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Research Article

Analysis and Numerical Solutions of Positive and Dead Core Solutions of Singular Sturm-Liouville Problems

Gernot Pulverer,[1] Svatoslav Staněk,[2] and Ewa B. Weinmüller[1]

[1] *Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 6-10, 1040 Vienna, Austria*
[2] *Department of Mathematical Analysis, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic*

Correspondence should be addressed to Svatoslav Staněk, stanek@inf.upol.cz

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In this paper, we investigate the singular Sturm-Liouville problem $u'' = \lambda g(u)$, $u'(0) = 0$, $\beta u'(1) + \alpha u(1) = A$, where λ is a nonnegative parameter, $\beta \geq 0$, $\alpha > 0$, and $A > 0$. We discuss the existence of multiple positive solutions and show that for certain values of λ, there also exist solutions that vanish on a subinterval $[0, \rho] \subset [0, 1)$, the so-called dead core solutions. The theoretical findings are illustrated by computational experiments for $g(u) = 1/\sqrt{u}$ and for some model problems from the class of singular differential equations $(\phi(u'))' + f(t, u') = \lambda g(t, u, u')$ discussed in Agarwal et al. (2007). For the numerical simulation, the collocation method implemented in our MATLAB code `bvpsuite` has been applied.

1. Introduction

In the theory of diffusion and reaction (see, e.g., [1]), the reaction-diffusion phenomena are described by the equation

$$\Delta v = \phi^2 h(x, v), \tag{1.1}$$

where $x \in \Omega \subset \mathbb{R}^N$. Here $v \geq 0$ is the concentration of one of the reactants and ϕ is the Thiele modulus. In case that h is radial symmetric with respect to x, the radial solutions of the above equation satisfying the boundary conditions

$$\beta\frac{\delta v}{\delta n} + \alpha v = A \tag{1.2}$$

are solutions to a boundary value problem of the type

$$\begin{gathered} u''(t) + f(t, u'(t)) = \phi^2 h(t, u(t)), \\ u'(0) = 0, \qquad \beta u'(1) + \alpha u(1) = A, \quad \beta \geq 0, \ \alpha, A > 0, \end{gathered} \tag{1.3}$$

where t denotes the radial coordinate. Baxley and Gersdorff [2] discussed problem (1.3), where f and h were continuous and h was allowed to be unbounded for $u \to 0^+$. They proved the existence of positive solutions and dead core solutions (vanishing on a subinterval $[0, t_0]$, $0 < t_0 < 1$) of problem (1.3), and also covered the case of the function h approximated by some regular function h_κ.

Problem (1.3) was a motivation for discussing positive, pseudo dead core, and dead core solutions to the singular boundary value problem with a ϕ-Laplacian,

$$\left(\phi(u'(t))\right)' + f(t, u'(t)) = \lambda g(t, u(t), u'(t)), \quad \lambda > 0, \tag{1.4a}$$

$$u'(0) = 0, \qquad \beta u'(T) + \alpha u(T) = A, \quad \beta \geq 0, \ \alpha, A > 0, \tag{1.4b}$$

see [3]. Here λ is a parameter, the function f is non-negative and satisfies the Carathéodory conditions on $(0, T] \times [0, \infty)$, $f(t, 0) = 0$ for a.e. $t \in [0, T]$, and g is positive and satisfies the Carathéodory conditions on $(0, T] \times \mathfrak{D}$, $\mathfrak{D} = (0, A/\alpha] \times [0, \infty)$. Moreover, the function $f(t, x)$ is singular at $t = 0$ and $g(t, x, y)$ is singular at $x = 0$.

Let us denote by $AC_{\text{loc}}(0, T]$ the set of functions $x : (0, T] \to \mathbb{R}$ which are absolutely continuous on $[\varepsilon, T]$ for arbitrary small $\varepsilon > 0$.

A function $u \in C^1[0, T]$ is called *a positive solution of problem* (1.4a)-(1.4b) if $u > 0$ on $[0, T]$, $\phi(u') \in AC_{\text{loc}}(0, T]$, u satisfies (1.4b) and (1.4a) holds for a.e. $t \in [0, T]$. We say that $u \in C^1[0, T]$ satisfying (1.4b) is *a dead core solution of problem* (1.4a)-(1.4b) if there exists a point $\rho \in (0, T)$ such that $u = 0$ on $[0, \rho]$, $u > 0$ on $(\rho, T]$, $\phi(u') \in AC[\rho, T]$ and (1.4a) holds for a.e. $t \in [\rho, T]$. The interval $[0, \rho]$ is called the *dead core of* u. If $u(0) = 0$, $u > 0$ on $(0, T]$, $\phi(u') \in AC_{\text{loc}}(0, T]$, u satisfies (1.4b) and (1.4a) holds a.e. on $[0, T]$, then u is called *a pseudo dead core solution of problem* (1.4a)-(1.4b).

Since problem (1.4a)-(1.4b) is singular, the existence results in [3] are proved by a combination of the method of lower and upper functions with regularization and sequential techniques. Therefore, the notion of a sequential solution of problem (1.4a)-(1.4b) was introduced. In [3], conditions on the functions ϕ, f, and g were specified which guarantee that for each $\lambda > 0$, problem (1.4a)-(1.4b) has a sequential solution and that any sequential solution is either a positive solution, a pseudo dead core solution, or a dead core solution. Also, it was shown that all sequential solutions of (1.4a)-(1.4b) are positive solutions for sufficiently small positive values of λ and dead core solutions for sufficiently large values of λ.

The differential equation (1.5a) of the following boundary value problem satisfies all conditions specified in [3]:

$$\left(\left(u'(t)\right)^{\gamma}\right)' + \frac{u'(t)}{t^{\rho}} = \lambda\left(\frac{1}{\sqrt{u(t)}} + \left(u'(t)\right)^{\nu}\right), \tag{1.5a}$$

$$u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0,\ \beta > 0. \tag{1.5b}$$

Here, $\gamma, \rho \in (0, \infty)$, and $\nu \in [0, \gamma + 1]$. We note that in papers [2, 3] no information on the number of positive and dead core solutions of the underlying problem is given.

In this paper, we discuss the singular boundary value problem

$$u''(t) = \lambda g(u(t)), \quad \lambda \geq 0, \tag{1.6a}$$

$$u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0,\ \beta > 0, \tag{1.6b}$$

where λ is a non-negative parameter, and the function $g \in C(0, \infty)$ becomes unbounded at $u = 0$. Problem (1.6a)-(1.6b) is the special case of problem (1.4a)-(1.4b).

A function $u \in C^2[0,1]$ is *a positive solution of problem* (1.6a)-(1.6b) if u satisfies the boundary conditions (1.6b), $u > 0$ on $[0,1]$ and (1.6a) holds for $t \in [0,1]$. A function $u : [0,1] \to [0,\infty)$ is called *a dead core solution of problem* (1.6a)-(1.6b) if there exists a point $\rho \in (0,1)$ such that $u(t) = 0$ for $t \in [0,\rho]$, $u \in C^1[0,1] \cap C^2(\rho,1]$, u satisfies (1.6b) and (1.6a) holds for $t \in (\rho, 1]$. The interval $[0,\rho]$ is called *the dead core of* u. If $\rho = 0$, then u is called *a pseudo dead core solution* of problem (1.6a)-(1.6b).

The aim of this paper is twofold.

(1) First of all, we analyze relations between the values of the parameter λ and the number and types of solutions to problem (1.6a)-(1.6b), provided that

$$\begin{gathered} g \in C(0,\infty), \quad g \text{ is positive}, \quad \lim_{u \to 0^+} g(u) = \infty, \\ \int_0^a g(s)\mathrm{d}s < \infty \quad \forall a > 0 \end{gathered} \tag{1.7}$$

or

$$\begin{gathered} g \in C^1(0,\infty), \quad g \text{ is positive and decreasing}, \quad \lim_{u \to 0^+} g(u) = \infty, \\ \int_0^a g(s)\mathrm{d}s < \infty \quad \forall a > 0. \end{gathered} \tag{1.8}$$

(2) Moreover, we compute solutions u to the singular boundary value problem

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad \lambda \geq 0, \tag{1.9a}$$

$$u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0,\ \beta > 0, \tag{1.9b}$$

and the singular problem (1.5a), (1.9b). Note that (1.9a) is the special case of (1.6a) with g satisfying (1.8).

In [4] similar questions in context of (1.6a) and the Dirichlet boundary conditions $u(0) = 1$, $u(1) = 1$ have been discussed. For further results on existence of positive and dead core solutions to differential equations of the types $u'' = \lambda g(t,u)$ and $(\phi(u'))' = g(t,u,u')$, we refer the reader to [5–9]. The Dirichlet conditions have been discussed in [5–7, 9], while [8] deals with the Robin conditions $-u'(-1) + \alpha u(-1) = a$, $u'(1) + \alpha u(1) = a$, α, $a > 0$.

We now recapitulate the main analytical results formulated in Theorems 2.10, 2.12, and 2.13. First, we introduce the auxiliary function

$$H(x,y) := \begin{cases} \alpha y + \beta \displaystyle\int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} \sqrt{\int_x^y g(v)\mathrm{d}v}, & 0 \leq x < y, \\ \alpha y, & 0 \leq x = y, \end{cases} \tag{1.10}$$

where g satisfies (1.7). By Lemma 2.2, the equation $H(x, \gamma(x)) = 1$ has a unique continuous solution $\gamma \in C[0, 1/\alpha]$, and the function

$$\chi(x) := \begin{cases} \displaystyle\int_x^{\gamma(x)} \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}}, & x \in \left[0, \dfrac{1}{\alpha}\right), \\ 0, & x = \dfrac{1}{\alpha} \end{cases} \tag{1.11}$$

is continuous on $[0, 1/\alpha]$. Let $\mathcal{M} := \{(\chi(x))^2/2 : 0 < x \leq 1/\alpha\}$. Then the following statements hold.

(i) Problem (1.6a)-(1.6b) has a positive solution if and only if $\lambda \in \mathcal{M}$. In addition, for each $a \in (0, 1/\alpha]$, problem (1.6a)-(1.6b) with $\lambda = (\chi(a))^2/2$ has a unique positive solution such that $u(0) = a$, $u(1) = \gamma(a)$.

(ii) Problem (1.6a)-(1.6b) has a pseudo dead core solution if and only if

$$\lambda = \frac{1}{2}\left(\int_0^{\gamma(0)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}}\right)^2. \tag{1.12}$$

This solution is unique.

(iii) Problem (1.6a)-(1.6b) has a dead core solution if and only if

$$\lambda > \frac{1}{2}\left(\int_0^{\gamma(0)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}}\right)^2. \tag{1.13}$$

In addition, for all such λ, problem (1.6a)-(1.6b) has a unique dead core solution.

The final result concerning the multiplicity of positive solutions to problem (1.6a)-(1.6b) is given in Theorem 2.11. Let (1.8) hold and let $\Gamma := \max\{\tau : \tau \in \mathcal{M}\}$. Then $\Gamma > (\chi(0))^2/2$ and for each $\lambda \in ((\chi(0))^2/2, \Gamma)$, there exist multiple positive solutions of problem (1.6a)-(1.6b).

In Section 2 analytical results are presented. Here, we formulate the existence and uniqueness results for the solutions of the boundary value problem (1.6a)-(1.6b) and study the dependance of the solution on the parameter values λ. The numerical treatment of problems (1.9a)-(1.9b) and (1.5a)-(1.5b) based on the collocation method is discussed in Section 3, where for different values of λ, we study positive, pseudo dead core, and dead core solutions of problem (1.9a)-(1.9b) and positive solutions of problem (1.5a)-(1.5b).

2. Analytical Results

2.1. Auxiliary Functions

Let assumption (1.7) hold, and let us introduce auxiliary functions φ_a, H, and h as

$$\varphi_a(x) := \begin{cases} \displaystyle\int_a^x \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}}, & x \in (a,\infty), \\ 0, & x = a, \end{cases} \tag{2.1}$$

where $a \in [0,\infty)$,

$$H(x,y) := \begin{cases} \alpha y + \beta \displaystyle\int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} \sqrt{\int_x^y g(v)\mathrm{d}v}, & 0 \le x < y, \\ \alpha y, & 0 \le x = y, \end{cases} \tag{2.2}$$

$$h(t,y) := \alpha y + \frac{\beta}{1-t}\int_0^y \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} \sqrt{\int_0^y g(v)\mathrm{d}v}, \quad (t,y) \in [0,1) \times (0,\infty). \tag{2.3}$$

Here, the positive constants α and β are identical with those used in boundary conditions (1.6b). Note that the function H is used in the analysis of positive and pseudo dead core solutions of problem (1.6a)-(1.6b), while the function h for its dead core solutions.

Properties of φ_a are described in the following lemma.

Lemma 2.1. *Let assumption* (1.7) *hold and let* $a \in [0,\infty)$. *Then* $\varphi_a \in C[a,\infty) \cap C^1(a,\infty)$, *and* φ_a *is increasing on* $[a,\infty)$.

Proof. Let c be arbitrary, $c > a$. Then $\varphi_a \in C[a,c] \cap C^1(a,c]$, and φ_a is increasing on $[a,c]$ by [4, Lemma 2.3 (where 1 is replaced by c)]. Since $c > a$ is arbitrary, the result immediately follows. □

In the following lemma, we introduce functions γ and χ and discuss their properties.

Lemma 2.2. *Let assumption* (1.7) *hold. Then the following statements follow.*

(i) *The function H is continuous on $\Delta = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y\}$, and $(\partial H/\partial y)(x,y) > 0$ for $0 \le x < y$.*

(ii) *For each $x \in [0, 1/\alpha]$, there exists a unique $\gamma(x) \in [a, 1/\alpha]$ such that*

$$H(x, \gamma(x)) = 1 \quad \text{for } x \in \left[0, \frac{1}{\alpha}\right], \tag{2.4}$$

and $\gamma \in C[0,1/\alpha]$, $\gamma(x) > x$ for $x \in [0,1/\alpha)$, $\gamma(1/\alpha) = 1/\alpha$.

(iii) *The function*

$$\chi(x) := \begin{cases} \displaystyle\int_x^{\gamma(x)} \frac{ds}{\sqrt{\int_x^s g(v)dv}}, & x \in \left[0, \dfrac{1}{\alpha}\right), \\ 0, & x = \dfrac{1}{\alpha} \end{cases} \tag{2.5}$$

is continuous on $[0, 1/\alpha]$.

