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Research Article

Positive Decreasing Solutions of Higher-Order Nonlinear Difference Equations

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It is shown that the decay rates of the positive, monotone decreasing solutions approaching the zero equilibrium of higher-order nonlinear difference equations are related to the positive characteristic values of the corresponding linearized equation. If the nonlinearity is sufficiently smooth, this result yields an asymptotic formula for the positive, monotone decreasing solutions.

1. Introduction and the Main Results

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ be the set of real and complex numbers and the set of integers, respectively. The symbol $\mathbb{Z}^+$ denotes the set of nonnegative integers.

Recently, Aprahamian et al. [1] have studied the second-order nonlinear difference equation

$$\Delta^2 x(n-1) + c\Delta x(n) + f(x(n)) = 0, \quad n \in \mathbb{Z}^+, \tag{1.1}$$

where $c > 0$, $\Delta x(n) = x(n+1) - x(n)$ so that $\Delta^2 x(n-1) = x(n+1) - 2x(n) + x(n-1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $f > 0$ in $(0,1)$ and $f = 0$ is identically in $(-\infty, 0] \cup [1, \infty)$. As noted in [1], (1.1) is related to the discretization of traveling wave solutions of the Fisher-Kolmogorov partial differential equation. The main result of [1] is the following theorem about the existence of positive, decreasing solutions of (1.1) (see [1, Theorem 1.2] and its proof).

Theorem 1.1. *In addition to the above hypotheses on f, suppose that there exist $\epsilon \in (0,1)$ and $k > 0$ such that*

$$\int_0^y f(s)ds \le \frac{k}{2}y^2, \quad y \in [0,\epsilon), \tag{1.2}$$

$$k + \sqrt{k^2 + 4k} < c. \tag{1.3}$$

Then (1.1) *has a strictly decreasing solution $x : \mathbb{Z}^+ \to (0,\epsilon)$ such that*

$$\lim_{n\to\infty} x(n) = 0, \tag{1.4}$$

$$\sum_{n=1}^{\infty}(1+c)^n|\Delta x(n-1)|^2 < \infty. \tag{1.5}$$

Note that in [1], solutions of (1.1) satisfying conditions (1.4) and (1.5) of Theorem 1.1 are called *fast solutions*.

In this paper, we give an asymptotic description of the decreasing fast solutions of (1.1) described in Theorem 1.1. We will also consider a similar problem for the more general higher-order equation

$$x(n+1) = g(x(n), x(n-1), \ldots, x(n-k)), \quad n \in \mathbb{Z}^+, \tag{1.6}$$

where $k \in \mathbb{Z}^+$ and $g : \Omega \to \mathbb{R}$, with Ω being a convex open neighborhood of 0 in $\mathbb{R}^{k+1}$. Throughout the paper, we will assume that the partial derivatives $D_j g$, $1 \le j \le k+1$, are continuous on Ω and $g(0) = 0$ so that (1.6) has the zero equilibrium; moreover, the mild technical assumption

$$D_{k+1}g(0) \ne 0 \tag{1.7}$$

holds. We are interested in the asymptotic properties of those positive, monotone decreasing solutions of (1.6) which tend to the zero equilibrium, that is,

$$\lim_{n\to\infty} x(n) = 0. \tag{1.8}$$

We will show that under appropriate assumptions the decay rates of these solutions are equal to the characteristic values of the corresponding linearized equation belonging to the interval $(0,1]$. Our result, combined with asymptotic theorems from [2] or [3], yields asymptotic formulas for the positive, monotone decreasing solutions of (1.6).

Before we formulate our main theorems, we introduce some notations and definitions. Associated with (1.6) is the *linearization* about the zero equilibrium, namely, the linear homogeneous equation

$$y(n+1) = \sum_{j=0}^{k} a_j y(n-j), \quad n \in \mathbb{Z}^+, \tag{1.9}$$

with coefficients

$$a_j = D_{j+1}g(0), \quad 0 \le j \le k. \tag{1.10}$$

By a *characteristic value* of (1.9), we mean a complex root of the *characteristic polynomial*

$$P(z) = z^{k+1} - \sum_{j=0}^{k} a_j z^{k-j}. \tag{1.11}$$

Thus, the set of all characteristic values of (1.9) is given by

$$Z(P) = \{\zeta \in \mathbb{C} \mid P(\zeta) = 0\}. \tag{1.12}$$

To each $\zeta \in Z(P)$, there corresponds solutions of (1.9) of the form

$$y(n) = q(n)\zeta^n, \quad n \in \mathbb{Z}^+, \tag{1.13}$$

where q is a polynomial of degree less than $m(\zeta)$, the multiplicity of ζ as a root of P. Such solutions are called *characteristic solutions* corresponding to ζ. If $S \subset Z(P)$ is a nonempty set of characteristic values, then by a *characteristic solution corresponding to the set* S, we mean a finite sum of characteristic solutions corresponding to values $\zeta \in S$.

