation criteria for (1.1). In [15, 19, 25], the authors established some general oscillation criteria for second-order neutral delay differential equation

$$\left[a(t)\left(x(t)+p(t)x(t-\tau)\right)'\right]'+q(t)f(x(t-\sigma))=0, \tag{1.25}$$

where $0 \le p(t) \le 1$. In 2002, Tanaka [27] studied the even-order neutral delay differential equation

$$[x(t)+h(t)x(t-\tau)]^{(n)}+f\left(t,x\left(g(t)\right)\right)=0 \tag{1.26}$$

where $0 \le \mu \le h(t) \le \lambda < 1$ or $1 < \lambda \le h(t) \le \mu$. The author established some comparison theorems for the oscillation of (1.26). Xu and Xia [28] investigated the second-order neutral differential equation

$$\left[x(t)+p(t)x(t-\tau)\right]''+q(t)f(x(t-\sigma))=0, \tag{1.27}$$

and obtained that if $0 \le p(t) \le p_0 < \infty$, $f(x)/x \ge k > 0$, for $x \neq 0$, and $q(t) \ge M > 0$, then (1.27) is oscillatory. We note that the result given in [28] fails to apply the cases $q(t)=\gamma/t$, or $q(t)=\gamma/t^2$ for $\gamma > 0$. To the best of our knowledge nothing is known regarding the qualitative behavior of (1.1) when $p(t) > 1, 0 < q(t) \le M$.

Motivated by [10, 21], for the sake of convenience, we give the following definitions.

Definition 1.1. Assume that $\Phi(t,s,l) \in X$. The operator is defined by $T_n[\cdot;l,t]$ by

$$T_n\left[g;l,t\right]=\int_l^t \Phi^n(t,s,l)g(s)ds, \tag{1.28}$$

for $n \ge 1$, $t \ge s \ge l \ge t_0$ and $g \in C([t_0,\infty),\mathbb{R})$.

Definition 1.2. The function $\varphi=\varphi(t,s,l)$ is defined by

$$\frac{\partial\Phi(t,s,l)}{\partial s}=\varphi(t,s,l)\Phi(t,s,l). \tag{1.29}$$

It is easy to verify that $\mathrm{T}_n[\cdot;l,t]$ is a linear operator and that it satisfies

$$T_n\left[g';l,t\right]=-nT_n\left[g\varphi;l,t\right], \quad \text{for } g \in C^1([t_0,\infty),\mathbb{R}). \tag{1.30}$$

In this paper, we obtain some new oscillation criteria for (1.1). The paper is organized as follows. In the next section, we will use the generalized Riccati transformation technique to give some sufficient conditions for the oscillation of (1.1), and we will give two examples to illustrate the main results. The key idea in the proofs makes use of the idea used in [23]. The method used in this paper is different from that of [27].

2. Main Results

In this section, we give some new oscillation criteria for (1.1). We start with the following oscillation result.

Theorem 2.1. *Assume that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$. Further, suppose that there exists a function $g \in C^1([t_0, \infty), \mathbb{R})$ such that for some $\beta \geq 1$ and some $H \in \mathbb{H}$, one has*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\varphi(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\beta}{4\sigma'(s)} u(s) h^2(t,s) \right] ds = \infty, \tag{2.1}$$

where $\varphi(t) := u(t)\{kQ(t) + (1 + p_0/\tau_0)[\sigma'(t)g^2(t) - g'(t)]\}$, $Q(t) := \min\{q(t), q(\tau(t))\}$, $u(t) := \exp\{-2\int_{t_0}^{t} \sigma'(s)g(s)ds\}$. Then every solution of (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, for all $t \geq t_1$. Define $z(t) = x(t) + p(t)x(\tau(t))$ for $t \geq t_0$, then $z(t) > 0$ for $t \geq t_1$. From (1.1), we have

$$z''(t) \leq -kq(t)x(\sigma(t)) \leq 0, \qquad t \geq t_1. \tag{2.2}$$

It is obvious that $z''(t) \leq 0$ and $z(t) > 0$ for $t \geq t_1$ imply $z'(t) > 0$ for $t \geq t_1$. Using (2.2) and condition (b), there exists $t_2 \geq t_1$ such that for $t \geq t_2$, we get

$$\begin{aligned}
0 &= z''(t) + q(t)f(x(\sigma(t))) \\
&= z''(t) + q(t)f(x(\sigma(t))) + p_0\left[z''(\tau(t)) + q(\tau(t))f(x(\sigma(\tau(t))))\right] \\
&= \left[z(t) + p_0 z(\tau(t))\right]'' + q(t)f(x(\sigma(t))) + p_0 q(\tau(t))f(x(\sigma(\tau(t)))) \\
&\geq \left[z(t) + \frac{p_0}{\tau_0} z(\tau(t))\right]'' + k\left[q(t)x(\sigma(t)) + p_0 q(\tau(t))x(\tau(\sigma(t)))\right] \\
&\geq \left[z(t) + \frac{p_0}{\tau_0} z(\tau(t))\right]'' + kQ(t)\left[x(\sigma(t)) + p_0 x(\tau(\sigma(t)))\right] \\
&\geq \left[z(t) + \frac{p_0}{\tau_0} z(\tau(t))\right]'' + kQ(t)z(\sigma(t)).
\end{aligned} \tag{2.3}$$