Proof. (i) Let us define S, P on Δ by

$$S(x,y) := \sqrt{\int_x^y g(v)\mathrm{d}v},$$
$$P(x,y) := \begin{cases} \displaystyle\int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}}, & 0 \le x < y, \\ 0, & 0 \le x = y. \end{cases} \tag{2.6}$$

Then $S \in C(\Delta)$. Let $x \ge 0$ and define $m := \min\{g(s) : 0 < s \le x+1\}$. Then, by (1.7), $m > 0$. Hence

$$0 < \int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} \le \frac{1}{\sqrt{m}} \int_x^y \frac{\mathrm{d}s}{\sqrt{s-x}} = 2\sqrt{\frac{y-x}{m}}, \quad y \in (x, x+1], \tag{2.7}$$

and consequently $\lim_{(x,y)\in\Delta, y\to x} P(x,y) = 0$, which means that P is continuous at (x,x). Let $0 \le x_0 < y_0$. We now show that P is continuous at the point (x_0, y_0). Let us choose an arbitrary y_* in the interval (x_0, y_0). Then $P(x,y) = I_1(x) + I_2(x,y)$ for $x \in [0, y_*]$ and $y > y_*$, where

$$I_1(x) = \int_x^{y_*} \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}}, \qquad I_2(x,y) = \int_{y_*}^{y} \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}}. \tag{2.8}$$

Since $I_1 \in C[0, y_*]$ by [4, Lemma 2.1 (where 1 was replaced by y_*)], it follows that I_1 is continuous at $x = x_0$. The continuity of P at (x_0, y_0) now follows from the fact that I_2 is continuous at this point. Hence P is continuous on Δ, and from $H(x,y) = \alpha y + \beta P(x,y) S(x,y)$ we conclude $H \in C(\Delta)$. Since

$$\frac{\partial H}{\partial y}(x,y) = \alpha + \beta + \frac{\beta q(y)}{2\sqrt{\int_x^y g(v)\mathrm{d}v}} \int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}}, \quad 0 \le x < y, \tag{2.9}$$

we have $(\partial H/\partial y)(x,y) > 0$ for $0 \le x < y$.

(ii) Consider the equation $H(x,y) = 1$, that is,

$$\alpha y + \beta \int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} \sqrt{\int_x^y g(v)\mathrm{d}v} = 1. \tag{2.10}$$

The function $H(x,\cdot)$ is increasing on $[x,\infty)$, $H(1/\alpha, 1/\alpha) = 1$, and, for $x \in [0, 1/\alpha)$, $H(x, 1/\alpha) > 1$. Hence, for each $x \in [0, 1/\alpha]$, there exists a unique $\gamma(x)$ such that $H(x, \gamma(x)) = 1$ and $\gamma(1/\alpha) = 1/\alpha$. Clearly, $\gamma(x) > x$ for $x \in [0, 1/\alpha)$. In order to prove that $\gamma \in C[0, 1/\alpha]$, suppose the contrary, that is, suppose that γ is discontinuous at a point $x = x_0$, $x_0 \in [0, 1/\alpha]$. Then there exist sequences $\{\nu_n\}, \{\mu_n\}$ in $[0, 1/\alpha]$ such that $\lim_{n\to\infty} \nu_n = x_0 = \lim_{n\to\infty} \mu_n$, and the sequences $\{\gamma(\nu_n)\}, \{\gamma(\mu_n)\}$ are convergent, $\lim_{n\to\infty} \gamma(\nu_n) = c_1$, $\lim_{n\to\infty} \gamma(\mu_n) = c_2$, $c_1 \ne c_2$. Let $n \to \infty$ in $H(\nu_n, \gamma(\nu_n)) = 1$ and in $H(\mu_n, \gamma(\mu_n)) = 1$. This means $H(x_0, c_j) = 1$, $j = 1, 2$, and $c_1 = c_2 = \gamma(x_0)$ by the definition of the function γ, which contradicts $c_1 \ne c_2$.

(iii) By (ii),

$$\alpha\gamma(x) + \beta\chi(x) \sqrt{\int_x^{\gamma(x)} g(v)\mathrm{d}v} = 1, \quad x \in \left[0, \frac{1}{\alpha}\right), \tag{2.11}$$

$\gamma \in C[0, 1/\alpha]$ and $\gamma(x) > x$ for $x \in [0, 1/\alpha)$. Hence, the function $\sqrt{\int_x^{\gamma(x)} g(v)\mathrm{d}v}$ is continuous on $[0, 1/\alpha]$ and positive on $[0, 1/\alpha)$. From

$$\chi(x) = \frac{1 - \alpha\gamma(x)}{\beta\sqrt{\int_x^{\gamma(x)} g(v)\mathrm{d}v}}, \quad x \in \left[0, \frac{1}{\alpha}\right), \tag{2.12}$$

we now deduce that $\chi \in C[0, 1/\alpha)$. Since

$$\chi(x) \leq \frac{1}{\sqrt{m}} \int_x^{\gamma(x)} \frac{\mathrm{d}s}{\sqrt{s-x}} = 2\sqrt{\frac{\gamma(x)-x}{m}}, \quad x \in \left[0, \frac{1}{\alpha}\right), \tag{2.13}$$

where $m := \min\{g(u) : 0 < u \leq 1/\alpha\} > 0$, and $\chi > 0$ on $[0, 1/\alpha)$, $\gamma(1/\alpha) = 1/\alpha$, we conclude $\lim_{x \to (1/\alpha)^-} \chi(x) = 0$. Hence χ is continuous at $x = 1/\alpha$, and consequently $\gamma \in C[1, 1/\alpha]$. □

Let γ be the function from Lemma 2.2(ii) defined on the interval $[0, 1/\alpha]$. From now on, Λ denotes the value of γ at $x = 0$, that is,

$$\Lambda = \gamma(0). \tag{2.14}$$

In the following lemma, we prove a property of χ which is crucial for discussing multiple positive solutions of problem (1.6a)-(1.6b).

Lemma 2.3. *Let assumption* (1.8) *hold and let the function* χ *be given by* (2.5). *Then there exists* $\varepsilon > 0$ *such that*

$$\chi(x) > \chi(0), \quad \text{for } x \in (0, \varepsilon). \tag{2.15}$$

Proof. Note that $\chi(0) = \int_0^\Lambda (1/\sqrt{\int_0^s g(v)\mathrm{d}v})\mathrm{d}s$. We deduce from [4, Lemma 2.2 (with 1 replaced by Λ)] that there exists an $\varepsilon > 0$ such that

$$\int_x^\Lambda \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} > \chi(0) \quad \text{for } x \in (0, \varepsilon). \tag{2.16}$$

If $\gamma(x) > \Lambda$ for some $x \in (0, \varepsilon)$, then (2.16) yields

$$\chi(x) = \int_x^{\gamma(x)} \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} > \int_x^\Lambda \frac{\mathrm{d}s}{\sqrt{\int_x^s g(v)\mathrm{d}v}} > \chi(0). \tag{2.17}$$

Consequently, inequality (2.15) holds for such an x. If the statement of the lemma were false, then some $x_* \in (0, \varepsilon)$ would exist such that $\gamma(x_*) \leq \Lambda$ and

$$\chi(x_*) \leq \chi(0). \tag{2.18}$$

From the following equalities, compare (2.4),

$$1 = \alpha\Lambda + \beta\chi(0)\sqrt{\int_0^\Lambda g(v)\mathrm{d}v},$$
$$1 = \alpha\gamma(x_*) + \beta\chi(x_*)\sqrt{\int_{x_*}^{\gamma(x_*)} g(v)\mathrm{d}v}, \tag{2.19}$$

and from $\gamma(x_*) \le \Lambda$, we conclude that

$$\chi(x_*) \ge \chi(0)\sqrt{\frac{\int_0^\Lambda g(v)\mathrm{d}v}{\int_{x_*}^{\gamma(x_*)} g(v)\mathrm{d}v}}. \tag{2.20}$$

Finally, from

$$\int_0^\Lambda g(v)\mathrm{d}v > \int_{x_*}^{\gamma(x_*)} g(v)\mathrm{d}v, \tag{2.21}$$

we have $\chi(x_*) > \chi(0)$, which contradicts (2.18). □

In order to discuss dead core solutions of problem (1.6a)-(1.6b) and their dead cores, we need to introduce two additional functions μ and p related to h and study their properties.

Lemma 2.4. *Assume that* (1.7) *holds and let* h *be given by* (2.3)*. Then for each* $t \in [0,1)$*, there exists a unique* $\mu(t) \in (0, 1/\alpha)$ *such that*

$$h\big(t, \mu(t)\big) = 1 \quad \textit{for } t \in [0,1). \tag{2.22}$$

The function μ *is continuous and decreasing on* $[0,1)$*, and the function*

$$p(t) := \frac{1}{1-t}\int_0^{\mu(t)} \frac{ds}{\sqrt{\int_0^s g(v)dv}}, \quad t \in [0,1), \tag{2.23}$$

is continuous and increasing on $[0,1)$*. Moreover,* $\lim_{t\to 1^-} p(t) = \infty$*.*

Proof. It follows from (1.7) that $h \in C([0,1)\times(0,\infty))$. Also, h is increasing w.r.t. both variables, $\lim_{t\to 1^-} h(t,y) = \infty$ for any $y \in (0, 1/\alpha]$, and $\lim_{y\to 0^+} h(t,y) = 0$, $\lim_{y\to 1/\alpha} h(t,y) > 1$ for any $t \in [0,1)$. Hence, for each $t \in [0,1)$, there exists a unique $\mu(t) \in (0,1/\alpha)$ such that $h(t,\mu(t)) = 1$. In order to prove that μ is decreasing on $[0,1)$, assume on the contrary that $\mu(t_1) \le \mu(t_2)$ for some $0 \le t_1 < t_2 < 1$. Then $h(t_1,\mu(t_1)) < h(t_2,\mu(t_2))$ which contradicts $h(t_j,\mu(t_j)) = 1$ for $j = 1,2$. Hence, μ is decreasing on $[0,1)$. If μ was discontinuous at a point $t_0 \in [0,1)$, then there would exist sequences $\{\nu_n\}$ and $\{\tau_n\}$ in $[0,1)$ such that $\lim_{n\to\infty}\nu_n = t_0 = \lim_{n\to\infty}\tau_n$ and $\{\mu(\nu_n)\}$, $\{\mu(\tau_n)\}$ are convergent, $\lim_{n\to\infty}\mu(\nu_n) = c_1$, and $\lim_{n\to\infty}\mu(\tau_n) = c_2$ with $c_1 \neq c_2$. Taking the limits $n \to \infty$ in $h(\nu_n, \mu(\nu_n)) = 1$ and $h(\tau_n, \mu(\tau_n)) = 1$, we obtain $h(t_0, c_j) = 1$, $j = 1, 2$. Consequently, $c_1 = c_2 = \mu(x_0)$ by the definition of the function μ, which is not possible.

By (2.22),

$$\alpha\mu(t) + \frac{\beta}{1-t}\int_0^{\mu(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}}\sqrt{\int_0^{\mu(t)} g(v)\mathrm{d}v} = 1 \quad \text{for } t \in [0,1), \tag{2.24}$$

and therefore,

$$p(t) = \frac{1-\alpha\mu(t)}{\beta\sqrt{\int_0^{\mu(t)} g(v)\mathrm{d}v}}, \quad t \in [0,1). \tag{2.25}$$

It follows from the properties of μ that the functions $1-\alpha\mu(t)$, $1/\sqrt{\int_0^{\mu(t)} g(v)\mathrm{d}v}$ are continuous, positive, and increasing on $[0,1)$. Hence (2.25) implies that $p \in C[0,1)$ and p is increasing. Moreover, $\lim_{t\to 1^-} p(t) = \infty$ since $\int_0^{\mu(t)}(1/\sqrt{\int_0^s g(v)\mathrm{d}v})\ \mathrm{d}s$ is bounded on $[0,1)$. □

Corollary 2.5. *Let assumption* (1.7) *hold. Then*

$$\frac{1}{1-t}\int_0^{\mu(t)} \frac{ds}{\sqrt{\int_0^s g(v)dv}} > \int_0^{\Lambda} \frac{ds}{\sqrt{\int_0^s g(v)dv}} \quad \textit{for } t \in (0,1), \tag{2.26}$$

and for each λ satisfying the inequality

$$\lambda > \frac{1}{2}\left(\int_0^{\Lambda} \frac{ds}{\sqrt{\int_0^s g(v)dv}}\right)^2, \tag{2.27}$$

there exists a unique $\rho \in (0,1)$ such that

$$\int_0^{\mu(\rho)} \frac{ds}{\sqrt{\int_0^s g(v)dv}} = (1-\rho)\sqrt{2\lambda}. \tag{2.28}$$

Proof. The equalities $h(0,y) = H(0,y)$ for $y \in (0,\infty)$ and $H(0,\Lambda) = 1$ imply that $\mu(0) = \Lambda$. Since the function p defined by (2.23) is continuous and increasing on $[0,1)$, it follows that $p(t) > p(0)$ for $t \in (0,1)$; see (2.26). Let us choose an arbitrary λ satisfying (2.27). Then $\sqrt{2\lambda} > p(0)$. Now, the properties of p guarantee that equation $\sqrt{2\lambda} = p(t)$ has a unique solution $\rho \in (0,1)$. This means that (2.28) holds for a unique $\rho \in (0,1)$. □

2.2. Dependence of Solutions on the Parameter λ

The following two lemmas characterize the dependence of positive and dead core solutions of problem (1.6a)-(1.6b) on the parameter λ.

Lemma 2.6. *Let assumption* (1.7) *hold and let u be a positive solution of problem* (1.6a)-(1.6b) *for some $\lambda > 0$. Also, let $a := \min\{u(t) : 0 \le t \le 1\}$, and $Q := \max\{u(t) : 0 \le t \le 1\}$. Then $a = u(0)$, $Q = u(1)$,*

$$\int_a^Q \frac{ds}{\sqrt{\int_a^s g(v)dv}} = \sqrt{2\lambda}, \tag{2.29}$$

$$\int_a^{u(t)} \frac{ds}{\sqrt{\int_a^s g(v)dv}} = \sqrt{2\lambda}t \quad \text{for } t \in [0,1], \tag{2.30}$$

$$H(a,Q) = 1, \tag{2.31}$$

where the function H is given by (2.2).