Now we can formulate our main results. The first theorem applies to the positive, monotone decreasing solutions of (1.6) provided that zero is a hyperbolic equilibrium of (1.6). Recall that the zero equilibrium of (1.6) is *hyperbolic* if the linearized equation (1.9) has no characteristic values on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Theorem 1.2. *In addition to the above hypotheses on g, suppose that the partial derivatives $D_j g$, $1 \le j \le k+1$, are Lipschitz continuous on compact subsets of Ω. Assume also that zero is a hyperbolic equilibrium of* (1.6)*. Let x be a positive, monotone decreasing solution of* (1.6) *satisfying* (1.8)*. Then the limit*

$$\lambda = \lim_{n\to\infty} \sqrt[n]{x(n)} \tag{1.14}$$

exists and λ is a characteristic value of the linearized equation (1.9) *belonging to the interval* $(0,1)$*. Moreover, there exists $\epsilon \in (0,\lambda)$ such that the asymptotic representation*

$$x(n) = y(n) + O\big((\lambda - \epsilon)^n\big), \quad n \longrightarrow \infty, \tag{1.15}$$

holds, where y is a positive characteristic solution of the linearized equation (1.9) *corresponding to the set*

$$S = \{\zeta \in Z(P) \mid |\zeta| = \lambda\}. \tag{1.16}$$

In contrast to Theorem 1.2, the next result applies also in some cases when the zero equilibrium of (1.6) is not hyperbolic.

Theorem 1.3. *In addition to the hypotheses on g, suppose that the linearized equation* (1.9) *has exactly one characteristic value λ in the interval* $(0,1]$. *Assume also that λ is the only characteristic value of* (1.9) *on the circle* $\{z \in \mathbb{C} \mid |z| = \lambda\}$ *and*

$$m(\lambda) \leq 2, \tag{1.17}$$

where $m(\lambda)$ is the multiplicity of λ as a root of the characteristic polynomial P. Let x be a positive, monotone decreasing solution of (1.6) *satisfying* (1.8). *Then*

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \lambda. \tag{1.18}$$

Note that conclusion (1.18) of Theorem 1.3 is stronger than (1.14).

For the second-order equation (1.1), we have the following theorem which provides new information about the decreasing fast solutions obtained by Aprahamian et al. [1].

Theorem 1.4. *Adopt the hypotheses of Theorem 1.1. Let $x : \mathbb{Z}^+ \to (0,\epsilon)$ be a monotone decreasing solution of* (1.1) *satisfying conditions* (1.4) *and* (1.5). *If the (finite) right-hand derivative $f'_+(0)$ exists, then*

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \lambda, \tag{1.19}$$

where λ is the unique root of the equation

$$z^2 - \frac{2+c-f'_+(0)}{1+c} z + \frac{1}{1+c} = 0, \tag{1.20}$$

in the interval $(0, 1/\sqrt{1+c})$.

If f' is Lipschitz continuous on $[0,1]$, *then* (1.19) *can be replaced with the stronger conclusion*

$$\lim_{n\to\infty} \frac{x(n)}{\lambda^n} = L \quad \text{for some } L \in (0,\infty). \tag{1.21}$$

The proofs of the above theorems are given in Section 3.

2. Preliminary Results

In this section, we establish some preliminary results on linear difference equations with asymptotically constant coefficients which will be useful in the proof of our main theorems.