We introduce a generalized Riccati transformation

$$\omega(t) = u(t)\left[\frac{z'(t)}{z(\sigma(t))} + g(t)\right]. \tag{2.4}$$

Differentiating (2.4) from (2.2), we have $z'(\sigma(t)) \geq z'(t)$. Thus, there exists $t_3 \geq t_1$ such that for all $t \geq t_3$,

$$\begin{aligned}\omega'(t) &\leq -2\sigma'(t)g(t)\omega(t) + u(t)\left\{\frac{z''(t)}{z(\sigma(t))} - \sigma'(t)\left[\frac{\omega(t)}{u(t)} - g(t)\right]^2 + g'(t)\right\} \\ &= u(t)\frac{z''(t)}{z(\sigma(t))} + u(t)\left[-\sigma'(t)g^2(t) + g'(t)\right] - \sigma'(t)\frac{\omega^2(t)}{u(t)}.\end{aligned} \tag{2.5}$$

Similarly, we introduce another generalized Riccati transformation

$$\nu(t) = u(t)\left[\frac{z'(\tau(t))}{z(\sigma(t))} + g(t)\right]. \tag{2.6}$$

Differentiating (2.6), note that $\sigma(t) \leq \tau(t)$, by (2.2), we have $z'(\sigma(t)) \geq z'(\tau(t))$, then for all sufficiently large t, one has

$$\begin{aligned}\nu'(t) &\leq -2\sigma'(t)g(t)\nu(t) + u(t)\left\{\tau_0\frac{z''(\tau(t))}{z(\sigma(t))} - \sigma'(t)\left[\frac{\nu(t)}{u(t)} - g(t)\right]^2 + g'(t)\right\} \\ &= \tau_0 u(t)\frac{z''(\tau(t))}{z(\sigma(t))} + u(t)\left[-\sigma'(t)g^2(t) + g'(t)\right] - \sigma'(t)\frac{\nu^2(t)}{u(t)}.\end{aligned} \tag{2.7}$$

From (2.5) and (2.7), we have

$$\begin{aligned}\left[\omega(t) + \frac{p_0}{\tau_0}\nu(t)\right]' \leq{}& \frac{u(t)}{z(\sigma(t))}\left[z(t) + \frac{p_0}{\tau_0}z(\tau(t))\right]'' + \left(1 + \frac{p_0}{\tau_0}\right)u(t)\left[-\sigma'(t)g^2(t) + g'(t)\right] \\ &- \frac{\sigma'(t)\omega^2(t)}{u(t)} - \frac{p_0}{\tau_0}\frac{\sigma'(t)\nu^2(t)}{u(t)}.\end{aligned} \tag{2.8}$$

By (2.3) and the above inequality, we obtain

$$\left[\omega(t) + \frac{p_0}{\tau_0}\nu(t)\right]' \leq -\varphi(t) - \frac{\sigma'(t)\omega^2(t)}{u(t)} - \frac{p_0}{\tau_0}\frac{\sigma'(t)\nu^2(t)}{u(t)}. \tag{2.9}$$

Multiplying (2.9) by $H(t,s)$ and integrating from T to t, we have, for any $\beta \geq 1$ and for all $t \geq T \geq t_3$,

$$\begin{aligned}
\int_T^t H(t,s)\psi(s)ds \leq{}& -\int_T^t H(t,s)\omega'(s)ds - \int_T^t H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}\mathrm{d}s \\
& - \frac{p_0}{\tau_0}\int_T^t H(t,s)\nu'(s)ds - \frac{p_0}{\tau_0}\int_T^t H(t,s)\frac{\sigma'(s)\nu^2(s)}{u(s)}\mathrm{d}s \\
={}& -H(t,s)\omega(s)\big|_T^t - \int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\omega(s) + H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}\right]\mathrm{d}s \\
& -\frac{p_0}{\tau_0}H(t,s)\nu(s)\bigg|_T^t - \frac{p_0}{\tau_0}\int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\nu(s) + H(t,s)\frac{\sigma'(s)\nu^2(s)}{u(s)}\right]\mathrm{d}s \\
={}& H(t,T)\omega(T) - \int_T^t\left[h(t,s)\sqrt{H(t,s)}\omega(s) + H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}\right]\mathrm{d}s \\
& + \frac{p_0}{\tau_0}H(t,T)\nu(T) - \frac{p_0}{\tau_0}\int_T^t\left[h(t,s)\sqrt{H(t,s)}\nu(s) + H(t,s)\frac{\sigma'(s)\nu^2(s)}{u(s)}\right]\mathrm{d}s \\
={}& H(t,T)\omega(T) - \int_T^t\left[\sqrt{\frac{H(t,s)\sigma'(s)}{\beta u(s)}}\omega(s) + \sqrt{\frac{\beta u(s)}{4\sigma'(s)}}h(t,s)\right]^2\mathrm{d}s \\
& + \int_T^t \frac{\beta u(s)}{4\sigma'(s)}h^2(t,s)ds - \int_T^t \frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\omega^2(s)\mathrm{d}s \\
& + \frac{p_0}{\tau_0}H(t,T)\nu(T) - \frac{p_0}{\tau_0}\int_T^t\left[\sqrt{\frac{H(t,s)\sigma'(s)}{\beta u(s)}}\nu(s) + \sqrt{\frac{\beta u(s)}{4\sigma'(s)}}h(t,s)\right]^2\mathrm{d}s \\
& + \frac{p_0}{\tau_0}\int_T^t \frac{\beta u(s)}{4\sigma'(s)}h^2(t,s)ds - \frac{p_0}{\tau_0}\int_T^t \frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\nu^2(s)\mathrm{d}s.
\end{aligned} \tag{2.10}$$