Proof. Since $u'(0) = 0$ and $u''(t) = \lambda g(u(t)) > 0$ for $t \in [0,1]$, we conclude that $u' > 0$ on $(0,1]$ and $a = u(0)$, $Q = u(1)$. By integrating the equality $u''(t)u'(t) = \lambda g(u(t))u'(t)$ over $[0,t] \subset [0,1]$, we obtain

$$\left(u'(t)\right)^2 = 2\lambda \int_a^{u(t)} g(v)\mathrm{d}v, \tag{2.32}$$

and consequently, since $u' > 0$ on $(0,1]$,

$$u'(t) = \sqrt{2\lambda}\sqrt{\int_a^{u(t)} g(v)\mathrm{d}v}, \quad t \in [0,1]. \tag{2.33}$$

Finally, integrating

$$\frac{u'(t)}{\sqrt{\int_a^{u(t)} g(v)\mathrm{d}v}} = \sqrt{2\lambda}, \quad t \in (0,1], \tag{2.34}$$

over $[0,t]$ yields (2.30). Now we set $t = 1$ in (2.30) and obtain (2.29). Equality (2.31) follows from $\alpha u(1) + \beta u'(1) = 1$ and from

$$u(1) = Q, \qquad u'(1) = \sqrt{2\lambda}\sqrt{\int_a^Q g(v)\mathrm{d}v} = \int_a^Q \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}}\sqrt{\int_a^Q g(v)\mathrm{d}v}. \tag{2.35}$$

□

Remark 2.7. Let (1.7) hold and let u be a pseudo dead core solution of problem (1.6a)-(1.6b). Then, by the definition of pseudo dead core solutions, $u(0) = 0$. We can proceed analogously to the proof of Lemma 2.6 in order to show that

$$\int_0^Q \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}} = \sqrt{2\lambda}, \tag{2.36}$$

where $Q = u(1)$, and

$$\int_0^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}} = \sqrt{2\lambda}t \quad \text{for } t \in [0,1], \tag{2.37}$$

$$H(0,Q) = 1. \tag{2.38}$$

From (2.38), we finally have $Q = \Lambda$. Consequently, $u(1) = \Lambda$.

Remark 2.8. If $\lambda = 0$, then $u(t) = 1/\alpha$, $t \in [0,1]$, is the unique solution of problem (1.6a)-(1.6b). This solution is positive.

Lemma 2.9. *Let assumption* (1.7) *hold and let* u *be a dead core solution of problem* (1.6a)-(1.6b) *for some* $\lambda = \lambda_0$. *Moreover, let* $Q := \max\{u(t) : 0 \le t \le 1\}$. *Then* $Q = u(1)$ *and there exists a point* $\rho \in (0,1)$ *such that* $u(t) = 0$ *for* $t \in [0,\rho]$,

$$\int_0^{u(t)} \frac{ds}{\sqrt{\int_0^s g(v)dv}} = \sqrt{2\lambda_0}(t-\rho) \quad \text{for } t \in [\rho,1], \tag{2.39}$$

$$\int_0^{Q} \frac{ds}{\sqrt{\int_0^s g(v)dv}} = \sqrt{2\lambda_0}(1-\rho), \tag{2.40}$$

$$h(\rho,Q) = 1, \tag{2.41}$$

where the function h *is given by* (2.3). *Furthermore,* u *is the unique dead core solution of problem* (1.6a)-(1.6b) *with* $\lambda = \lambda_0$.

Proof. Since u is a dead core solution of problem (1.6a)-(1.6b) with $\lambda = \lambda_0$, there exists by definition, a point $\rho \in (0,1)$ such that $u \in C^1[0,1] \cap C^2(\rho,1]$, $u(t) = 0$ for $t \in [0,\rho]$ and $u > 0$ on $(\rho,1]$. Consequently, $u' > 0$ on $(\rho,1]$, and $Q = u(1)$. We can now proceed analogously to the proof of Lemma 2.6 to show that

$$u'(t) = \sqrt{2\lambda_0}\sqrt{\int_0^{u(t)} g(v)\mathrm{d}v}, \quad t \in [\rho,1], \tag{2.42}$$

and (2.39) holds. Setting $t = 1$ in (2.39), we obtain (2.40). Also, from (1.6b), $u(1) = Q$,

$$u'(1) = \sqrt{2\lambda_0}\sqrt{\int_0^{Q} g(v)\mathrm{d}v} = \frac{1}{1-\rho}\int_0^{Q} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}}\sqrt{\int_0^{Q} g(v)\mathrm{d}v}, \tag{2.43}$$

equality (2.41) follows.

It remains to verify that u is the unique dead core solution of problem (1.6a)-(1.6b) with $\lambda = \lambda_0$. Let us suppose that w is another dead core solution of the above problem. Let $w(t) = 0$ for $t \in [0,\rho_1]$ and $w > 0$ on $(\rho_1,1]$ for some $\rho_1 \in (0,1)$. Then $w''(t) = \lambda_0 g(w(t)) > 0$

for $t \in (\rho_1, 1]$, and consequently $w' > 0$ on $(\rho_1, 1]$ and $Q_1 := \max\{w(t) : 0 \le t \le 1\} = w(1)$. Hence, compare (2.40) and (2.41),

$$\int_0^{Q_1} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} = \sqrt{2\lambda_0}(1-\rho_1), \tag{2.44}$$

$$\alpha Q_1 + \beta\sqrt{2\lambda_0}\sqrt{\int_0^{Q_1} g(v)\mathrm{d}v} = 1. \tag{2.45}$$

Since

$$\alpha Q + \beta\sqrt{2\lambda_0}\sqrt{\int_0^{Q} g(v)\mathrm{d}v} = 1 \tag{2.46}$$

by (2.41), and the function $p(r) := \alpha r + \beta\sqrt{2\lambda_0}\sqrt{\int_0^r g(v)\mathrm{d}v}$ is increasing and continuous on $(0,\infty)$, we deduce from (2.45) and (2.46) that $Q = Q_1$. Then (2.40) and (2.44) yield $\rho = \rho_1$. Therefore, $\int_0^{w(t)}(1/\sqrt{\int_0^s g(v)\mathrm{d}v})\ \mathrm{d}s = \sqrt{2\lambda_0}(t-\rho)$ for $t \in [\rho, 1]$. Finally, since $w(t) = 0$ for $t \in [0,\rho]$ and since by Lemma 2.1 the function φ_0 is increasing on $[0,\infty)$, $u = w$ follows. This completes the proof. □

2.3. Main Results

Let the function χ be given by (2.5) and let us denote by $\mathcal{M}$ the range of the function $\chi^2/2$ restricted to the interval $(0, 1/\alpha]$,

$$\mathcal{M} := \left\{ \frac{(\chi(x))^2}{2} : 0 < x \le \frac{1}{\alpha} \right\}. \tag{2.47}$$

Since $\chi \in C[0, 1/\alpha]$ by Lemma 2.2(iii), $\chi(x) > 0$ for $x \in [0, 1/\alpha)$ and $\chi(1/\alpha) = 0$, we can have either (i) $\chi(x) < \chi(0)$ for $x \in (0, 1/\alpha]$, or (ii) $\chi(x_1) \ge \chi(0)$ for some $x_1 \in (0, 1/\alpha]$. For (i), we have $\mathcal{M} = [0, (\chi(0))^2/2)$, while in case of (ii), $\mathcal{M} = [0, \Gamma]$ with

$$\Gamma := \max\{\tau : \tau \in \mathcal{M}\} \tag{2.48}$$

holds. Clearly, $\Gamma \ge (\chi(0))^2/2$.

Positive solutions of problem (1.6a)-(1.6b) are analyzed in the following theorem.

Theorem 2.10. *Let assumption* (1.7) *hold. Then problem* (1.6a)-(1.6b) *has a positive solution if and only if $\lambda \in \mathcal{M}$. Additionally, for each $a \in (0, 1/\alpha]$, problem* (1.6a)-(1.6b) *with $\lambda = (\chi(a))^2/2$ has a unique positive solution u such that $u(0) = a$ and $u(1) = \gamma(a)$.*

Proof. Let u be a positive solution of problem (1.6a)-(1.6b) for $\lambda > 0$. By Lemma 2.6, (2.31) holds with $a = u(0) > 0$ and $Q = u(1)$. Furthermore, by Lemmas 2.2(ii) and 2.6, $Q = \gamma(a)$, which together with (2.29) implies that $\sqrt{2\lambda} = \chi(a)$. Consequently, $\lambda \in \mathcal{M}$. For $\lambda = 0$, problem

(1.6a)-(1.6b) has the unique positive solution $u = 1/\alpha$; see Remark 2.8. Since $\chi(1/\alpha) = 0$, $0 \in \mathcal{M}$. Consequently, if problem (1.6a)-(1.6b) has a positive solution, then $\lambda \in \mathcal{M}$.

We now show that for each $\lambda \in \mathcal{M}$, problem (1.6a)-(1.6b) has a positive solution, and if $\lambda = (\chi(a))^2/2$ for some $a \in (0, 1/\alpha]$, then problem (1.6a)-(1.6b) has a unique positive solution u such that $u(0) = a$ and $u(1) = \gamma(a)$. Let us choose $\lambda \in \mathcal{M}$. Then $\sqrt{2\lambda} = \chi(a)$ for some $a \in (0, 1/\alpha]$. If $a = 1/\alpha$, then $\chi(a) = 0$. Consequently, $\lambda = 0$ and $u = 1/\alpha$ is the unique solution of problem (1.6a)-(1.6b). Clearly, $u(0) = a$ and $u(1) = \gamma(a)$ since $a = \gamma(a) = 1/\alpha$. Let us suppose that $a \in (0, 1/\alpha)$. If u is a positive solution of problem (1.6a)-(1.6b) and $u(0) = a$, then, by Lemma 2.6; see (2.30), the equality $\varphi_a(u(t)) = \sqrt{2\lambda}t$ holds for $t \in [0, 1]$, where φ_a is given by (2.1). Hence, in order to prove that for $\lambda = (\chi(a))^2/2$ problem (1.6a)-(1.6b) has a unique positive solution u such that $u(0) = a$ and $u(1) = \gamma(a)$, we have to show that the equation

$$\varphi_a(u(t)) = \sqrt{2\lambda}t, \quad t \in [0, 1], \tag{2.49}$$

has a unique solution u; this solution is a positive solution of problem (1.6a)-(1.6b), and $u(0) = a$, $u(1) = \gamma(a)$. Since $\varphi_a \in C[a, \infty) \cap C^1(a, \infty)$, φ_a is increasing by Lemma 2.1, and $\varphi_a(\gamma(a)) = \chi(a)$, (2.49) has a unique solution $u \in C[0, 1]$. It follows from $\varphi_a(a) = 0$ and $\varphi_a(u(1)) = \sqrt{2\lambda} = \chi(a)$ that $u(a) = a$ and $u(1) = \gamma(a)$. In addition,

$$u'(t) = \frac{\sqrt{2\lambda}}{\varphi_a'(u(t))} = \sqrt{2\lambda \int_a^{u(t)} g(v)\mathrm{d}v}, \quad t \in (0, 1]. \tag{2.50}$$

Hence, $u \in C^1(0, 1]$ and $\lim_{t \to 0^+} u'(t) = 0$. In order to show that u' is continuous at $t = 0$, we set $M = \max\{g(s) : a \le s \le 1/\alpha\} > 0$. Then, compare (2.49),

$$\sqrt{2\lambda}t = \int_a^{u(t)} \frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}} \ge \frac{1}{\sqrt{M}} \int_a^{u(t)} \frac{\mathrm{d}s}{\sqrt{s-a}} = 2\sqrt{\frac{u(t)-a}{M}}, \tag{2.51}$$

and therefore,

$$0 < \frac{u(t)-u(0)}{t} = \frac{u(t)-a}{t} \le \frac{M\lambda t}{2}, \quad t \in (0, 1]. \tag{2.52}$$

Consequently, $u'(0) = \lim_{t \to 0^+}((u(t) - a)/t) = 0$, and so u' is continuous at $t = 0$, or equivalently, $u \in C^1[0, 1]$. Now (2.50) indicates that $u \in C^2(0, 1]$ and

$$u''(t) = \sqrt{2\lambda} \frac{g(u(t))u'(t)}{2\sqrt{\int_a^{u(t)} g(v)\mathrm{d}v}} = \lambda g(u(t)) \quad \text{for } t \in (0, 1]. \tag{2.53}$$

Moreover, by the de L'Hospital rule,

$$\begin{aligned}\lim_{t\to 0^+}\frac{u'(t)-u'(0)}{t} &= \lim_{t\to 0^+}\frac{u'(t)}{t} = \lim_{t\to 0^+}\frac{1}{t}\sqrt{2\lambda\int_a^{u(t)} g(v)\mathrm{d}v} \\ &= \sqrt{2\lambda}\lim_{t\to 0^+}\frac{g(u(t))u'(t)}{2\sqrt{\int_a^{u(t)} g(v)\mathrm{d}v}} = \lambda\lim_{t\to 0^+} g(u(t)) \\ &= \lambda g(u(0)).\end{aligned} \tag{2.54}$$

As a result $u \in C^2[0,1]$ and $u''(t) = \lambda g(u(t))$ for $t \in [0,1]$. Since $u(1) = \gamma(a)$ and, by (2.50), $u'(1) = \chi(a)\sqrt{\int_a^{\gamma(a)} g(v)\mathrm{d}v}$, we have

$$\alpha u(1) + \beta u'(1) = \alpha\gamma(a) + \beta\int_a^{\gamma(a)}\frac{\mathrm{d}s}{\sqrt{\int_a^s g(v)\mathrm{d}v}}\sqrt{\int_a^{\gamma(a)} g(v)\mathrm{d}v} = H\big(a,\gamma(a)\big) = 1 \tag{2.55}$$

by Lemma 2.2(ii). Thus, u satisfies (1.6b), and therefore u is a unique positive solution of problem (1.6a)-(1.6b) such that $u(0) = a$ and $u(1) = \gamma(a)$. □

The following theorem deals with multiple positive solutions of problem (1.6a)-(1.6b).

Theorem 2.11. *Let assumption* (1.8) *hold. Then* $\Gamma > (\chi(0))^2/2$, *with* Γ *given by* (2.48), *and for each* $\lambda \in ((\chi(0))^2/2, \Gamma)$, *there exist multiple positive solutions of problem* (1.6a)-(1.6b).