Consider the linear homogeneous difference equation

$$x(n+1)=\sum_{j=0}^{k}b_j(n)x(n-j),\quad n\in\mathbb{Z}^+,\tag{2.1}$$

where the coefficients $b_j:\mathbb{Z}^+\longrightarrow\mathbb{R}$, $0\le j\le k$, are *asymptotically constant*, that is, the (finite) limits

$$a_j=\lim_{n\to\infty}b_j(n),\quad 0\le j\le k,\tag{2.2}$$

exist. We will assume that

$$a_k\neq 0.\tag{2.3}$$

If we replace the coefficients in (2.1) with their limits, we obtain the *limiting equation*

$$u(n+1)=\sum_{j=0}^{k}a_ju(n-j),\quad n\in\mathbb{Z}^+.\tag{2.4}$$

Theorem 1.2 will be deduced from the following proposition.

Proposition 2.1. *Suppose* (2.2) *and* (2.3) *hold. Assume that the convergence in* (2.2) *is exponential, that is, there exists a constant* $\eta\in(0,1)$ *such that*

$$b_j(n)=a_j+O(\eta^n),\quad n\longrightarrow\infty.\tag{2.5}$$

Let x be a positive, monotone decreasing solution of (2.1)*. Then the limit*

$$\lambda=\lim_{n\to\infty}\sqrt[n]{x(n)}\tag{2.6}$$

exists and λ is a characteristic value of the limiting equation (2.4) *belonging to the interval* $(0,1]$*. Moreover, there exists $\epsilon\in(0,\lambda)$ such that the asymptotic representation*

$$x(n)=u(n)+O((\lambda-\epsilon)^n),\quad n\longrightarrow\infty,\tag{2.7}$$

holds, where u is a positive characteristic solution of the limiting equation (2.4) *corresponding to the set of characteristic values*

$$S=\{\zeta\in Z(P)\mid |\zeta|=\lambda\},\tag{2.8}$$

where P is the characteristic polynomial corresponding to (2.4)*.*

Proof. Equation (2.1) can be written in the form

$$x(n+1) = \sum_{j=0}^{k} a_j x(n-j) + h(n), \quad n \in \mathbb{Z}^+, \tag{2.9}$$

where

$$h(n) = \sum_{j=0}^{k} \left(b_j(n) - a_j\right) x(n-j), \quad n \in \mathbb{Z}^+. \tag{2.10}$$

Let $\tilde{x}$ be the z-transform of x defined by

$$\tilde{x}(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{for } z \in \mathbb{C} \text{ with } |z| > r, \tag{2.11}$$

where r is the radius of convergence given by

$$r = \limsup_{n \to \infty} \sqrt[n]{x(n)}. \tag{2.12}$$

The boundedness of x implies that $r \leq 1$. By the application of a Perron-type theorem (see [4, Theorem 2] or [5, Theorem 8.47]), we conclude that $r = |\zeta|$ for some characteristic value ζ of (2.4). By virtue of (2.3), the characteristic values of (2.4) are nonzero. Therefore, $r = |\zeta| > 0$.

From (2.5) and (2.12), we see that the radius of convergence R of the z-transform $\tilde{h}$ of h given by

$$R = \limsup_{n \to \infty} \sqrt[n]{|h(n)|} \tag{2.13}$$

satisfies the inequality

$$R \leq \eta r < r. \tag{2.14}$$

Therefore, $\tilde{h}$ is holomorphic in the region $|z| > \eta r$. Taking the z-transform of (2.9) and using the shifting properties

$$\begin{gathered} \sum_{n=0}^{\infty} x(n+1) z^{-n} = z\tilde{x}(z) - x(0)z, \quad |z| > r, \\ \sum_{n=0}^{\infty} x(n-j) z^{-n} = z^{-j}\tilde{x}(z) + \sum_{n=-j}^{-1} x(n) z^{-n-j}, \quad |z| > r, \ 1 \leq j \leq k, \end{gathered} \tag{2.15}$$

it follows by easy calculations that

$$P(z)\tilde{x}(z) = F(z) \quad \text{for } z \in \mathbb{C} \text{ with } |z| > r, \tag{2.16}$$

where P is the polynomial given by (1.11) and

$$F(z) = Q(z) + z^k \tilde{h}(z) \tag{2.17}$$

with Q given by

$$Q(z) = x(0)z^{k+1} + \sum_{j=1}^{k} a_j \sum_{n=-j}^{-1} x(n) z^{k-n-j}. \tag{2.18}$$

Since the coefficients of the z-transform $\tilde{x}$ are positive, according to Prinsheim's theorem (see [6, Theorem 18.3] or [7, Theorem 2.1, page 262]) $\tilde{x}$ has a singularity at $z = r$. Since $\tilde{h}$ and hence F is holomorphic in the region $|z| > \eta r$, this implies that $P(r) = 0$. Otherwise, (2.16) would imply that $\tilde{x}$ can be extended as a holomorphic function to a neighborhood of $z = r$ by $\tilde{x} = F/P$. Thus, r is a characteristic value of (2.4).