From the above inequality and using monotonicity of H, for all $t \geq t_3$, we obtain

$$\int_{t_3}^t\left[H(t,s)\psi(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)\right]\mathrm{d}s \leq H(t,t_0)|\omega(t_3)| + \frac{p_0}{\tau_0}H(t,t_0)|\nu(t_3)|, \tag{2.11}$$

and, for all $t \geq t_3$,

$$\begin{aligned} &\int_{t_0}^{t}\left[H(t,s)\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)\right]\mathrm{d}s \\ &\quad\leq H(t,t_0)\left[\int_{t_0}^{t_3}|\varphi(s)|\mathrm{d}s+|\omega(t_3)|+\frac{p_0}{\tau_0}|\nu(t_3)|\right]. \end{aligned} \tag{2.12}$$

By (2.12),

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)\right]\mathrm{d}s \\ &\quad\leq \int_{t_0}^{t_3}|\varphi(s)|\mathrm{d}s+|\omega(t_3)|+\frac{p_0}{\tau_0}|\nu(t_3)|<\infty, \end{aligned} \tag{2.13}$$

which contradicts (2.1). This completes the proof. □

Remark 2.2. We note that it suffices to satisfy (2.1) in Theorem 2.1 for any $\beta \geq 1$, which ensures a certain flexibility in applications. Obviously, if (2.1) is satisfied for some $\beta_0 \geq 1$, it well also hold for any $\beta_1 > \beta_0$. Parameter β introduced in Theorem 2.1 plays an important role in the results that follow, and it is particularly important in the sequel that $\beta > 1$.

With an appropriate choice of the functions H and h, one can derive from Theorem 2.1 a number of oscillation criteria for (1.1). For example, consider a Kamenev-type function $H(t,s)$ defined by

$$H(t,s) = (t-s)^{n-1}, \qquad (t,s)\in\mathbb{D}, \tag{2.14}$$

where $n > 2$ is an integer. It is easy to see that $H \in \mathbb{H}$, and

$$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \qquad (t,s)\in\mathbb{D}. \tag{2.15}$$

As a consequence of Theorem 2.1, we have the following result.

Corollary 2.3. *Suppose that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$. Furthermore, assume that there exists a function $g \in C^1([t_0,\infty),\mathbb{R})$ such that for some integer $n > 2$ and some $\beta \geq 1$,*

$$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-3}\left[(t-s)^2\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)\right]ds=\infty, \tag{2.16}$$

where u and φ are as in Theorem 2.1. Then every solution of (1.1) is oscillatory.

For an application of Corollary 2.3, we give the following example.

Example 2.4. Consider the second-order neutral differential equation

$$[x(t) + (3 + \sin t)x(t - \tau)]'' + \frac{\gamma}{t^2}x(t - \sigma) = 0, \quad t \geq 1, \tag{2.17}$$

where $\sigma \geq \tau$, $\gamma > 0$. Let $p(t) = 3 + \sin t$, $q(t) = \gamma/t^2$, $f(x) = x$, and $g(t) = -1/(2t)$. Then $u(t) = t$, $\varphi(t) = (\gamma - 5/4)/t$. Take $k = 1$, $p_0 = 4$. Applying Corollary 2.3 with $n = 3$, for any $\beta \geq 1$,

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} (t-s)^{n-3}\left[\varphi(s)(t-s)^2 - \frac{\beta(n-1)^2(1+p_0)}{4}u(s)\right]\mathrm{d}s \\
&\quad = \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t\left[\left(\gamma - \frac{5}{4}\right)\frac{(t-s)^2}{s} - 5\beta s\right]\mathrm{d}s = \infty,
\end{aligned} \tag{2.18}$$

for $\gamma > 5/4$. Hence, (2.17) is oscillatory for $\gamma > 5/4$.

Remark 2.5. Corollary 2.3 can be applied to the second-order Euler differential equation

$$x''(t) + \frac{\gamma}{t^2}x(t) = 0, \quad t \geq 1, \tag{2.19}$$

where $\gamma > 0$. Let $p(t) = 0$, $q(t) = \gamma/t^2, f(x) = x$, and $g(t) = -1/(2t)$. Then $u(t) = t$, $\varphi(t) = (\gamma - 1/4)/t$. Take $k = 1$, $p_0 = 0$. Applying Corollary 2.3 with $n = 3$, for any $\beta \geq 1$,

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} (t-s)^{n-3}\left[\varphi(s)(t-s)^2 - \frac{\beta(n-1)^2(1+p_0)}{4}u(s)\right]\mathrm{d}s \\
&\quad = \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t\left[\left(\gamma - \frac{1}{4}\right)\frac{(t-s)^2}{s} - \beta s\right]\mathrm{d}s = \infty,
\end{aligned} \tag{2.20}$$

for $\gamma > 1/4$. Hence, (2.19) is oscillatory for $\gamma > 1/4$.

It may happen that assumption (2.1) is not satisfied, or it is not easy to verify, consequently, that Theorem 2.1 does not apply or is difficult to apply. The following results provide some essentially new oscillation criteria for (1.1).