Proof. By Lemmas 2.2(iii) and 2.3, $\chi \in C[0, 1/\alpha]$, $\chi(1/\alpha) = 0$, and $\chi(x) > \chi(0)$ in a right neighbourhood of $x = 0$. Hence, $\Gamma > (\chi(0))^2/2$. Let us choose $\lambda \in ((\chi(0))^2/2, \Gamma)$. Then there exist $0 < x_1 < x_2 < 1/\alpha$ such that $\lambda = (\chi(x_j))^2/2$ for $j = 1, 2$. Now Theorem 2.10 guarantees that problem (1.6a)-(1.6b) has positive solutions u_1 and u_2 such that $u_j(0) = x_j$, $j = 1, 2$. Since $x_1 \neq x_2$, we have $u_1 \neq u_2$ and therefore, for each $\lambda \in ((\chi(0))^2/2, \Gamma)$, problem (1.6a)-(1.6b) has multiple positive solutions. □

Next, we present results for pseudo dead core solutions of problem (1.6a)-(1.6b). Note that here $\Lambda = \gamma(0)$.

Theorem 2.12. *Let assumption* (1.7) *hold. Then problem* (1.6a)-(1.6b) *has a pseudo dead core solution if and only if*

$$\lambda = \frac{1}{2}\left(\int_0^{\Lambda}\frac{ds}{\sqrt{\int_0^s g(v)dv}}\right)^2. \tag{2.56}$$

Moreover, for λ *given by* (2.56), *problem* (1.6a)-(1.6b) *has a unique pseudo dead core solution such that* $u(1) = \Lambda$.

Proof. Let us assume that u is a pseudo dead core solution of problem (1.6a)-(1.6b) and let $Q := u(1)$. Then, by Remark 2.7, equalities (2.36), (2.38) hold, and $Q = \Lambda$. Also, (2.37) implies that u is a solution of the equation

$$\varphi_0(u(t)) = \sqrt{2\lambda}t, \quad t \in [0,1], \tag{2.57}$$

where φ_0 and λ are given by (2.1) and (2.56), respectively. The result follows by showing that equation (2.57) has a unique solution and that this solution is a pseudo dead core solution of problem (1.6a)-(1.6b). We verify these facts for solutions of (2.57) arguing as in the proof of Theorem 2.10, with a replaced by 0. □

In the final theorem below, we deal with dead core solutions of problem (1.6a)-(1.6b).

Theorem 2.13. *Let assumption* (1.7) *hold and let μ be the function defined in Lemma 2.4. Then the following statements hold.*

(i) *Problem* (1.6a)-(1.6b) *has a dead core solution if and only if*

$$\lambda > \frac{1}{2}\left(\int_0^\Lambda \frac{ds}{\sqrt{\int_0^s g(v)dv}}\right)^2. \tag{2.58}$$

(ii) *For each λ satisfying* (2.58), *problem* (1.6a)-(1.6b) *has a unique dead core solution.*

(iii) *If the subinterval $[0,\rho]$ is the dead core of a dead core solution u of problem* (1.6a)-(1.6b), *then* $\max\{u(t) : 0 \le t \le 1\} = \mu(\rho)$ *and*

$$\int_0^{\mu(\rho)} \frac{ds}{\sqrt{\int_0^s g(v)dv}} = (1-\rho)\sqrt{2\lambda}. \tag{2.59}$$

Proof. (i) Let u be a dead core solution of problem (1.6a)-(1.6b) for some $\lambda = \lambda_0$ and let $Q := u(1)$. Then there exists a point $\rho \in (0,1)$ such that $u(t) = 0$ for $t \in [0,\rho]$, and equalities (2.39), (2.40), and (2.41) are satisfied by Lemma 2.9. We deduce from (2.41) and from Lemma 2.4 that $Q = \mu(\rho)$. Therefore, compare (2.40),

$$\frac{1}{1-\rho}\int_0^{\mu(\rho)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} = \sqrt{2\lambda_0}. \tag{2.60}$$

Since

$$\frac{1}{1-\rho}\int_0^{\mu(\rho)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} > \int_0^\Lambda \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} \tag{2.61}$$

by Corollary 2.5, we have

$$\lambda_0 > \frac{1}{2}\left(\int_0^{\Lambda} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}}\right)^2. \tag{2.62}$$

Hence, if problem (1.6a)-(1.6b) has a dead core solution, then λ satisfies inequality (2.58).

We now prove that for each λ satisfying (2.58), problem (1.6a)-(1.6b) has a dead core solution. Let us choose λ satisfying (2.58). Then, by Corollary 2.5, there exists a unique $\rho \in (0,1)$ such that

$$\int_0^{\mu(\rho)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} = (1-\rho)\sqrt{2\lambda}. \tag{2.63}$$

Let us now consider, compare (2.39),

$$\varphi_0(w(t)) = (t-\rho)\sqrt{2\lambda}, \quad t \in [\rho, 1], \tag{2.64}$$

where φ_0 is given by (2.1). Since $\varphi_0 \in C[0,\infty) \cap C^1(0,\infty)$ and φ_0 is increasing on $[0,\infty)$ by Lemma 2.1, $\varphi_0(0) = 0$, and, by (2.63), $\varphi_0(\mu(\rho)) = (1-\rho)\sqrt{2\lambda}$, there exists a unique solution $w \in C[\rho, 1]$ of (2.64) and $w(\rho) = 0$, $w(1) = \mu(\rho)$. In addition,

$$w'(t) = \frac{\sqrt{2\lambda}}{\varphi_0'(w(t))} = \sqrt{2\lambda\int_0^{w(t)} g(v)\mathrm{d}v}, \quad t \in (\rho, 1], \tag{2.65}$$

and consequently, $w \in C^1(\rho, 1]$ and $\lim_{t\to\rho^+} w'(t) = 0$. Since

$$(t-\rho)\sqrt{2\lambda} = \int_0^{w(t)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} = \frac{w(t)}{\sqrt{\int_0^{\xi(t)} g(v)\mathrm{d}v}}, \quad t \in (\rho, 1], \tag{2.66}$$

by the Mean Value Theorem for integrals, where $0 < \xi(t) < w(t)$, we have

$$\frac{w(t) - w(\rho)}{t-\rho} = \frac{w(t)}{t-\rho} = \sqrt{2\lambda\int_0^{\xi(t)} g(v)\mathrm{d}v}. \tag{2.67}$$

Therefore,

$$\lim_{t\to\rho^+} \frac{w(t) - w(\rho)}{t-\rho} = \lim_{t\to\rho^+} \sqrt{2\lambda\int_0^{\xi(t)} g(v)\mathrm{d}v} = 0 \tag{2.68}$$

since $\lim_{t\to\rho^+}\xi(t) = 0$. Hence, w' is continuous at $t = \rho$, and $w \in C^1[\rho, 1]$. Furthermore,

$$w''(t) = \frac{\sqrt{2\lambda}\, g(w(t))w'(t)}{2\sqrt{\int_0^{w(t)} g(v)\mathrm{d}v}} = \lambda g(w(t)), \quad t \in (\rho, 1]. \tag{2.69}$$

Let

$$u(t) := \begin{cases} 0, & \text{for } t \in [0, \rho), \\ w(t), & \text{for } t \in [\rho, 1]. \end{cases} \tag{2.70}$$

Then $u \in C^1[0,1] \cap C^2(\rho, 1]$, $u''(t) = \lambda g(u(t))$ for $t \in (\rho, 1]$, $u(\rho) = u'(\rho) = 0$, $u(1) = \mu(\rho)$, and

$$u'(1) = \sqrt{2\lambda \int_0^{u(1)} g(v)\mathrm{d}v} = \frac{1}{1-\rho}\int_0^{\mu(\rho)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} \sqrt{\int_0^{\mu(\rho)} g(v)\mathrm{d}v}. \tag{2.71}$$

Thus,

$$\alpha u(1) + \beta u'(1) = \alpha\mu(\rho) + \frac{\beta}{1-\rho}\int_0^{\mu(\rho)} \frac{\mathrm{d}s}{\sqrt{\int_0^s g(v)\mathrm{d}v}} \sqrt{\int_0^{\mu(\rho)} g(v)\mathrm{d}v} = h(\rho, \mu(\rho)), \tag{2.72}$$

where h is given by (2.3). Since $h(\rho, \mu(\rho)) = 1$ by Lemma 2.4, u satisfies the boundary conditions (1.6b). Consequently, u is a dead core solution of problem (1.6a)-(1.6b).

(ii) Let us choose an arbitrary λ satisfying (2.58). By (i), problem (1.6a)-(1.6b) has a dead core solution which is unique by Lemma 2.9.

(iii) Let the subinterval $[0, \rho]$ be the dead core of a dead core solution u of problem (1.6a)-(1.6b). Then, by Lemma 2.9, equalities (2.40) and (2.41) hold with λ_0 replaced by λ and $Q = \max\{u(t) : 0 \le t \le 1\}$. Since $h(\rho, \mu(\rho)) = 1$ by the definition of the function μ, we have $\mu(\rho) = Q$. Equality (2.59) now follows from (2.40) with Q and λ_0 replaced by $\mu(\rho)$ and λ, respectively. □

Example 2.14. We now turn to the case study of the boundary value problem (1.9a)-(1.9b),

$$u''(t) = \frac{\lambda}{\sqrt{u(t)}}, \quad u'(0) = 0, \quad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0,\ \beta > 0. \tag{2.73}$$

Note that (1.9a)-(1.9b) is a special case of (1.6a)-(1.6b) with $g(u) = 1/\sqrt{u}$ satisfying (1.8).

Since

$$\int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s (1/\sqrt{v})\mathrm{d}v}} = \frac{4}{3\sqrt{2}}\sqrt{\sqrt{y}-\sqrt{x}}\left(\sqrt{y}+2\sqrt{x}\right), \quad 0 \le x < y, \tag{2.74}$$

we have

$$H(x,y) = \alpha y + \beta \int_x^y \frac{\mathrm{d}s}{\sqrt{\int_x^s (1/\sqrt{v})\mathrm{d}v}} \sqrt{\int_x^y \frac{\mathrm{d}v}{\sqrt{v}}} = \alpha y + \frac{4\beta}{3}\left(\sqrt{y}-\sqrt{x}\right)\left(\sqrt{y}+2\sqrt{x}\right) \tag{2.75}$$

for $0 \le x < y$, and $H(x,x) = \alpha x$ for $x \ge 0$. By Lemma 2.2, the equation $H(x,y) = 1$ has a unique solution $y = \gamma(x)$ for $x \in [0, 1/\alpha]$, $\gamma \in C[0,1/\alpha]$, $\gamma(x) > x$ for $x \in [0,1/\alpha)$, and $\gamma(1/\alpha) = 1/\alpha$. Let

$$k(x) := \frac{x}{\gamma(x)} \quad \text{for } x \in \left[0, \frac{1}{\alpha}\right]. \tag{2.76}$$

Then $k \in C[0,1/\alpha]$, $k(0) = 0$, and $k(1/\alpha) = 1$. In order to show that k is increasing on $[0,1/\alpha]$ it is sufficient to verify that k is injective. Let us assume that this is not the case, then there exist $x_1, x_2 \in [0,1/\alpha]$, $x_1 \neq x_2$, such that $k(x_1) = k(x_2)$. From $H(x_j, \gamma(x_j)) = 1$, $j = 1,2$, or equivalently, from

$$3\alpha + 4\beta\left(1 - \sqrt{k(x_j)}\right)\left(1 + 2\sqrt{k(x_j)}\right) = \frac{3}{\gamma(x_j)}, \qquad j = 1,2, \tag{2.77}$$

it follows that $\gamma(x_1) = \gamma(x_2)$, and $x_1 = x_2$, which is a contradiction. Hence, k is increasing on $[0,1/\alpha]$ and therefore, there exists the inverse function k^{-1} mapping $[0,1]$ onto $[0,1/\alpha]$. Since

$$H(k(x)\gamma(x), \gamma(x)) = \gamma(x)\left[\alpha + \frac{4\beta}{3}\left(1 - \sqrt{k(x)}\right)\left(1 + 2\sqrt{k(x)}\right)\right] \tag{2.78}$$

and $H(k(x)\gamma(x), \gamma(x)) = 1$ for $x \in [0,1/\alpha]$, we have

$$\gamma(x) = \frac{3}{3\alpha + 4\beta\left(1 - \sqrt{k(x)}\right)\left(1 + 2\sqrt{k(x)}\right)}. \tag{2.79}$$

Consequently,

$$\begin{aligned}
(\chi(x))^2 &= \left(\int_x^{\gamma(x)} \frac{\mathrm{d}s}{\sqrt{\int_x^s (1/\sqrt{v})\mathrm{d}v}}\right)^2 \\
&= \frac{8}{9}\left(\sqrt{\gamma(x)}-\sqrt{x}\right)\left(\sqrt{\gamma(x)}+2\sqrt{x}\right)^2 \\
&= \frac{8}{9}(\gamma(x))^{3/2}\left(1-\sqrt{k(x)}\right)\left(1+2\sqrt{k(x)}\right)^2 \\
&= \frac{8}{9}\left(\frac{3}{3\alpha+4\beta\left(1-\sqrt{k(x)}\right)\left(1+2\sqrt{k(x)}\right)}\right)^{3/2}\left(1-\sqrt{k(x)}\right)\left(1+2\sqrt{k(x)}\right)^2.
\end{aligned} \tag{2.80}$$

In order to discuss the range $\mathcal{M}$ of the function $\chi^2/2$ and the value of $(\chi(x))^2/2$ for $x \in [0,1/\alpha]$, we first consider properties of the function

$$\delta(x) := \frac{4}{9}\left(\frac{3}{3\alpha+4\beta(1-\sqrt{x})(1+2\sqrt{x})}\right)^{3/2}(1-\sqrt{x})(1+2\sqrt{x})^2 \tag{2.81}$$

defined on $[0,1]$. Let

$$f(x) := \delta\left(\left(\frac{x-1}{2}\right)^2\right) \quad \text{for } x \in [1,3]. \tag{2.82}$$

Then

$$\begin{aligned}
f(x) &= \frac{2}{9}\left(\frac{3}{3\alpha+2\beta x(3-x)}\right)^{3/2} x^2(3-x), \\
f'(x) &= \frac{2x}{3}\left(\frac{3}{3\alpha+2\beta x(3-x)}\right)^{3/2}\frac{p(x)}{3\alpha+2\beta x(3-x)},
\end{aligned} \tag{2.83}$$

where $p(x) = 6\alpha + 3(\beta-\alpha)x - \beta x^2$. The function p vanishes only at point

$$x_* = \frac{1}{2\beta}\left(3(\beta-\alpha)+\sqrt{9(\beta-\alpha)^2+24\alpha\beta}\right) \tag{2.84}$$

in the interval $[1,3]$, and $2 < x_* < 3$, because $p(2) > 0$ and $p(3) < 0$. Since $f'(x_*) = 0$, $p > 0$ on $[1,x_*)$, $p < 0$ on $(x_*,3]$ and

$$\frac{2x}{3}\left(\frac{3}{3\alpha+2\beta x(3-x)}\right)^{3/2}\frac{1}{3\alpha+2\beta x(3-x)} > 0 \tag{2.85}$$

for $x \in [1,3]$, we have $f' > 0$ on $[1, x_*)$ and $f' < 0$ on $(x_*, 3]$. Let us define $k_* := ((x_* - 1)/2)^2$. Then $k_* \in (1/4, 1)$, and it follows from $f'(x) = ((x-1)/2)\delta'(((x-1)/2)^2)$ that $\delta' > 0$ on $(0, k_*)$ and $\delta' < 0$ on $(k_*, 1]$. Consequently, δ is increasing on $[0, k_*]$ and decreasing on $[k_*, 1]$. It follows from the equality $(\chi(x))^2 = 2\delta(k(x))$ for $x \in [0, 1/\alpha]$ and from the properties of the functions δ and k that χ^2 is increasing on $[0, k^{-1}(k_*)]$ and decreasing on $[k^{-1}(k_*), 1/\alpha]$. Hence, $\mathcal{M} = [0, M]$, where $M := \max\{\delta(x) : 0 \le x \le 1\}$. Also,

$$(\chi(0))^2 = \frac{8}{9}\left(\frac{3}{3\alpha + 4\beta}\right)^{3/2}, \quad \chi(1) = 0. \tag{2.86}$$

Using properties of the function χ and the results of Theorems 2.10–2.13, we can now characterize the structure of the solution u.