According to [4, Theorems 2 and 3], we have

$$\lim_{n\to\infty} \sqrt[n]{x(n) + x(n+1) + \cdots + x(n+k)} = \limsup_{n\to\infty} \sqrt[n]{x(n)} = r. \tag{2.19}$$

Since x is monotone decreasing, it follows for $n \in \mathbb{Z}^+$,

$$x(n) + x(n+1) + \cdots + x(n+k) \le (k+1)x(n) \tag{2.20}$$

and hence

$$\sqrt[n]{x(n)} \ge \frac{1}{\sqrt[n]{k+1}} \sqrt[n]{x(n) + x(n+1) + \cdots + x(n+k)}. \tag{2.21}$$

Letting $n \to \infty$ in the last inequality, and using (2.19), we find that

$$\liminf_{n\to\infty} \sqrt[n]{x(n)} \ge \lim_{n\to\infty} \sqrt[n]{x(n) + x(n+1) + \cdots + x(n+k)} = \limsup_{n\to\infty} \sqrt[n]{x(n)} = r. \tag{2.22}$$

This proves the existence of the limit (2.6) with $\lambda = r$. Finally, conclusion (2.7) is an immediate consequence of [2, Theorem 2.3 and Remark 2.5]. □

Theorem 1.3 can be regarded as a corollary of the following result.

Proposition 2.2. *Suppose* (2.2) *and* (2.3) *hold. Assume that the limiting equation* (2.4) *has exactly one characteristic value* λ *in the interval* $(0,1]$. *Assume also that* λ *is the only characteristic value of* (2.4) *on the circle* $\{z \in \mathbb{C} \mid |z| = \lambda\}$ *and*

$$m(\lambda) \leq 2, \tag{2.23}$$

where $m(\lambda)$ *is the multiplicity of* λ *as a root of the characteristic polynomial corresponding to* (2.4). *Let* x *be a positive, monotone decreasing solution of* (2.1). *Then*

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \lambda. \tag{2.24}$$

Before we give a proof of Proposition 2.2, we establish two lemmas.

Lemma 2.3. *Suppose* (2.2) *and* (2.3) *hold. Let* x *be a positive, monotone decreasing solution of* (2.1). *Then there exists* $\alpha \in (0,1)$ *such that*

$$\alpha \leq \frac{x(n+1)}{x(n)} \leq 1, \quad n \in \mathbb{Z}^+. \tag{2.25}$$

Proof. The second inequality in (2.25) follows from the monotonicity of x. In order to prove the first inequality, suppose by the way of contradiction that there exists a strictly increasing sequence $\{n_i\}_{i=0}^{\infty}$ in $\mathbb{Z}^+$ such that

$$\lim_{i\to\infty} \frac{x(n_i+1)}{x(n_i)} = 0. \tag{2.26}$$

From (2.1), we obtain for $n \in \mathbb{Z}^+$,

$$b_k(n) = \frac{x(n+1)}{x(n-k)} - \sum_{j=0}^{k-1} b_j(n) \frac{x(n-j)}{x(n-k)}. \tag{2.27}$$

From this and from the fact that x is monotone decreasing, we find for $n \in \mathbb{Z}^+$,

$$|b_k(n)| \leq \frac{x(n+1)}{x(n-k)} + \sum_{j=0}^{k-1} \left|b_j(n)\right| \frac{x(n-j)}{x(n-k)} \leq \frac{x(n-k+1)}{x(n-k)} \left(1 + \sum_{j=0}^{k-1} \left|b_j(n)\right|\right). \tag{2.28}$$

Writing $n = n_i + k$ in the last inequality, letting $i \to \infty$, and using (2.2), (2.3), and (2.26), we obtain

$$0 < |a_k| \leq 0, \tag{2.29}$$

a contradiction. Thus, (2.25) holds for some $\alpha \in (0,1)$. □

The following lemma will play a key role in the proof of Proposition 2.2.