Theorem 2.6. *Assume that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$, and for some $H \in \mathbb{H}$,*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty. \tag{2.21}$$

Further, suppose that there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ *and* $m \in C([t_0,\infty),\mathbb{R})$ *such that for all* $T \geq t_0$ *and for some* $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\varphi(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\beta}{4\sigma'(s)} u(s) h^2(t,s) \right] ds \geq m(T), \tag{2.22}$$

where u, φ *are as in Theorem 2.1. Suppose further that*

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\sigma'(s) m_+^2(s)}{u(s)} ds = \infty, \tag{2.23}$$

where $m_+(t) := \max\{m(t), 0\}$. *Then every solution of* (1.1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 2.1, assuming, without loss of generality, that there exists a solution x of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \geq t_1$. We define the functions ω and ν as in Theorem 2.1; we arrive at inequality (2.10), which yields for $t > T \geq t_1$, sufficiently large

$$\begin{aligned} &\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\varphi(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\beta}{4\sigma'(s)} u(s) h^2(t,s) \right] \mathrm{d}s \\ &\quad\leq \omega(T) - \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\sigma'(s) H(t,s)}{\beta u(s)} \omega^2(s) \mathrm{d}s \\ &\qquad + \frac{p_0}{\tau_0} \nu(T) - \frac{p_0}{\tau_0} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\sigma'(s) H(t,s)}{\beta u(s)} \nu^2(s) \mathrm{d}s. \end{aligned} \tag{2.24}$$

Therefore, for $t > T \geq t_1$, sufficiently large

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\varphi(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\beta}{4\sigma'(s)} u(s) h^2(t,s) \right] \mathrm{d}s \\ &\quad\leq \omega(T) + \frac{p_0}{\tau_0} \nu(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\sigma'(s) H(t,s)}{\beta u(s)} \left(\omega^2(s) + \frac{p_0}{\tau_0} \nu^2(s) \right) \mathrm{d}s. \end{aligned} \tag{2.25}$$

It follows from (2.22) that

$$\omega(T) + \frac{p_0}{\tau_0} \nu(T) \geq m(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\sigma'(s) H(t,s)}{\beta u(s)} \left(\omega^2(s) + \frac{p_0}{\tau_0} \nu^2(s) \right) \mathrm{d}s, \tag{2.26}$$

for all $T \geq t_1$ and for any $\beta > 1$. Consequently, for all $T \geq t_1$, we obtain

$$\omega(T) + \frac{p_0}{\tau_0}\nu(T) \geq m(T),$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}\left(\omega^2(s) + \frac{p_0}{\tau_0}\nu^2(s)\right)\mathrm{d}s \leq \frac{\beta}{\beta-1}\left(\omega(t_1) + \frac{p_0}{\tau_0}\nu(t_1) - m(t_1)\right) < \infty. \tag{2.27}$$

In order to prove that

$$\int_{t_1}^{\infty}\frac{\sigma'(s)\left(\omega^2(s) + (p_0/\tau_0)\nu^2(s)\right)}{u(s)}\mathrm{d}s < \infty, \tag{2.28}$$

suppose the contrary, that is,

$$\int_{t_1}^{\infty}\frac{\sigma'(s)\left(\omega^2(s) + (p_0/\tau_0)\nu^2(s)\right)}{u(s)}\mathrm{d}s = \infty. \tag{2.29}$$

Assumption (2.21) implies the existence of a $\rho > 0$ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > \rho. \tag{2.30}$$

By (2.30), we have

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > \rho > 0, \tag{2.31}$$

and there exists a $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \rho$, for all $t \geq T_2$. On the other hand, by virtue of (2.29), for any positive number κ, there exists a $T_1 \geq t_1$ such that, for all $t \geq T_1$,

$$\int_{t_1}^{t}\frac{\sigma'(s)\left(\omega^2(s) + (p_0/\tau_0)\nu^2(s)\right)}{u(s)}\mathrm{d}s \geq \frac{\kappa}{\rho}. \tag{2.32}$$

Using integration by parts, we conclude that, for all $t \geq T_1$,

$$\begin{aligned}
&\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}\left(\omega^2(s) + \frac{p_0}{\tau_0}\nu^2(s)\right)\mathrm{d}s \\
&\quad = \frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{t_1}^{s}\frac{\sigma'(v)\left(\omega^2(v) + (p_0/\tau_0)\nu^2(v)\right)}{u(v)}\mathrm{d}v\right]\mathrm{d}s \\
&\quad \geq \frac{\kappa}{\rho}\frac{1}{H(t,t_1)}\int_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\mathrm{d}s = \frac{\kappa H(t,T_1)}{\rho H(t,t_1)}.
\end{aligned} \tag{2.33}$$

It follows from (2.33) that, for all $t \geq T_2$,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}\left(\omega^2(s)+\frac{p_0}{\tau_0}\nu^2(s)\right)\mathrm{d}s \geq \kappa. \tag{2.34}$$