(i) For each $\lambda \in (M, \infty)$, there exists only a unique dead core solution of problem (1.9a)-(1.9b).

(ii) For $\lambda = M$, there exist a unique dead core solution and a unique positive solution of problem (1.9a)-(1.9b).

(iii) For each $\lambda \in (4/9(3/(3\alpha + 4\beta))^{3/2}, M)$, there exist a unique dead core solution and exactly two positive solutions of problem (1.9a)-(1.9b).

(iv) For $\lambda = 4/9(3/(3\alpha + 4\beta))^{3/2}$, there exist the unique pseudo dead core solution $u(t) = (3/(3\alpha + 4\beta))t^{4/3}$ and a unique positive solution of problem (1.9a)-(1.9b).

(v) For each $\lambda \in [0, 4/9(3/(3\alpha + 4\beta))^{3/2})$, there exist only a unique positive solution of problem (1.9a)-(1.9b).

Using Theorem 2.10, Lemma 2.6, and the properties of the function δ, we can specify further properties of positive solutions of problem (1.9a)-(1.9b).

(i) If u is the (unique) positive solution of problem (1.9a)-(1.9b) with $\lambda \in [0, 4/9(3/(3\alpha + 4\beta))^{3/2}) \cup \{M\}$, then $u(0) = x_0 u(1)$, where $x_0 \neq 0$ is the root of the equation $\delta(x) - \lambda = 0$.

(ii) If u_1, u_2 are the (unique) positive solutions of problem (1.9a)-(1.9b) with $\lambda \in (4/9(3/(3\alpha + 4\beta))^{3/2}, M)$, then $u_j(0) = x_j u_j(1)$, $j = 1, 2$, where $0 < x_1 < k_* < x_2 < 1$, are the roots of the equation $\delta(x) - \lambda = 0$.

We are also able to give some more information on the dead core solutions of problem (1.9a)-(1.9b). Since

$$h(t, y) = \alpha y + \frac{\beta}{1-t}\int_0^y \frac{\mathrm{d}s}{\sqrt{\int_0^s (1/\sqrt{v})\mathrm{d}v}}\sqrt{\int_0^y \frac{\mathrm{d}v}{\sqrt{v}}} = \alpha y + \frac{4\beta y}{3(1-t)}, \tag{2.87}$$

the function $\mu(t) = 3(1-t)/(3\alpha(1-t) + 4\beta)$, $t \in [0, 1)$, is the solution of the equation $h(t, y) = 1$. Let us choose an arbitrary $\lambda > 4/9(3/(3\alpha + 4\beta))^{3/2}$. By Corollary 2.5, the equation; see (2.28),

$$\frac{4}{3\sqrt{2}}\left(\frac{3(1-t)}{3\alpha(1-t) + 4\beta}\right)^{3/4} = (1-t)\sqrt{2\lambda}, \quad t \in [0, 1), \tag{2.88}$$

has a unique solution $\rho \in (0,1)$. Consequently,

$$\lambda = \frac{4}{9\sqrt{1-\rho}}\left(\frac{3}{3\alpha(1-\rho)+4\beta}\right)^{3/2}. \tag{2.89}$$

One can easily show that the function

$$u(t) = \begin{cases} 0, & \text{for } t \in [0,\rho], \\ \left(\dfrac{3}{2}\sqrt{\lambda}(t-\rho)\right)^{4/3}, & \text{for } t \in (\rho,1], \end{cases} \tag{2.90}$$

is the unique dead core solution of problem (1.9a)-(1.9b). Additionally, it follows from Theorem 2.13(iii) that $\max\{u(t) : 0 \le t \le 1\} = 3(1-\rho)/(3\alpha(1-\rho)+4\beta)$ since $\max\{u(t) : 0 \le t \le 1\} = \mu(\rho)$.

3. Numerical Treatment

We now aim at the numerical approximation to the solution of the following two-point boundary value problem:

$$\begin{gathered} u''(t) = f(t,u(t)), \quad t \in [0,1], \\ u'(0) = 0, \quad \beta u'(1) + \alpha u(1) = A, \quad \beta \ge 0, \quad \alpha, A > 0. \end{gathered} \tag{3.1}$$

For the numerical solution of (3.1), we are using the collocation method implemented in our Matlab code `bvpsuite`. It is a new version of the general purpose Matlab code `sbvp`, compare [10–12]. This code has already been used to treat a variety of problems relevant in application; see, for example, [13–17]. Collocation is a widely used and well-studied standard solution method for two-point boundary value problems, compare [18] and the references therein. It can also be successfully applied to boundary value problems with singularities.

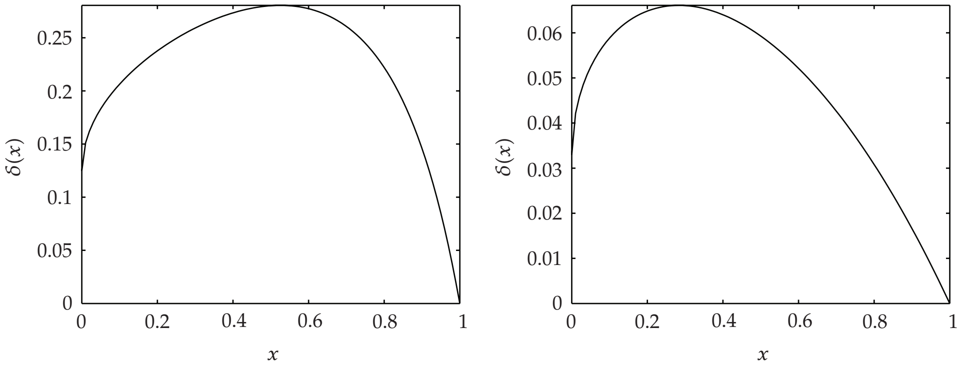
In the scope of the code are systems of ordinary differential equations of arbitrary order. For simplicity of notation we present a problem of maximal order four which can be given in a fully implicit form,

$$F\left(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t)\right) = 0, \quad 0 \le t \le 1, \tag{3.2a}$$

$$b\left(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)\right) = 0. \tag{3.2b}$$

In order to compute the numerical approximation, we first introduce a mesh

$$\Delta_h := \{\tau_i : i = 0, \ldots, N\}, \quad 0 = \tau_0 < \tau_1 \cdots < \tau_N = 1. \tag{3.3}$$



(a) (b)

Figure 1: $\delta(x)$ for $\alpha = \beta = 1$ (a) and for $\alpha = 5, \beta = 0.5$ (b).

The approximation for u is a collocation function

$$p(t) := p_i(t), \quad t \in [\tau_i, \tau_{i+1}], \; i = 0, \ldots, N-1, \tag{3.4}$$

where we require $p \in C^{q-1}[0,1]$ in case that the order of the underlying differential equation is q. Here, p_i are polynomials of maximal degree $m - 1 + q$ which satisfy the system (3.2a) at m inner *collocation points*

$$\left\{ t_{i,j} = \tau_i + \rho_j(\tau_{i+1} - \tau_i), \; i = 0, \ldots, N-1, \; j = 1, \ldots, m \right\}, \quad 0 < \rho_1 < \cdots < \rho_m < 1, \tag{3.5}$$

and the associated boundary conditions (3.2b).

Classical theory, compare [18], predicts that the convergence order for the global error of the method is at least $O(h^{\mathrm{m}})$, where h is the maximal stepsize, $h := \max_i(\tau_{i+1} - \tau_i)$. To increase efficiency, an adaptive mesh selection strategy based on an a posteriori estimate for the global error of the collocation solution is utilized. A more detailed description of the numerical approach can be found in [4].

The code `bvpsuite` also allows to follow a path in the parameter-solution space. This means that in the following problem setting, parameter ϑ is unknown:

$$F\left(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t), \vartheta\right) = 0, \quad 0 \le t \le 1, \tag{3.6a}$$

$$b\left(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)\right) = 0, \quad \vartheta = \lambda, \tag{3.6b}$$

where λ is given. The path following strategy can also cope with turning points in the path. The theoretical justification for the path following strategy implemented in `bvpsuite` has been given in [19].

We first study the boundary problem (1.9a)-(1.9b). Positive solutions of problem (1.5a)-(1.5b) will be discussed in Section 3.4.

The above analytical discussion indicates that depending on the values of α, β, λ, the problem has one or more positive solutions, a pseudo dead core solution or a dead core solution. All numerical approximations have been calculated on a fixed mesh with $N = 500$ subintervals and collocation degree $m = 4$. Figure 1 shows $\delta(x)$ for our choice of parameters used in the following sections. Here, δ is given by (2.81).

3.1. Positive Solutions

For $\lambda \in [0, \delta(0))$, $\delta(0) = (4/9)(3/(3\alpha + 4\beta))^{3/2}$, there exist a unique positive solution. This solution was found numerically by using the original problem formulation (1.9a)-(1.9b). For $\alpha = \beta = 1$ we obtain $\delta(0) \approx 0.12469$. In Figure 2 we display the numerical solution, the error estimate and the residual for $\lambda = 0.05$. The residual $r(t)$ is calculated by substituting the numerical solution $p(t)$ into the differential equation,

$$r(t) := p''(t) - \frac{\lambda}{\sqrt{p(t)}}. \tag{3.7}$$

Due to the very small size of the error estimate and residual, it is obvious that the numerical approximation is very accurate. According to the analytical results, a solution to the problem satisfies $|u(0) - x_0 u(1)| = 0$ where x_0 is a root of $\delta(x) - \lambda = 0$. Here, we have $x_0 = 0.972608$ and $|u(0) - x_0 u(1)| = 6.4\,10^{-8}$ which again shows the high quality of the numerical solution. In Figure 3 we depict the results for the parameter $\alpha = 5$, $\beta = 0.5$ and $\lambda = 0.02 < \delta(0) \approx 0.03294$. For this choice of parameters $x_0 = 0.877692$ and $|u(0) - x_0 u(1)| = 3.5\,10^{-8}$.

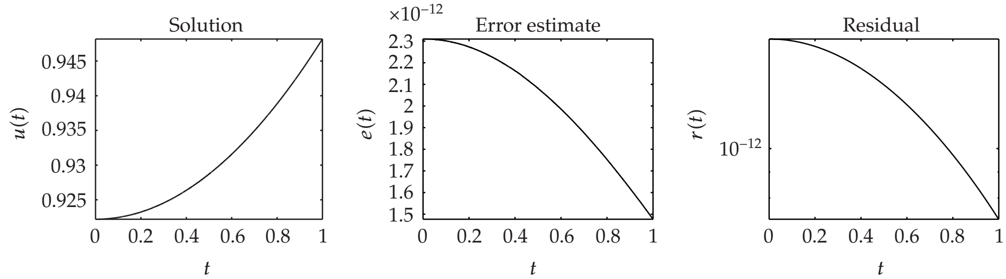
For $\lambda = \delta(0) = (4/9)(3/(3\alpha+4\beta))^{3/2}$ there exists a unique positive solution. To compute its numerical approximation, we rewrite the problem (1.9a)-(1.9b) and consider

$$\begin{gathered} u''(t)\sqrt{u(t)} = \lambda = \frac{4}{9}\left(\frac{3}{3\alpha + 4\beta}\right)^{3/2}, \quad t \in [0,1], \\ u'(0) = 0, \qquad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0, \ \beta > 0. \end{gathered} \tag{3.8}$$

The numerical results related to parameter sets $\alpha = 1$, $\beta = 1$, and $\alpha = 5$, $\beta = 0.5$ are shown in Figure 4 and Figure 5, respectively.

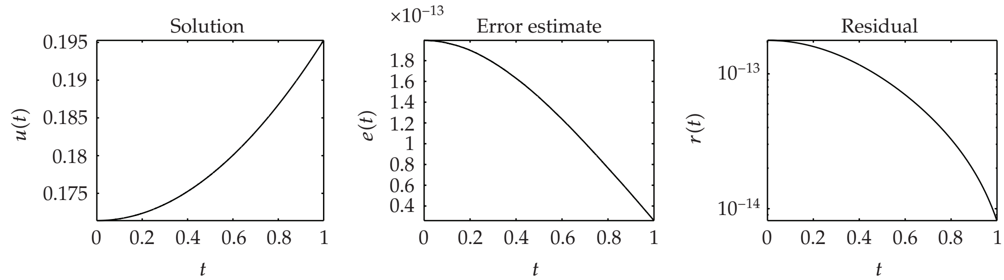
Again, the error estimate and the residual are both very small and $x_0 = 0.919315$, so $|u(0) - x_0 u(1)| = 3.5\,10^{-7}$. Moreover, for the second set of parameters, $x_0 = 0.783283$ and $|u(0) - x_0 u(1)| = 1.7\,10^{-8}$.

For $\lambda \in (\delta(0), M)$ with $M = \max\{\delta(x) : 0 \le x \le 1\}$ there exist two positive solutions. These two different solutions for a fixed value of λ can be characterized via the roots $x_{1,2}$ of $\delta(x) - \lambda = 0$ for $x \in [0,1]$. The choice of parameters remains the same. For $\alpha = 1$, $\beta = 1$ and $\lambda = 0.15$ the solution corresponding to $x_1 \approx 0.009159$ is shown in Figure 6. The solution corresponding to $x_2 \approx 0.896054$ is depicted in Figure 7. Note that for these values of α and β we have $M \approx 0.28049$.