Lemma 2.4. *Suppose that* (2.4) *has exactly one characteristic value* λ *in the interval* $(0,1]$. *Assume also that* λ *is the only characteristic value of* (2.4) *on the circle* $\{z \in \mathbb{C} \mid |z| = \lambda\}$ *and* (2.23) *holds. Let* $u : \mathbb{Z} \to (0,\infty)$ *be a positive, monotone decreasing biinfinite sequence satisfying* (2.4) *on* $\mathbb{Z}$, *that is,*

$$u(n+1) = \sum_{j=0}^{k} a_j u(n-j), \quad n \in \mathbb{Z}. \tag{2.30}$$

Then

$$u(n) = u(0)\lambda^n, \quad n \in \mathbb{Z}. \tag{2.31}$$

Proof. We will prove the lemma by using a similar method as in the proof of [8, Lemma 3.2]. The radius of convergence r of the z-transform

$$\tilde{u}(z) = \sum_{n=0}^{\infty} u(n) z^{-n} \tag{2.32}$$

of u is given by

$$r = \limsup_{n\to\infty} \sqrt[n]{u(n)}. \tag{2.33}$$

Since u is bounded on $\mathbb{Z}^+$, $r \le 1$, and [4, Theorem 2] or Lemma 2.3 implies that $r > 0$. Taking the z-transform of (2.4), we obtain

$$P(z)\tilde{u}(z) = Q(z) \quad \text{for } z \in \mathbb{C} \text{ with } |z| > r, \tag{2.34}$$

where P is given by (1.11) and

$$Q(z) = u(0)z^{k+1} + \sum_{j=1}^{k} a_j \sum_{n=-j}^{-1} u(n) z^{k-n-j}. \tag{2.35}$$

Relation (2.34), combined with Pringsheim's theorem [6, Theorem 18.3], implies that $P(r) = 0$. (Otherwise, $\tilde{u}$ can be extended as a holomorphic function to a neighborhood of $z = r$ by $\tilde{u} = Q/P$.) Since $r \in (0,1]$ and the only root of P in $(0,1]$ is λ, we have that $r = \lambda$.

Define

$$U(z) = \sum_{n=1}^{\infty} u(-n) z^n \quad \text{for } z \in \mathbb{C} \text{ with } |z| < R, \tag{2.36}$$

where R is the radius of convergence of the above power series given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{u(-n)}}. \tag{2.37}$$

Since u is monotone decreasing, we have

$$\sqrt[n]{u(-n)} \geq \sqrt[n]{u(0)}, \quad n \in \mathbb{Z}^+. \tag{2.38}$$

Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{u(-n)} \geq 1 \tag{2.39}$$

and hence $R \leq 1$. As a solution of a constant coefficient equation, u is a sum of characteristic solutions. Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{u(-n)} < \infty \tag{2.40}$$

and hence $R > 0$. From (2.30), we find for $n \in \mathbb{Z}^+$ and $z \in \mathbb{C}$,

$$u(-(n-1))z^n = \sum_{j=0}^{k} a_j u(-(n+j))z^n. \tag{2.41}$$

Summation from $n = 1$ to infinity and the definition of U yield

$$P(z)U(z) = -Q(z) \quad \text{for } z \in \mathbb{C} \text{ with } |z| < R, \tag{2.42}$$

with Q as in (2.35). This, combined with Prinsheim's theorem, implies that $P(R) = 0$. Since $R \in (0,1]$ and the only root of P in $(0,1]$ is λ, we have that $R = \lambda$. Thus, $r = R = \lambda$. This, together with (2.34) and (2.42), implies that the holomorphic function H defined on the open set $W = \{z \in \mathbb{C} \mid |z| \neq \lambda\}$ by

$$H(z) = \begin{cases} \tilde{u}(z) & \text{if } |z| > \lambda, \\ -U(z) & \text{if } |z| < \lambda. \end{cases} \tag{2.43}$$

satisfies

$$P(z)H(z) = Q(z), \quad z \in W. \tag{2.44}$$

By hypotheses, $z = \lambda$ is the only root of P on the circle $|z| = \lambda$. Therefore, (2.44) implies that H can be extended as a holomorphic function to $\mathbb{C} \setminus \{\lambda\}$ by

$$H(z) = \frac{Q(z)}{P(z)} \quad \text{for } z \in \mathbb{C} \setminus \{\lambda\} \text{ with } |z| = \lambda. \tag{2.45}$$