Since κ is an arbitrary positive constant, we get

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}\left(\omega^2(s)+\frac{p_0}{\tau_0}\nu^2(s)\right)\mathrm{d}s = \infty, \tag{2.35}$$

which contradicts (2.17). Consequently, (2.28) holds, so

$$\int_{t_1}^{\infty}\frac{\sigma'(s)\omega^2(s)}{u(s)}\mathrm{d}s < \infty, \qquad \int_{t_1}^{\infty}\frac{\sigma'(s)\nu^2(s)}{u(s)}\mathrm{d}s < \infty, \tag{2.36}$$

and, by virtue of (2.27),

$$\begin{aligned}
&\int_{t_1}^{\infty}\frac{\sigma'(s)m_+^2(s)}{u(s)}\mathrm{d}s\\
&\quad\leq \int_{t_1}^{\infty}\frac{\sigma'(s)\omega^2(s)+(p_0/\tau_0)^2\sigma'(s)\nu^2(s)+(2p_0/\tau_0)\sigma'(s)\omega(s)\nu(s)}{u(s)}\mathrm{d}s\\
&\quad\leq \int_{t_1}^{\infty}\frac{\sigma'(s)\omega^2(s)+(p_0/\tau_0)^2\sigma'(s)\nu^2(s)+(p_0/\tau_0)\sigma'(s)\left[\omega^2(s)+\nu^2(s)\right]}{u(s)}\mathrm{d}s < \infty,
\end{aligned} \tag{2.37}$$

which contradicts (2.23). This completes the proof. □

Choosing H as in Corollary 2.3, it is easy to verify that condition (2.21) is satisfied because, for any $s \geq t_0$,

$$\lim_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty}\frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1. \tag{2.38}$$

Consequently, we have the following result.

Corollary 2.7. *Suppose that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$. Furthermore, assume that there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ and $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \geq t_0$, for some integer $n > 2$ and some $\beta \geq 1$,*

$$\limsup_{t\to\infty} t^{1-n}\int_{T}^{t}(t-s)^{n-3}\left[(t-s)^2\psi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)\right]ds \geq m(T), \tag{2.39}$$

where u and ψ are as in Theorem 2.1. Suppose further that (2.23) *holds, where m_+ is as in Theorem 2.6. Then every solution of* (1.1) *is oscillatory.*

From Theorem 2.6, we have the following result.

Theorem 2.8. *Assume that $\sigma(t) \le \tau(t)$ for $t \ge t_0$. Further, suppose that $H \in \mathbb{H}$ such that* (2.21) *holds, there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ and $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \ge t_0$ and for some $\beta > 1$,*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\psi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)\right]ds \ge m(T), \tag{2.40}$$

where u and ψ are as in Theorem 2.1. Suppose further that (2.23) *holds, where m_+ is as in Theorem 2.6. Then every solution of* (1.1) *is oscillatory.*

Theorem 2.9. *Assume that $\sigma(t) \le \tau(t)$ for $t \ge t_0$. Further, assume that there exists a function $\Phi \in X$, such that for each $l \ge t_0$, for some $n \ge 1$,*

$$\limsup_{t\to\infty} T_n\left[\psi(s)-\frac{n^2}{4}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\sigma'(s)};l,t\right] > 0, \tag{2.41}$$

where ψ, u are defined as in Theorem 2.1, the operator T_n is defined by (1.28), *and $\varphi = \varphi(t,s,l)$ is defined by* (1.29). *Then every solution of* (1.1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 2.1, assuming, without loss of generality, that there exists a solution x of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \ge t_1$. We define the functions ω and ν as in Theorem 2.1; we arrive at inequality (2.9). Applying $T_n[\cdot;l,t]$ to (2.9), we get

$$T_n\left[\left[\omega(s)+\frac{p_0}{\tau_0}\nu(s)\right]';l,t\right] \le T_n\left[-\psi(s)-\frac{\sigma'(s)\omega^2(s)}{u(s)}-\frac{p_0}{\tau_0}\frac{\sigma'(s)\nu^2(s)}{u(s)};l,t\right]. \tag{2.42}$$

By (1.30) and the above inequality, we obtain

$$T_n[\psi(s);l,t] \le T_n\left[n\varphi\omega(s)-\frac{\sigma'(s)\omega^2(s)}{u(s)}+n\frac{p_0}{\tau_0}\varphi\nu(s)-\frac{p_0}{\tau_0}\frac{\sigma'(s)\nu^2(s)}{u(s)};l,t\right]. \tag{2.43}$$

Hence, from (2.43) we have

$$T_n[\psi(s);l,t] \le T_n\left[\left(1+\frac{p_0}{\tau_0}\right)\frac{n^2u(s)\varphi^2(s)}{4\sigma'(s)};l,t\right], \tag{2.44}$$

that is,

$$T_n\left[\psi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{n^2u(s)\varphi^2(s)}{4\sigma'(s)};l,t\right] \le 0. \tag{2.45}$$

Taking the super limit in the above inequality, we get

$$\limsup_{t\to\infty} T_n\left[\varphi(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{n^2 u(s)\varphi^2(s)}{4\sigma'(s)}; l, t\right] \le 0, \tag{2.46}$$

which contradicts (2.41). This completes the proof. □

If we choose

$$\Phi(t,s,l) = \rho(s)(t-s)^{\alpha}(s-l)^{\beta} \tag{2.47}$$

for $\alpha, \beta > 1/2$ and $\rho \in C^1([t_0,\infty),(0,\infty))$, then we have

$$\varphi(t,s,l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}. \tag{2.48}$$

Thus by Theorem 2.9, we have the following oscillation result.