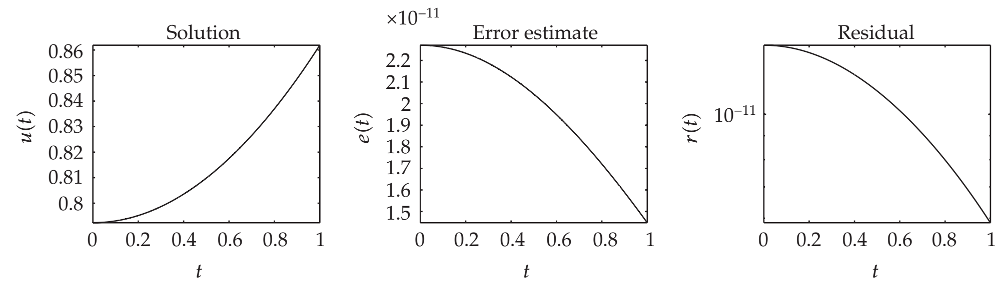
(a) (b) (c)

Figure 2: Problem (1.9a)-(1.9b): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = 0.05$.



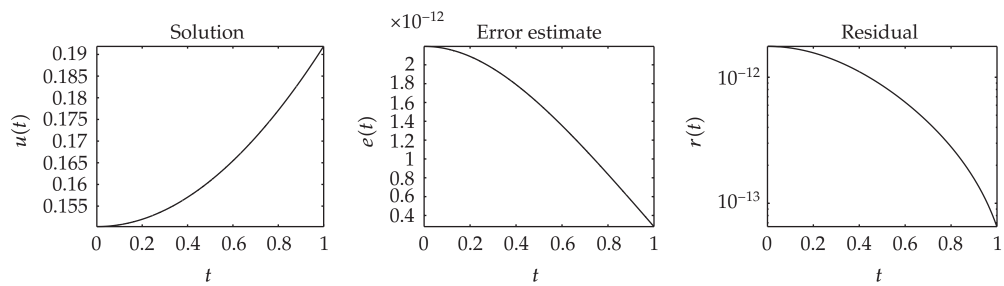
(a) (b) (c)

Figure 3: Problem (1.9a)-(1.9b): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = 0.02$.



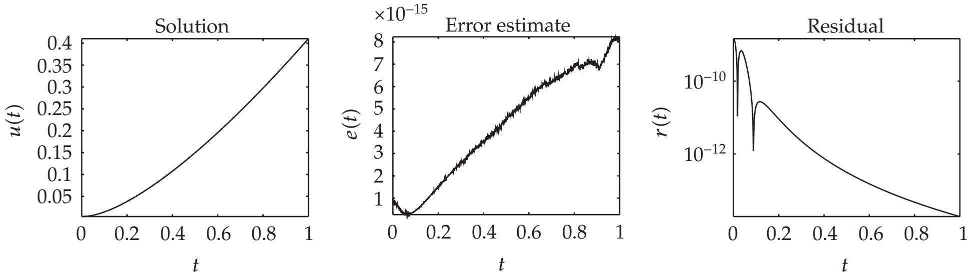
(a) (b) (c)

Figure 4: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = (4/9)(3/(3\alpha + 4\beta))^{3/2}$.



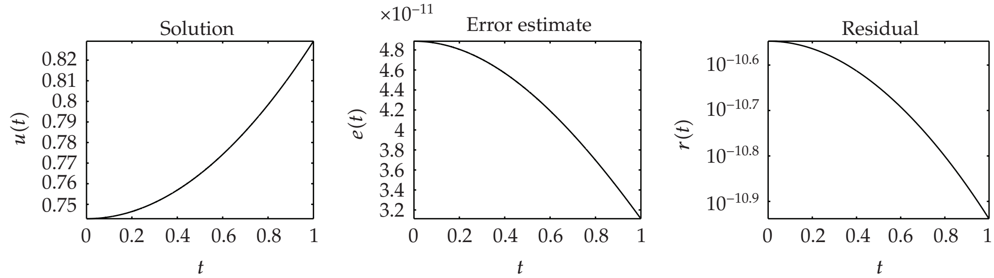
(a) (b) (c)

Figure 5: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = (4/9)(3/(3\alpha + 4\beta))^{3/2}$.



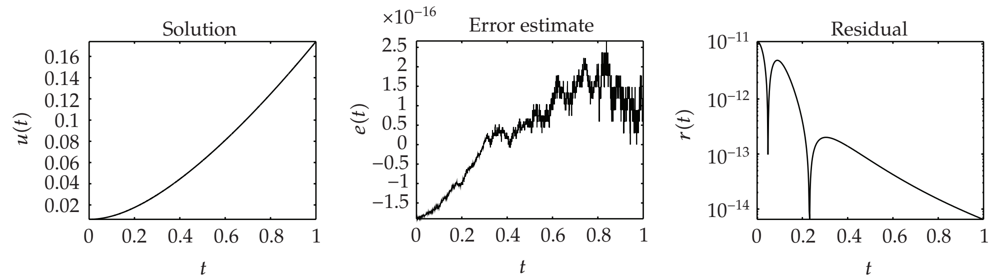
(a) (b) (c)

Figure 6: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = 0.15$. The associated root is x_1.



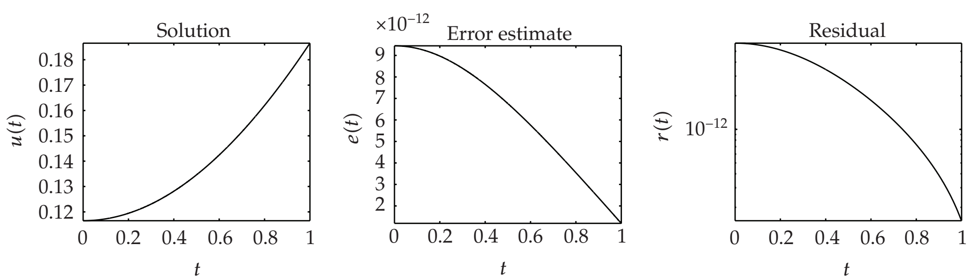
(a) (b) (c)

Figure 7: Problem (3.9): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = 0.15$. The associated root is x_2.



(a) (b) (c)

Figure 8: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = 0.05$. The associated root is x_1.



(a) (b) (c)

Figure 9: Problem (3.9): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = 0.05$. The associated root is x_2.

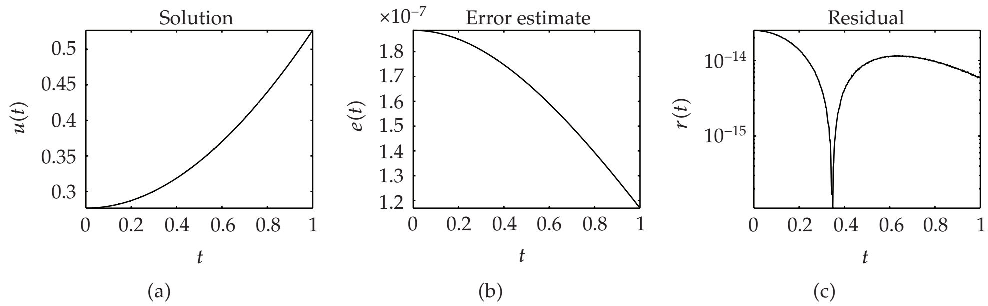


Figure 10: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = 0.28049410745840$.

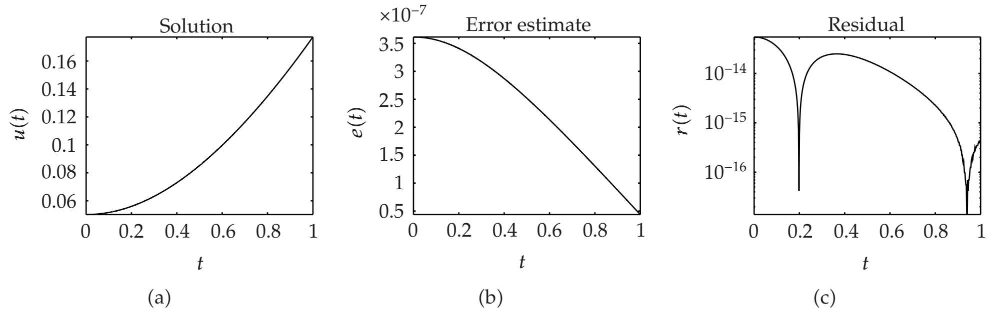


Figure 11: Problem (3.8): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = 0.06608546529011$.

The first of those two solutions was found using the reformulated problem (3.8) with λ as the right-hand side. For the second solution it was necessary to rewrite the problem again and use

$$u''(t)\sqrt{u(t)} = \frac{4}{9}\left(\frac{3}{3\alpha + 4\beta(1-\sqrt{x})(1+2\sqrt{x})}\right)^{3/2}\left(1-\sqrt{x}\right)\left(1+2\sqrt{x}\right)^2, \quad t \in [0,1],$$
$$u'(0) = 0, \quad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0,\ \beta > 0, \tag{3.9}$$

with x as a free unknown parameter and $x = x_2$ as a necessary additional boundary condition. Here, $x_1 = 0.009159$ and $|u(0) - x_1 u(1)| = 1.7\,10^{-15}$. For comparison, $x_2 = 0.896054$ and $|u(0) - x_2 u(1)| = 4.1\,10^{-7}$. In Figures 8 and 9, two different positive solutions for the second parameter set, $\alpha = 5$, $\beta = 0.5$, and $\lambda = 0.05$, are shown. Note that $M \approx 0.06608$, $x_1 \approx 0.037199$ and $x_2 \approx 0.624635$. For this example $x_1 = 0.037119$ and $|u(0) - x_1 u(1)| = 5.5\,10^{-17}$. Here, $x_2 = 0.624635$ and $|u(0) - x_2 u(1)| = 8.1\,10^{-8}$. Finally, for $\lambda = M$, there exists a unique positive solution. In Figures 10 and 11 we display the numerical results for $\alpha = 1$, $\beta = 1$ and for $\alpha = 5$, $\beta = 0.5$, respectively. In this example, $x_0 = 0.525260$ and $|u(0) - x_0 u(1)| = 2.5\,10^{-6}$. Using this latter set of parameters, we obtain $x_0 = 0.283205$ and $|u(0) - x_0 u(1)| = 4.1\,10^{-6}$. All positive solutions could be easily found and they all show a very satisfactory level of accuracy.

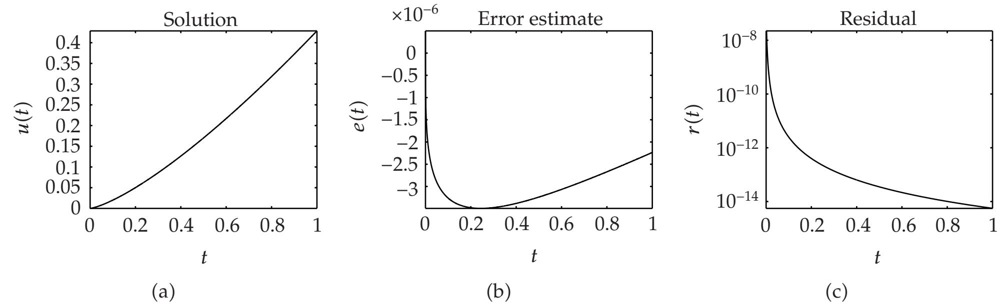


Figure 12: Problem (3.10): The numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $\lambda = (4/9)(3/(3\alpha + 4\beta))^{3/2}$.

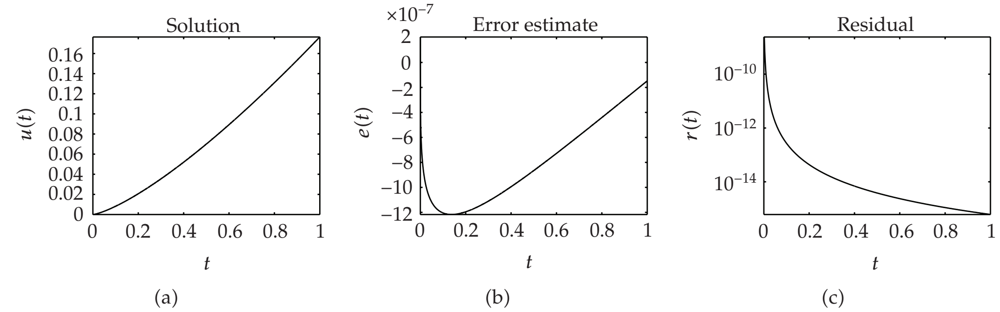


Figure 13: Problem (3.10): The numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $\lambda = (4/9)(3/(3\alpha + 4\beta))^{3/2}$.