Moreover, since $m(\lambda) \leq 2$, the function $E : \mathbb{C} \setminus \{\lambda\} \to \mathbb{C}$ defined by

$$E(z) = (z-\lambda)^2 H(z), \quad z \in \mathbb{C} \setminus \{\lambda\}, \tag{2.46}$$

has a removable singularity at $z = \lambda$. Thus, it can be regarded as an entire function. Using (1.11), (2.35), and (2.44), we obtain for $z \in \mathbb{C} \setminus Z(P)$,

$$H(z) = \frac{Q(z)}{P(z)} = O(1), \quad |z| \longrightarrow \infty. \tag{2.47}$$

Hence

$$E(z) = O\left(|z|^2\right), \quad |z| \longrightarrow \infty. \tag{2.48}$$

By the application of the Extended Liouville Theorem [6, Theorem 5.11], we conclude that E is a polynomial of degree at most 2, that is,

$$E(z) = az^2 + bz + c, \quad z \in \mathbb{C}, \tag{2.49}$$

for some $a, b, c \in \mathbb{C}$. Since $H(0) = U(0) = 0$, we have

$$c = E(0) = \lambda^2 H(0) = 0. \tag{2.50}$$

Therefore,

$$H(z) = \frac{E(z)}{(z-\lambda)^2} = \frac{az^2 + bz}{(z-\lambda)^2} = \frac{a[z(z-\lambda) + \lambda z] + bz}{(z-\lambda)^2}, \quad z \neq \lambda. \tag{2.51}$$

Hence

$$H(z) = d\frac{\lambda z}{(z-\lambda)^2} + a\frac{z}{z-\lambda}, \quad z \neq \lambda, \tag{2.52}$$

where $d = a + b\lambda^{-1}$. This, together with (2.43), implies

$$\tilde{u}(z) = d\frac{\lambda z}{(z-\lambda)^2} + a\frac{z}{z-\lambda}, \quad |z| > \lambda. \tag{2.53}$$

From this, in view of the uniqueness of the z-transform, we obtain

$$u(n) = dn\lambda^n + a\lambda^n, \quad n \in \mathbb{Z}^+. \tag{2.54}$$

From (2.43) and (2.52), we obtain

$$U(z) = -H(z) = -d\frac{\lambda z}{(z-\lambda)^2} - a\frac{z}{z-\lambda}, \quad |z| < \lambda. \tag{2.55}$$

From this and (2.36), in view of the uniqueness of the coefficients of the power series, we obtain

$$u(-n) = -dn\lambda^{-n} + a\lambda^{-n}, \quad n \in \mathbb{Z}^+. \tag{2.56}$$

This, together with (2.54), yields

$$u(n) = dn\lambda^n + a\lambda^n, \quad n \in \mathbb{Z}. \tag{2.57}$$

From this, taking into account that $u(n) > 0$ for all $n \in \mathbb{Z}$, we see that $d = 0$. Therefore, (2.57) reduces to (2.31). □

Now we are in a position to give a proof of Proposition 2.2.

Proof of Proposition 2.2. Let a be an arbitrary accumulation point of $\{x(n+1)/x(n)\}_{n=0}^{\infty}$ so that for some strictly increasing sequence $\{n_i\}_{i=0}^{\infty}$ in $\mathbb{Z}^+$,

$$\lim_{i\to\infty} \frac{x(n_i+1)}{x(n_i)} = a. \tag{2.58}$$

In order to prove (2.24), it is enough to show that $a = \lambda$. From conclusion (2.25) of Lemma 2.3 and the relation

$$\frac{x(n+m)}{x(n)} = \frac{x(n+1)}{x(n)}\frac{x(n+2)}{x(n+1)}\cdots\frac{x(n+m)}{x(n+m-1)}, \quad n, m \in \mathbb{Z}^+, \tag{2.59}$$

we obtain

$$\begin{gathered} \alpha^m \le \frac{x(n+m)}{x(n)} \le 1, \quad n, m \in \mathbb{Z}^+, \\ 1 \le \frac{x(n-m)}{x(n)} \le \alpha^{-m}, \quad n \ge m, \ n, m \in \mathbb{Z}^+. \end{gathered} \tag{2.60}$$