Corollary 2.10. *Suppose that $\sigma(t) \le \tau(t)$ for $t \ge t_0$. Further, assume that for each $l \ge t_0$, there exist a function $\rho \in C^1([t_0,\infty),(0,\infty))$ and two constants $\alpha,\ \beta > 1/2$ such that for some $n \ge 1$,*

$$\begin{aligned}
&\limsup_{t\to\infty}\int_l^t \rho^n(s)(t-s)^{n\alpha}(s-l)^{n\beta} \\
&\quad\times\left[\varphi(s) - \frac{n^2}{4}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\sigma'(s)}\left(\frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}\right)^2\right] ds > 0,
\end{aligned} \tag{2.49}$$

where φ, u are as in Theorem 2.1. Then every solution of (1.1) *is oscillatory.*

If we choose

$$\Phi(t,s,l) = \sqrt{H_1(s,l)H_2(t,s)}, \tag{2.50}$$

where $H_1, H_2 \in \mathscr{H}$, then we have

$$\varphi(t,s,l) = \frac{1}{2}\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}} - \frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right), \tag{2.51}$$

where $h_1^{(1)}(s,l)$, $h_2^{(2)}(t,s)$ are defined as the following:

$$\frac{\partial H_1(s,l)}{\partial s} = h_1^{(1)}(s,l)\sqrt{H_1(s,l)}, \qquad \frac{\partial H_2(t,s)}{\partial s} = -h_2^{(2)}(t,s)\sqrt{\mathrm{H}_2(t,s)}. \tag{2.52}$$

According to Theorem 2.9, we have the following oscillation result.

Corollary 2.11. *Suppose that $\sigma(t) \le \tau(t)$ for $t \ge t_0$. Further, assume that for each $l \ge t_0$, there exist two functions H_1, $H_2 \in \mathscr{H}$ such that for some $n \ge 1$,*

$$\limsup_{t\to\infty} \int_l^t \left(\sqrt{H_1(s,l)H_2(t,s)}\right)^n \times \left[\varphi(s) - \frac{n^2}{16}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\sigma'(s)}\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}} - \frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right)^2\right] ds > 0, \tag{2.53}$$

where φ, u are as in Theorem 2.1. Then every solution of (1.1) *is oscillatory.*

In the following, we give some new oscillation results for (1.1) when $\sigma(t) \ge \tau(t)$ for $t \ge t_0$.

Theorem 2.12. *Assume that $\sigma(t) \ge \tau(t)$ for $t \ge t_0$. Suppose that there exists a function $g \in C^1([t_0,\infty),\mathbb{R})$ such that for some $\beta \ge 1$ and for some $H \in \mathbb{H}$, one has*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\varphi(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\tau_0}u(s)h^2(t,s)\right] ds = \infty, \tag{2.54}$$

where $\varphi(t) = u(t)\{kQ(t) + (1+p_0/\tau_0)[\tau_0 g^2(t) - g'(t)]\}$, $u(t) = \exp\{-2\tau_0 \int_{t_0}^t g(s)ds\}$, and Q is as in Theorem 2.1. Then every solution of (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a solution x of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \ge t_1$. Proceeding as in the proof of Theorem 2.1, we obtain (2.2) and (2.3). In view of (2.2), we have $z'(t) > 0$ for $t \ge t_1$. We introduce a generalized Riccati transformation

$$\omega(t) = u(t)\left[\frac{z'(t)}{z(\tau(t))} + g(t)\right]. \tag{2.55}$$

Differentiating (2.55) from (2.2), we have $z'(\tau(t)) \ge z'(t)$. Thus, there exists $t_2 \ge t_1$ such that for all $t \ge t_2$,

$$\begin{aligned} \omega'(t) &\le -2\tau_0 g(t)\omega(t) + u(t)\left\{\frac{z''(t)}{z(\tau(t))} - \tau_0\left[\frac{\omega(t)}{u(t)} - g(t)\right]^2 + g'(t)\right\} \\ &= u(t)\frac{z''(t)}{z(\tau(t))} + u(t)\left[-\tau_0 g^2(t) + g'(t)\right] - \frac{\tau_0\omega^2(t)}{u(t)}. \end{aligned} \tag{2.56}$$

Similarly, we introduce another generalized Riccati transformation

$$\nu(t) = u(t)\left[\frac{z'(\tau(t))}{z(\tau(t))} + g(t)\right]. \tag{2.57}$$

Differentiating (2.57), then for all sufficiently large t, one has

$$\begin{aligned} \nu'(t) &= -2\tau_0 g(t)\nu(t) + u(t)\left\{\tau_0 \frac{z''(\tau(t))}{z(\tau(t))} - \tau_0\left[\frac{\nu(t)}{u(t)} - g(t)\right]^2 + g'(t)\right\} \\ &= \tau_0 u(t)\frac{z''(\tau(t))}{z(\tau(t))} + u(t)\left[-\tau_0 g^2(t) + g'(t)\right] - \frac{\tau_0\nu^2(t)}{u(t)}. \end{aligned} \tag{2.58}$$

From (2.56) and (2.58), we have

$$\begin{aligned} \left[\omega(t) + \frac{p_0}{\tau_0}\nu(t)\right]' \le{}& \frac{u(t)}{z(\tau(t))}\left[z(t) + \frac{p_0}{\tau_0}z(\tau(t))\right]'' \\ &+ \left(1 + \frac{p_0}{\tau_0}\right)u(t)\left[-\tau_0 g^2(t) + g'(t)\right] - \frac{\tau_0\omega^2(t)}{u(t)} - p_0\frac{\nu^2(t)}{u(t)}. \end{aligned} \tag{2.59}$$

Note that $z'(t) > 0$, then we have $z(\sigma(t)) \ge z(\tau(t))$. By (2.3) and the above inequality, we obtain

$$\left[\omega(t) + \frac{p_0}{\tau_0}\nu(t)\right]' \le -\varphi(t) - \frac{\tau_0\omega^2(t)}{u(t)} - p_0\frac{\nu^2(t)}{u(t)}. \tag{2.60}$$