3.2. Pseudo Dead Core Solutions

In order to calculate the pseudo dead core solutions, we solved the following problem:

$$\begin{gathered} u''(t)\sqrt{u(t)}u(t) = \lambda u(t), \quad t \in [0,1], \\ u'(0) = 0, \quad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0, \ \beta > 0, \end{gathered} \tag{3.10}$$

where the differential equation has been premultiplied by the factor u. Otherwise, the problem formulated as (3.1) or (3.8), would have not been well defined at all points $t \neq 0$ such that $u(t) = 0$. In Figures 12 and 13, we report on the pseudo dead core solutions for

$$\lambda = \frac{4}{9}\left(\frac{3}{3\alpha + 4\beta}\right)^{3/2}. \tag{3.11}$$

In this case, the analytical unique pseudo dead core solution is known,

$$u(t) = \frac{3}{3\alpha + 4\beta} t^{4/3}, \quad t \in [0,1]. \tag{3.12}$$

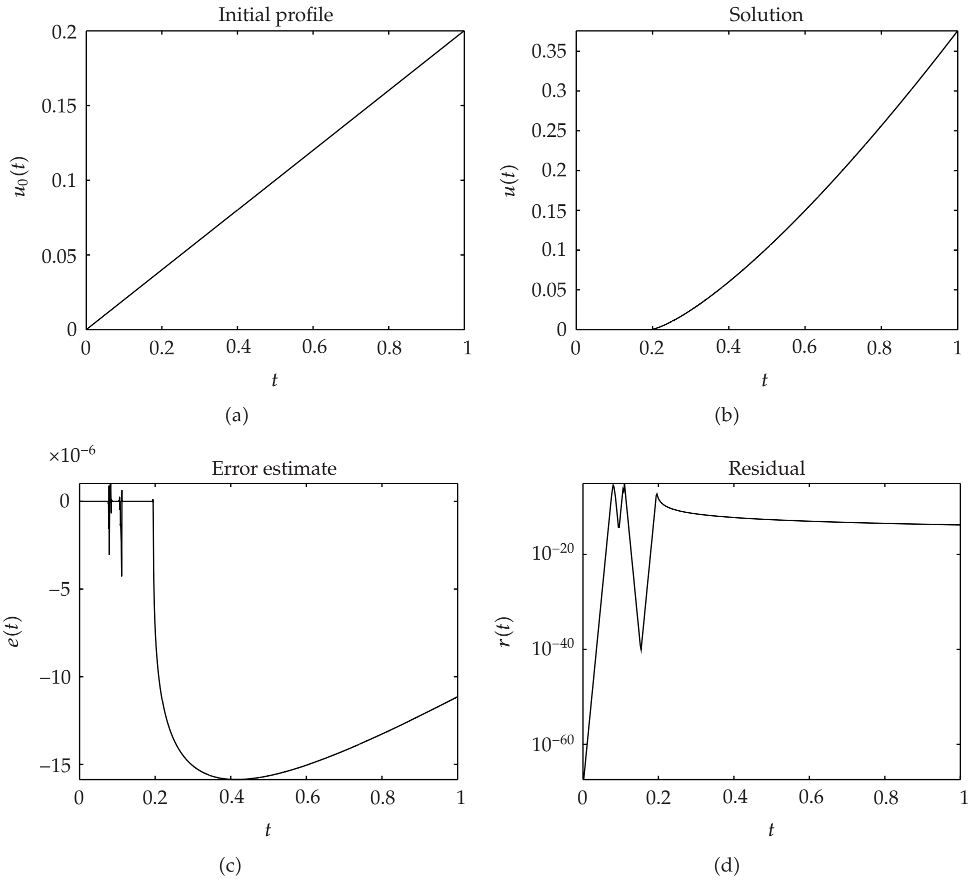


Figure 14: Problem (3.10): The initial profile, the numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $t_1 = 0.2$.

Table 1: Problem (3.10): Exact global error of the pseudo dead core solution.

α	β	$\max_{t\in[0,1]}\lvert u(t) - p(t)\rvert$
1	1	$1.5 \cdot 10^{-5}$
5	0.5	$5.1 \cdot 10^{-6}$

Therefore, the exact global error is accessible. In Table 1, we show the values for the global error, $\max_{0\le t\le 1}\lvert u(t) - p(t)\rvert$ where $p(t)$ is the numerical solution at t.

3.3. *Dead Core Solutions*

We now deal with the dead core solutions of the problem. Note that they only occur for

$$\lambda > \frac{4}{9}\left(\frac{3}{3\alpha + 4\beta}\right)^{3/2}. \tag{3.13}$$

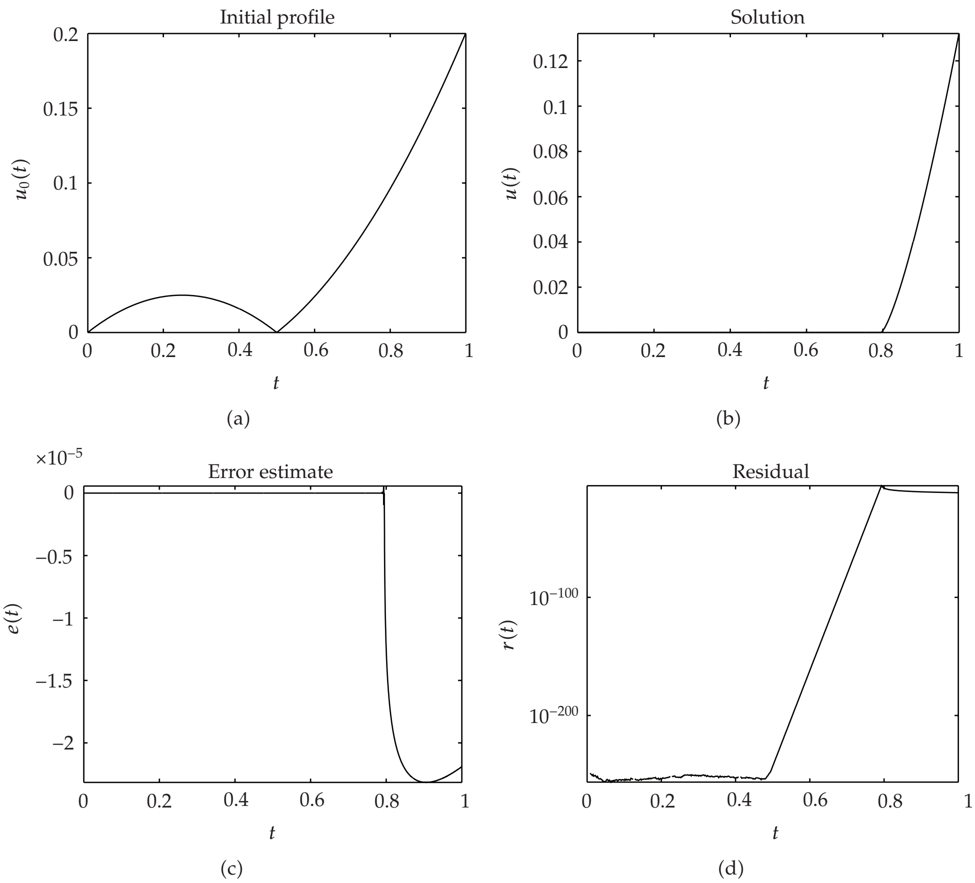


Figure 15: Problem (3.10): The initial profile, the numerical solution, the error estimate, and the residual for $\alpha = 1$, $\beta = 1$ and $t_1 = 0.8$.

Moreover, the relation between λ and t_1, where t_1 is such that the solution vanishes on $[0, t_1]$, is given by

$$\lambda = \frac{4}{9\sqrt{1-t_1}}\left(\frac{3}{3\alpha(1-t_1)+4\beta}\right)^{3/2}. \tag{3.14}$$

Also, the dead core solution is known,

$$u(t) = \left(\frac{3}{2}\sqrt{\lambda}(t-t_1)\right)^{4/3}, \quad t \in [t_1, 1]. \tag{3.15}$$

For the experiments, we used $t_1 = 0.2$ and $t_1 = 0.8$, in order to solve the problem,

$$\begin{gathered} u''(t)\sqrt{u(t)}u(t) = \lambda u(t), \quad t \in [t_1, 1], \\ u'(t_1) = 0, \quad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0, \ \beta > 0. \end{gathered} \tag{3.16}$$

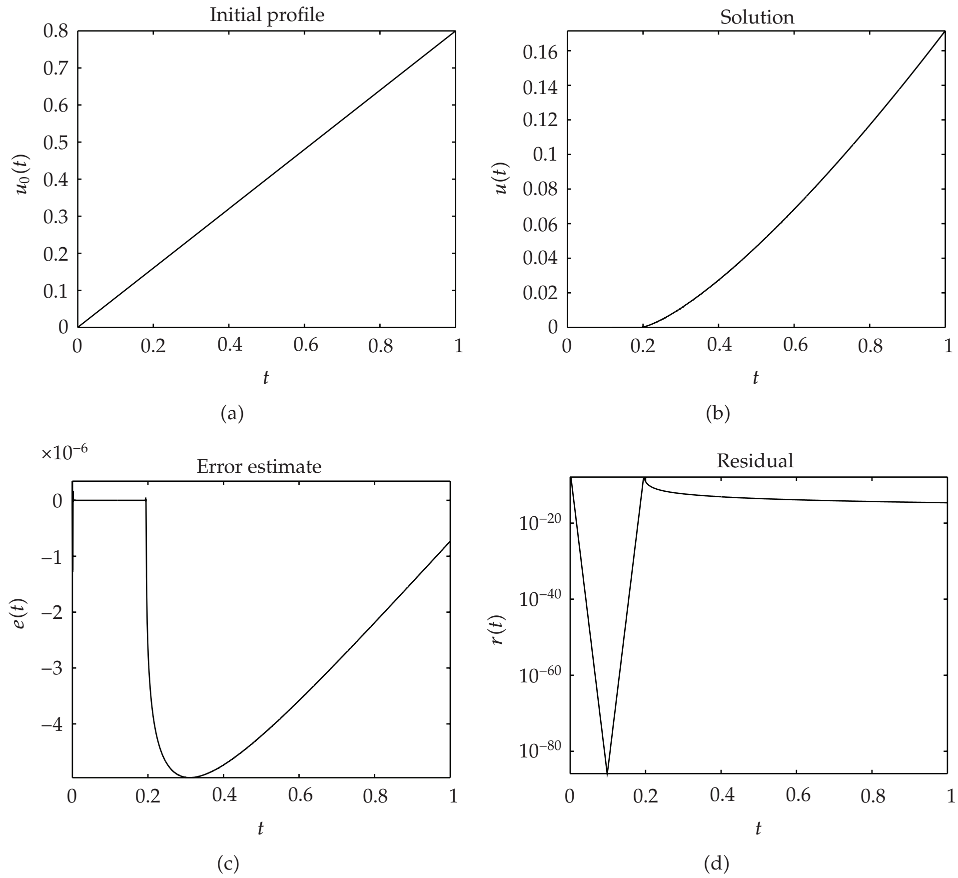


Figure 16: Problem (3.10): The initial profile, the numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $t_1 = 0.2$.

Clearly, if we approached the problem (3.16) directly, we had to use the knowledge of t_1 which is not available in general. Therefore, it is especially important to note that we were able to find the dead core solution without explicit knowledge of t_1 by treating the problem (3.10), formulated on the whole interval $[0, 1]$,

$$\begin{gathered}
u''(t)\sqrt{u(t)}u(t) = \lambda u(t), \quad t \in [0,1], \\
u'(0) = 0, \quad \alpha u(1) + \beta u'(1) = 1, \quad \alpha > 0, \ \beta > 0,
\end{gathered} \tag{3.17}$$

instead of solving (3.16). In Figures 14 and 15, we report on the numerical test runs for $\alpha = 1$, $\beta = 1$, and two values of t_1, $t_1 = 0.2$ and $t_1 = 0.8$, respectively. In Figures 16 and 17, analogous results for $\alpha = 5$, $\beta = 0.5$, and $t_1 = 0.2$, $t_1 = 0.8$, respectively, can be found.

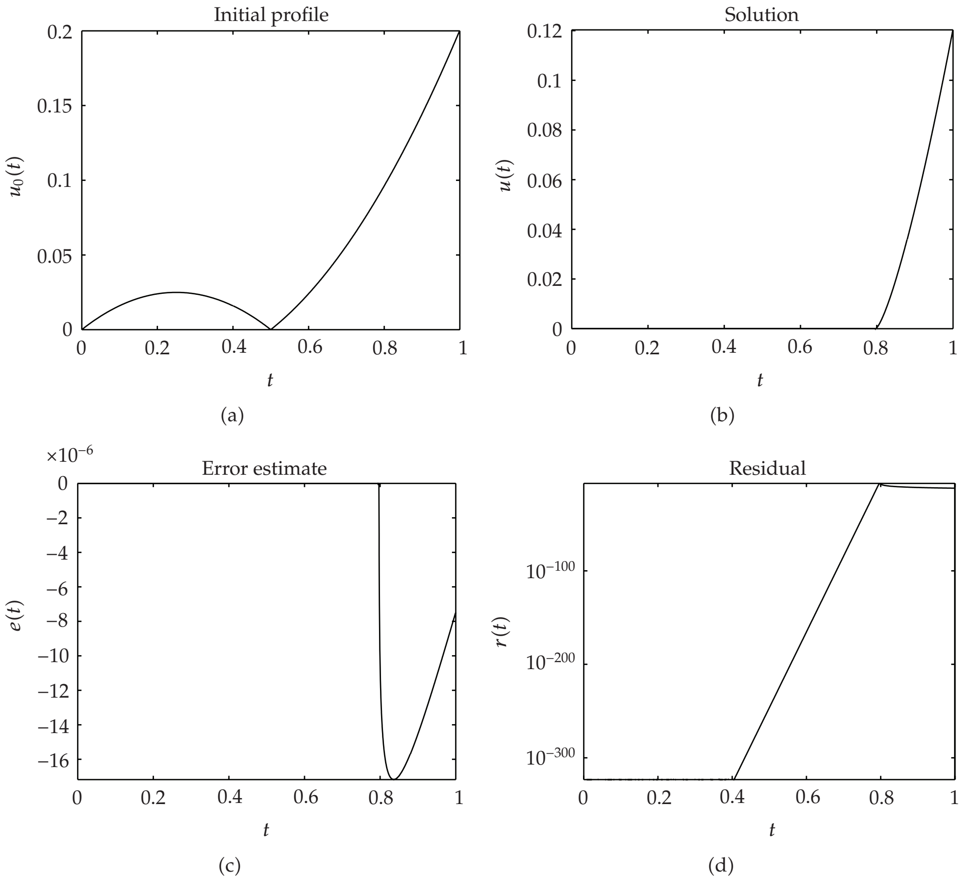


Figure 17: Problem (3.10): The initial profile, the numerical solution, the error estimate, and the residual for $\alpha = 5$, $\beta = 0.5$ and $t_1 = 0.8$.

Table 2 contains the information on the exact global error of the numerical dead core solution. We report on its maximal value $\max_{t\in[0,1]}|u(t) - p(t)|$ for a wide range of parameters. Obviously, dead core solutions can be found without exact use of the known solution structure, but the initial profile must be chosen carefully to guarantee the Newton iteration to convergence.

3.4. Positive Solutions of Problem (1.5a)-(1.5b)

In this section, we deal with problem (1.5a)-(1.5b). Since this problem is very involved, we decided to simulate it numerically first in order to provide some preliminary information about its solution. The numerical treatment of (1.5a)-(1.5b) turned out to be not at all straightforward, but nevertheless, for a certain choice of parameters, $\gamma = 3$, $\rho = 2$, $\nu = 2$, and $\alpha = 0.1$, $\beta = 1$, we were able to solve the problem and provide the error estimate and

Table 2: Maximum of the exact global error of the numerical dead core solution.