From (2.60), it follows by the standard diagonal choice (see the proof of [8, Theorem 1.3]) that there exists a subsequence $\{d_i\}_{i=0}^{\infty}$ of $\{n_i\}_{i=0}^{\infty}$ such that the limits

$$u(m) = \lim_{i\to\infty} \frac{x(d_i+m)}{x(d_i)}, \quad m \in \mathbb{Z}, \tag{2.61}$$

exist and are finite. Let $m \in \mathbb{Z}$ be fixed. Writing $n = d_i + m$ in (2.1) and dividing the resulting equation by $x(d_i)$, we obtain

$$\frac{x(d_i+m+1)}{x(d_i)} = \sum_{j=0}^{k} b_j(d_i+m)\frac{x(d_i+m-j)}{x(d_i)}, \quad i \geq i_0, \tag{2.62}$$

where $i_0 \in \mathbb{Z}^+$ is so large that $d_i \geq -m$ for $i \geq i_0$. From this, letting $i \to \infty$ and using (2.2) and (2.61), we see that the biinfinite sequence u satisfies (2.30). Further, it is easily seen that u inherits the monotone decreasing property of x. Finally, from estimates (2.60), we see that $u > 0$ on $\mathbb{Z}$. By the application of Lemma 2.4, we conclude that u has the form (2.31). Since $u(0) = 1$ (see (2.61)), (2.31) yields

$$u(n) = \lambda^n, \quad n \in \mathbb{Z}. \tag{2.63}$$

From this and (2.61), taking into account that $\{d_i\}_{i=0}^{\infty}$ is a subsequence of $\{n_i\}_{i=0}^{\infty}$, we obtain

$$a = \lim_{i\to\infty}\frac{x(n_i+1)}{x(n_i)} = \lim_{i\to\infty}\frac{x(d_i+1)}{x(d_i)} = u(1) = \lambda. \tag{2.64}$$

□

3. Proofs of the Main Theorems

Proof of Theorem 1.2. Since $g(0) = 0$, from (1.6), we find for $n \in \mathbb{Z}^+$,

$$\begin{aligned} x(n+1) &= \int_0^1 \frac{d}{ds} g(sx(n), sx(n-1), \dots, sx(n-k))ds \\ &= \sum_{i=1}^{k+1}\int_0^1 D_i g(sx(n), sx(n-1), \dots, sx(n-k))ds \; x(n-i+1). \end{aligned} \tag{3.1}$$

Thus, x is a solution of (2.1) with coefficients

$$b_j(n) = \int_0^1 D_{j+1} g(sx(n), sx(n-1), \dots, sx(n-k))ds, \quad 0 \leq j \leq k. \tag{3.2}$$

By virtue of (1.8), the limits in (2.2) exist and are given by (1.10), that is, the limiting equation (2.4) coincides with the linearized equation (1.9). By virtue of (1.7), hypothesis (2.3) of Proposition 2.1 is satisfied. Let

$$r = \limsup_{n\to\infty} \sqrt[n]{x(n)}. \tag{3.3}$$

By virtue of the boundedness of x, $r \le 1$. As noted in the proof of Proposition 2.1, according to [4, Theorem 2], $1 \ge r = |\zeta|$ for some $\zeta \in Z(P)$. Since zero is a hyperbolic equilibrium of (1.6), $|\zeta| \ne 1$ for all $\zeta \in Z(P)$. Therefore, $r < 1$. Choose $\eta \in (r, 1)$. By virtue of (3.3),

$$x(n) = O(\eta^n), \quad n \longrightarrow \infty. \tag{3.4}$$

From this, (1.10), (3.2) and the Lipschitz continuity of the partial derivatives $D_j g$, $1 \le j \le k+1$, it is easily shown that hypothesis (2.5) of Proposition 2.1 also holds. The conclusions of Theorem 1.2 follow from Proposition 2.1. □

Proof of Theorem 1.3. As noted in the proof of Theorem 1.2, x is a solution of (2.1) with the asymptotically constant coefficients given by (3.2) and the limiting equation of (2.1) is the linearized equation (1.9). Further, by virtue of (1.7), condition (2.3) holds. Thus, all hypotheses of Proposition 2.2 are satisfied and the result follows from Proposition 2.2. □

Proof of Theorem 1.4. Equation (1.1) can be written in the form

$$x(n+1) = \frac{2+c}{1+c}x(n) - \frac{1}{1+c}f(x(n)) - \frac{1}{1+c}x(n-1), \quad n \in \mathbb{Z}^+. \tag{3.5}$$