Multiplying (2.60) by $H(t,s)$ and integrating from T to t, we have, for any $\beta \ge 1$ and for all $t \ge T \ge t_2$,

$$\begin{aligned} \int_T^t H(t,s)\varphi(s)ds \le{}& -\int_T^t H(t,s)\omega'(s)ds - \int_T^t H(t,s)\frac{\tau_0\omega^2(s)}{u(s)}\mathrm{d}s \\ &- \frac{p_0}{\tau_0}\int_T^t H(t,s)\nu'(s)ds - \frac{p_0}{\tau_0}\int_T^t H(t,s)\frac{\tau_0\nu^2(s)}{u(s)}\mathrm{d}s \\ ={}& -H(t,s)\omega(s)|_T^t - \int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\omega(s) + H(t,s)\frac{\tau_0\omega^2(s)}{u(s)}\right]\mathrm{d}s \\ &-\frac{p_0}{\tau_0}H(t,s)\nu(s)\bigg|_T^t - \frac{p_0}{\tau_0}\int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\nu(s) + H(t,s)\frac{\tau_0\nu^2(s)}{u(s)}\right]\mathrm{d}s \\ ={}& H(t,T)\omega(T) - \int_T^t\left[h(t,s)\sqrt{H(t,s)}\omega(s) + H(t,s)\frac{\tau_0\omega^2(s)}{u(s)}\right]\mathrm{d}s \\ &+ \frac{p_0}{\tau_0}H(t,T)\nu(T) - \frac{p_0}{\tau_0}\int_T^t\left[h(t,s)\sqrt{H(t,s)}\nu(s) + H(t,s)\frac{\tau_0\nu^2(s)}{u(s)}\right]\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= H(t,T)\omega(T) - \int_T^t \left[\sqrt{\frac{H(t,s)\tau_0}{\beta u(s)}}\omega(s) + \sqrt{\frac{\beta u(s)}{4\tau_0}}h(t,s)\right]^2 \mathrm{d}s \\
&\quad + \int_T^t \frac{\beta u(s)}{4\tau_0}h^2(t,s)\mathrm{d}s - \int_T^t \frac{(\beta-1)\tau_0 H(t,s)}{\beta u(s)}\omega^2(s)\mathrm{d}s \\
&\quad + \frac{p_0}{\tau_0}H(t,T)\nu(T) - \frac{p_0}{\tau_0}\int_T^t \left[\sqrt{\frac{H(t,s)\tau_0}{\beta u(s)}}\nu(s) + \sqrt{\frac{\beta u(s)}{4\tau_0}}h(t,s)\right]^2 \mathrm{d}s \\
&\quad + \frac{p_0}{\tau_0}\int_T^t \frac{\beta u(s)}{4\tau_0}h^2(t,s)\mathrm{d}s - \frac{p_0}{\tau_0}\int_T^t \frac{(\beta-1)\tau_0 H(t,s)}{\beta u(s)}\nu^2(s)\mathrm{d}s.
\end{aligned}
\tag{2.61}
$$

The rest of the proof is similar to that of Theorem 2.1, we omit the details. This completes the proof. □

Take $H(t,s) = (t-s)^{n-1}$, $(t,s) \in \mathbb{D}$, where $n > 2$ is an integer. As a consequence of Theorem 2.12, we have the following result.

Corollary 2.13. *Suppose that $\sigma(t) \ge \tau(t)$ for $t \ge t_0$. Furthermore, assume that there exists a function $g \in C^1([t_0,\infty),\mathbb{R})$ such that for some integer $n > 2$ and some $\beta \ge 1$,*

$$
\limsup_{t\to\infty} t^{1-n}\int_{t_0}^t (t-s)^{n-3}\left[(t-s)^2\psi(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{\beta(n-1)^2}{4\tau_0}u(s)\right]ds = \infty,
\tag{2.62}
$$

where u and ψ are as in Theorem 2.12. Then every solution of (1.1) *is oscillatory.*

For an application of Corollary 2.13, we give the following example.

Example 2.14. Consider the second-order neutral differential equation

$$
\left[x(t) + p(t)x(\lambda t)\right]'' + \frac{\gamma}{t^2}f(x(\sigma(t))) = 0, \qquad t \ge 1,
\tag{2.63}
$$

where $\tau(t) = \lambda t$, $\sigma(t) \ge \lambda t$, $\sigma(\lambda t) = \lambda\sigma(t)$, $0 < \lambda < 1$, $\gamma > 0$, $0 \le p(t) \le p_0 < \infty$, and $f(x)/x \ge k > 0$, for $x \ne 0$. Let $\tau_0 = \lambda$, $q(t) = \gamma/t^2$, and $g(t) = -1/(2\lambda t)$. Then $u(t) = t$, $\psi(t) = (k\gamma - (1+p_0/\lambda)/(4\lambda))/t$. Applying Corollary 2.13 with $n = 3$, for any $\beta \ge 1$,

$$
\begin{aligned}
&\limsup_{t\to\infty} t^{1-n}\int_{t_0}^t (t-s)^{n-3}\left[\psi(s)(t-s)^2 - \left(1+\frac{p_0}{\tau_0}\right)\frac{\beta(n-1)^2}{4\tau_0}u(s)\right]\mathrm{d}s \\
&\quad = \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t\left[\left(k\gamma - \frac{1+(p_0/\lambda)}{4\lambda}\right)\frac{1}{s}(t-s)^2 - \frac{\beta}{\lambda}\left(1+\frac{p_0}{\lambda}\right)s\right]\mathrm{d}s = \infty,
\end{aligned}
\tag{2.64}
$$

for $\gamma > (1+p_0/\lambda)/(4k\lambda)$. Hence, (2.63) is oscillatory for $\gamma > (1+p_0/\lambda)/(4k\lambda)$.