α	β	t_1	$\max_{t\in[0,1]}\|u(t)-p(t)\|$
1	1	0.2	9.7×10^{-4}
1	1	0.5	1.2×10^{-3}
1	1	0.8	1.7×10^{-3}
0.5	1.5	0.2	1.5×10^{-3}
0.5	1.5	0.5	1.5×10^{-3}
0.5	1.5	0.8	1.2×10^{-3}
0.5	0.8	0.3	5.5×10^{-2}
0.5	0.8	0.5	1.8×10^{-3}
0.5	0.8	0.8	2.8×10^{-3}
0.3	5	0.2	7.4×10^{-4}
0.3	5	0.5	7.1×10^{-4}
0.3	5	0.8	6.2×10^{-4}
5	0.5	0.2	3.6×10^{-4}
5	0.5	0.5	6.2×10^{-4}
5	0.5	0.8	6.7×10^{-4}

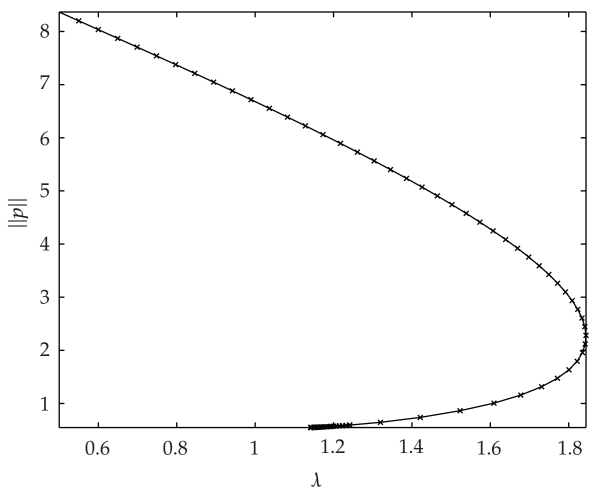


Figure 18: Graph of the $\|p\| - \lambda$ path obtained in 76 steps of the path following procedure, where $\|p\| = \max_{t\in[0,1]}|p(t)|$. The turning point has been found at $\lambda \approx 1.8442$.

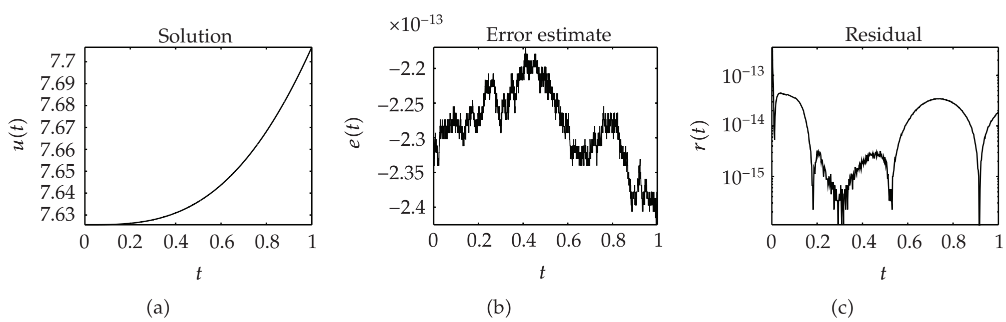


Figure 19: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 0.69901190254861$.

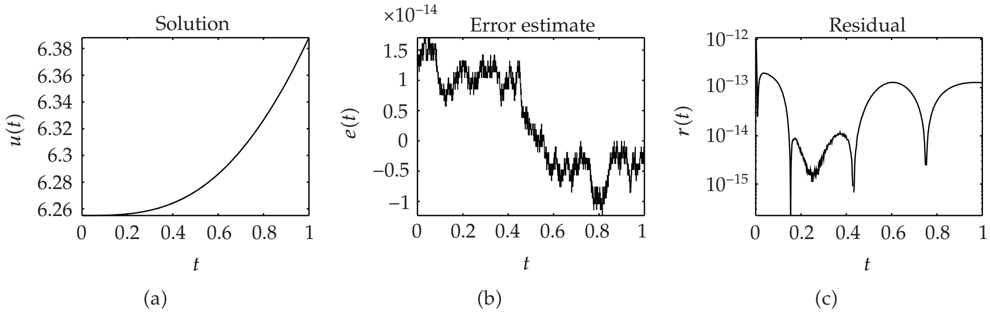


Figure 20: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 1.08259965025194$.

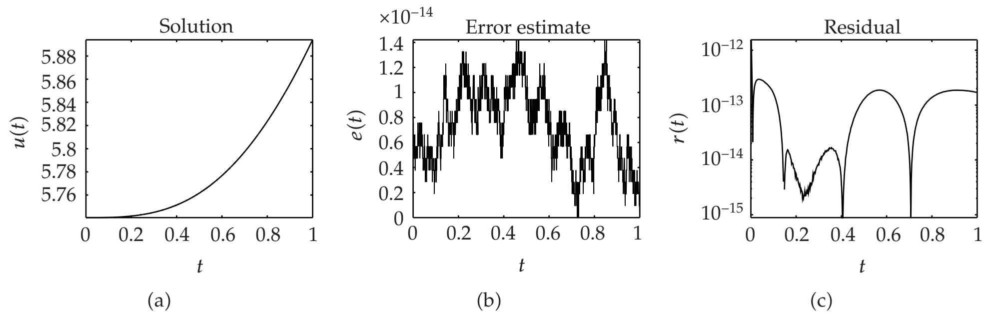


Figure 21: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 1.21752999971798$.

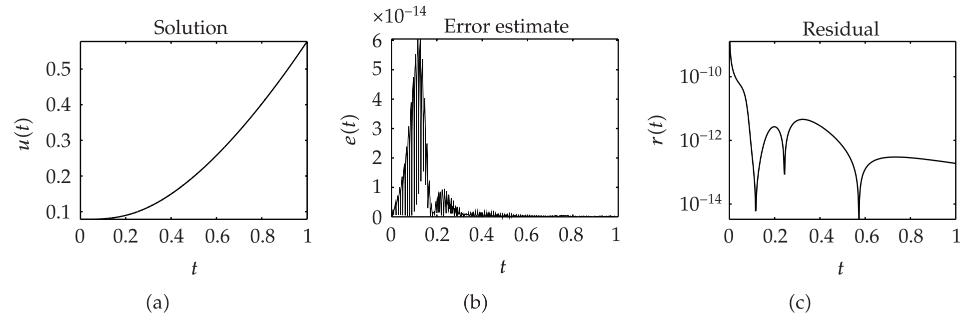


Figure 22: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 1.21476799699434$.

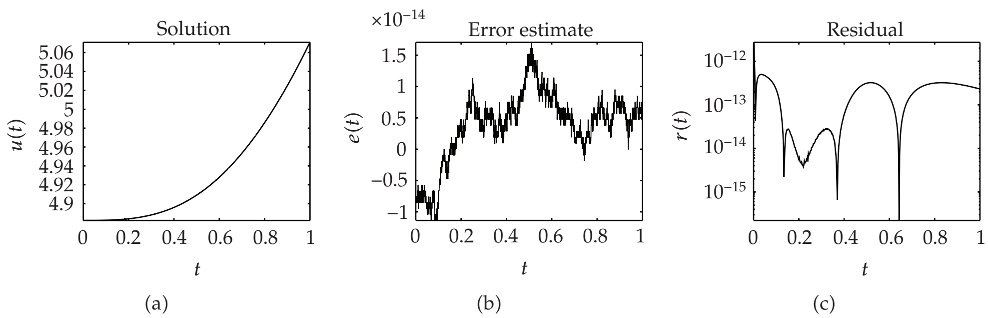


Figure 23: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 1.42604644036221$.

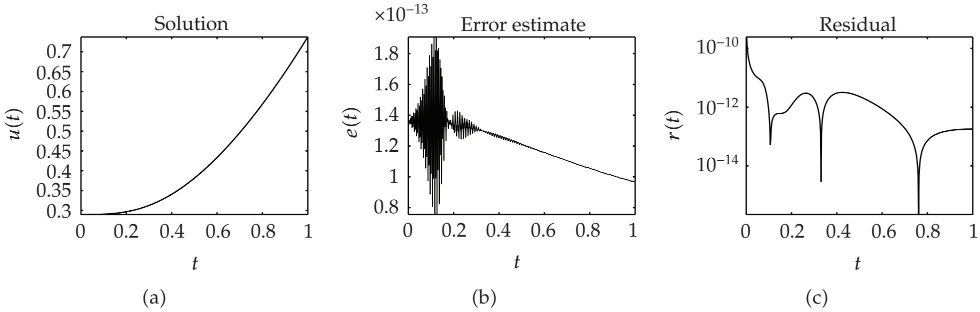


Figure 24: Problem (3.18): The numerical solution, the error estimate, and the residual for λ = 1.42139222684689.

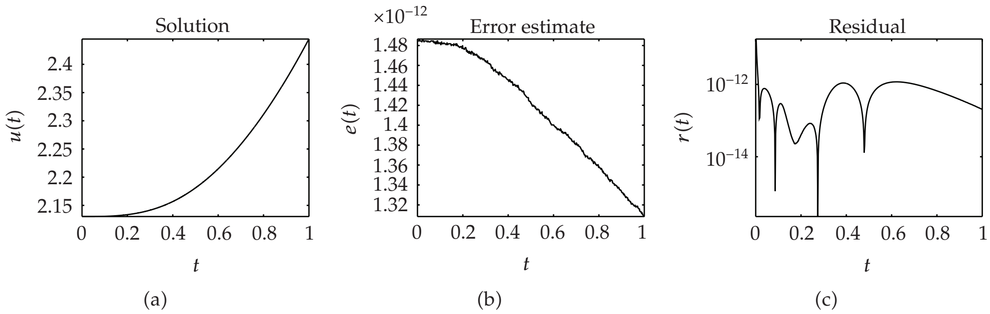


Figure 25: Problem (3.18): The numerical solution, the error estimate, and the residual for λ = 1.84118395344504.

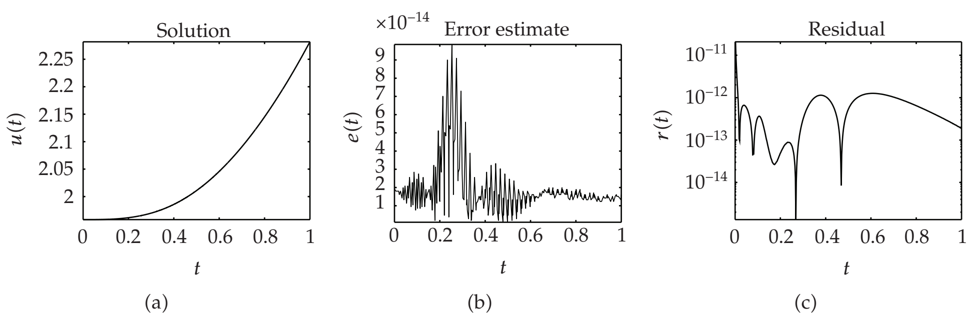


Figure 26: Problem (3.18): The numerical solution, the error estimate, and the residual for λ = 1.84416811671110.

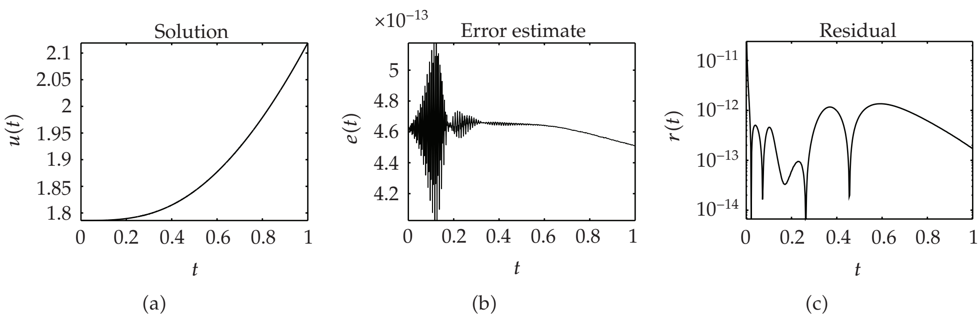


Figure 27: Problem (3.18): The numerical solution, the error estimate, and the residual for λ = 1.84240837502548.

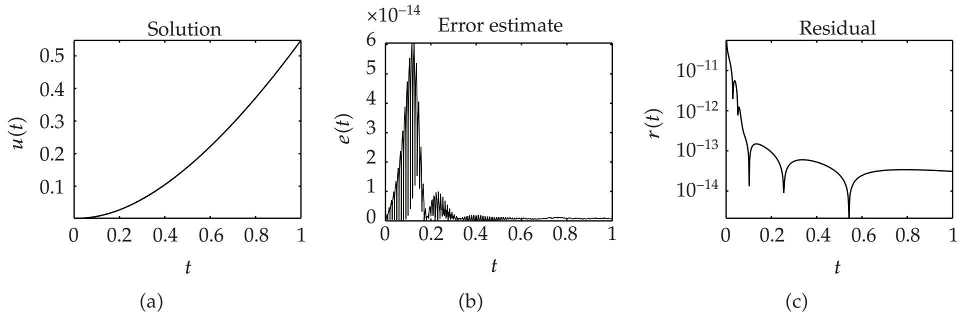


Figure 28: Problem (3.18): The numerical solution, the error estimate, and the residual for $\lambda = 1.14216524081032$.

the residual for its approximative solution. We have applied the path following strategy implemented in `bvpsuite` to the boundary value problem

$$\left(\left(u'(t)\right)^3\right)' + \frac{u'(t)}{t^2} = \vartheta\left(\frac{1}{\sqrt{u(t)}} + \left(u'(t)\right)^2\right), \quad 0 < t \leq 1,$$
$$u'(0) = 0, \quad 0.1u(1) + u'(1) = 1, \quad \vartheta = \lambda. \tag{3.18}$$

In Figures 19 to 28, we present numerical results for problem (3.18). The values of λ for which we were able to calculate the associated numerical solutions, are shown in Figure 18. According to Figure 18, we have found a turning point at $\lambda \approx 1.8442$. In a certain region below this value, there exist for any λ two different positive solutions.

In order to start the path following procedure we set $\lambda = 0.5$ and used $u \equiv 1$ as an initial profile. For each further step, we used the solution from the previous step as an initial profile. The solution corresponding to the values of λ shown in Figures 19 and 20 is unique. For $\lambda \approx 1.215$ we have found two different positive solutions, compare Figures 21 and 22. Also, for $\lambda \approx 1.425$, two different positive solutions exist; see Figures 23 and 24. Interestingly, solutions found in the vicinity of the turning point change rather fast, although the values of λ do not; see Figures 25 to 26. Finally, in the last step of the procedure, we obtained a solution which nearly reaches a pseudo dead core solution with $p(0) \approx u(0) \approx 0$.

Acknowledgments

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