Consequently, x is a solution of the equation

$$x(n+1) = \left[\frac{2+c}{1+c} - \frac{1}{1+c}\frac{f(x(n))}{x(n)}\right]x(n) - \frac{1}{1+c}x(n-1), \quad n \in \mathbb{Z}^+. \tag{3.6}$$

Suppose that $f'_+(0)$ is finite. Condition (1.4), together with the positivity of x and $f(0) = 0$, implies

$$\alpha = \lim_{n\to\infty}\frac{f(x(n))}{x(n)} = f'_+(0). \tag{3.7}$$

Therefore, the coefficients in (3.6) are asymptotically constant and the corresponding limiting equation is

$$u(n+1) = \frac{2+c-\alpha}{1+c}u(n) - \frac{1}{1+c}u(n-1), \quad n \in \mathbb{Z}^+. \tag{3.8}$$

Since $f \ge 0$ and $f(0) = 0$, we have $\alpha = f'_+(0) \ge 0$. From (1.2), it follows

$$k \ge \lim_{y\to 0+}\frac{2\int_0^y f(s)ds}{y^2} = \lim_{y\to 0+}\frac{f(y)}{y} = \alpha, \tag{3.9}$$

with the last but one equality being a consequence of L'Hospital's rule. This, together with (1.3), yields

$$\alpha + 2\sqrt{\alpha} \le k + 2\sqrt{k} < k + \sqrt{k^2+4k} < c. \tag{3.10}$$

Hence

$$c - \alpha > 2\sqrt{\alpha}. \tag{3.11}$$

The characteristic polynomial P corresponding to (3.8) is

$$P(z) = z^2 - \frac{2 + c - \alpha}{1 + c} z + \frac{1}{1 + c}. \tag{3.12}$$

We have

$$P(0) = \frac{1}{1 + c} > 0, \qquad P(1) = \frac{\alpha}{1 + c} \geq 0. \tag{3.13}$$

Further, it is easily shown that (3.11) implies that

$$P\left(\frac{1}{\sqrt{1 + c}}\right) = \frac{2}{1 + c} - \frac{2 + c - \alpha}{1 + c} \frac{1}{\sqrt{1 + c}} < 0. \tag{3.14}$$

Therefore, P has a root λ in $(0, 1/\sqrt{1 + c})$ and the second root μ of P belongs to the interval $(1/\sqrt{1 + c}, 1]$. By the application of Poincaré's theorem (see [5, Section 8.2] or [9, Section 2.13]), we conclude that the limit

$$l = \lim_{n \to \infty} \frac{x(n + 1)}{x(n)} \tag{3.15}$$

exists and $l = \lambda$ or $l = \mu$. Condition (1.5) implies that

$$\lim_{n \to \infty} (1 + c)^n |x(n) - x(n - 1)|^2 = 0. \tag{3.16}$$

Therefore,

$$x(n - 1) - x(n) = O\left(\frac{1}{\left(\sqrt{1 + c}\right)^n}\right), \quad n \longrightarrow \infty. \tag{3.17}$$

From this and (1.4), we find that

$$x(n) = \sum_{i=n}^{\infty} (x(i) - x(i + 1)) = O\left(\frac{1}{\left(\sqrt{1 + c}\right)^n}\right), \quad n \longrightarrow \infty. \tag{3.18}$$

Hence

$$l = \lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \lim_{n\to\infty} \sqrt[n]{x(n)} \leq \frac{1}{\sqrt{1+c}}. \tag{3.19}$$

Since $\mu > 1/\sqrt{1+c}$, we have that $l = \lambda$ and (1.19) holds.

Now suppose that f' is Lipschitz continuous on $[0,1]$. For all large n, we have

$$\left|f(x(n)) - \alpha x(n)\right| = \left|\int_0^1 \left[f'(sx(n)) - f'_+(0)\right] ds x(n)\right| \leq K x^2(n), \tag{3.20}$$

where K is the Lipschitz constant of f' on $[0,1]$. From this and (3.18), we see that

$$\frac{f(x(n))}{x(n)} = \alpha + O\left(\frac{1}{\left(\sqrt{1+c}\right)^n}\right), \quad n \longrightarrow \infty. \tag{3.21}$$

This shows that the convergence of the coefficients of (3.6) to their limits is exponentially fast. Thus, Proposition 2.1 applies and the limit relation (1.21) follows from the asymptotic formula (2.7). □

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