By (2.61), similar to the proof of Theorem 2.6, we have the following result.

Theorem 2.15. *Assume that $\sigma(t) \geq \tau(t)$ for $t \geq t_0$. Assume also that $H \in \mathbb{H}$ such that* (2.21) *holds. Moreover, suppose that there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ and $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \geq t_0$ and for some $\beta > 1$,*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\tau_0}u(s)h^2(t,s)\right]ds \geq m(T), \tag{2.65}$$

where u and φ are as in Theorem 2.12. Suppose further that

$$\limsup_{t\to\infty}\int_{t_0}^t\frac{m_+^2(s)}{u(s)}ds=\infty, \tag{2.66}$$

where m_+ is defined as in Theorem 2.6. Then every solution of (1.1) *is oscillatory.*

Choosing $H(t,s)=(t-s)^{n-1}$, $(t,s)\in\mathbb{D}$, where $n>2$ is an integer. By Theorem 2.15, we have the following result.

Corollary 2.16. *Suppose that $\sigma(t) \geq \tau(t)$ for $t \geq t_0$. Furthermore, assume that there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ and $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \geq t_0$, some integer $n > 2$ and some $\beta \geq 1$,*

$$\limsup_{t\to\infty}t^{1-n}\int_T^t(t-s)^{n-3}\left[(t-s)^2\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta(n-1)^2}{4\tau_0}u(s)\right]ds \geq m(T), \tag{2.67}$$

where u and φ are as in Theorem 2.12. Suppose further that (2.66) *holds, where m_+ is defined as in Theorem 2.6. Then every solution of* (1.1) *is oscillatory.*

From Theorem 2.15, we have the following result.

Theorem 2.17. *Assume that $\sigma(t) \geq \tau(t)$ for $t \geq t_0$. Assume also that $H \in \mathbb{H}$ such that* (2.21) *holds. Moreover, suppose that there exist functions $g \in C^1([t_0,\infty),\mathbb{R})$ and $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \geq t_0$ and for some $\beta > 1$,*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\varphi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\tau_0}u(s)h^2(t,s)\right]ds \geq m(T), \tag{2.68}$$

where u and φ are as in Theorem 2.12. Suppose further that (2.66) *holds, where m_+ is as in Theorem 2.6. Then every solution of* (1.1) *is oscillatory.*

Next, by (2.60), similar to the proof of Theorem 2.9, we have the following result.

Theorem 2.18. *Assume that* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *Further, assume that there exists a function* $\Phi \in X$, *such that for each* $l \geq t_0$, *for some* $n \geq 1$,

$$\limsup_{t\to\infty} T_n\left[\psi(s) - \frac{n^2}{4}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\tau_0};l,t\right] > 0, \tag{2.69}$$

where ψ, u *are defined as in Theorem 2.12, the operator* T_n *is defined by* (1.28), *and* $\varphi = \varphi(t,s,l)$ *is defined by* (1.29). *Then every solution of* (1.1) *is oscillatory.*

If we choose $\Phi(t,s,l)$ as (2.47), then from Theorem 2.18, we have the following oscillation result.

Corollary 2.19. *Suppose that* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *Further, assume that for each* $l \geq t_0$, *there exist a function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *and two constants* α, $\beta > 1/2$ *such that for some* $n \geq 1$,

$$\begin{aligned}\limsup_{t\to\infty} &\int_l^t \rho^n(s)(t-s)^{n\alpha}(s-l)^{n\beta} \\ &\times\left[\psi(s) - \frac{n^2}{4}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\tau_0}\left(\frac{\rho'(s)}{\rho(s)} + \frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)}\right)^2\right]ds > 0,\end{aligned} \tag{2.70}$$

where ψ, u *are as in Theorem 2.12. Then every solution of* (1.1) *is oscillatory.*

If we choose $\Phi(t,s,l)$ as (2.50), then from Theorem 2.18, we have the following oscillation result.

Corollary 2.20. *Suppose that* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *Further, assume that for each* $l \geq t_0$, *there exist two functions* H_1, $H_2 \in \mathscr{H}$ *such that for some* $n \geq 1$,

$$\begin{aligned}\limsup_{t\to\infty} &\int_l^t \left(\sqrt{H_1(s,l)H_2(t,s)}\right)^n \\ &\times\left[\psi(s) - \frac{n^2}{16}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\tau_0}\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}} - \frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right)^2\right]ds > 0,\end{aligned} \tag{2.71}$$

where ψ, u *are as in Theorem 2.12. Then every solution of* (1.1) *is oscillatory.*

Remark 2.21. The results of this paper can be extended to the more general equation of the form

$$\left(r(t)\left(x(t)+p(t)x(\tau(t))\right)'\right)' + q(t)f(x(\sigma(t))) = 0. \tag{2.72}$$

The statement and the formulation of the results are left to the interested reader.

Remark 2.22. One can easily see that the results obtained in [15, 16, 18, 19, 25, 28] cannot be applied to (2.17), (2.63), so our results are new.

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