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Research Article

Controllability for the Impulsive Semilinear Nonlocal Fuzzy Integrodifferential Equations in n-Dimensional Fuzzy Vector Space

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We study the existence and uniqueness of solutions and nonlocal controllability for the impulsive semilinear nonlocal fuzzy integrodifferential equations in n-dimensional fuzzy vector space $(E_N)^n$ by using short-term perturbations techniques and Banach fixed point theorem. This is an extension of the result of Kwun et al. (Kwun et al., 2009) to impulsive system.

1. Introduction

The theory of differential equations with discontinuous trajectories during the last twenty years has been to a great extent stimulated by their numerous applications to problem arising in mechanics, electrical engineering, the theory of automatic control, medicine and biology. For the monographs of the theory of impulsive differential equations, see the papers of Bainov and Simenov [1], Lakshmikantham et al. [2] and Samoileuko and Perestyuk [3], where numerous properties of their solutions are studied and detailed bibliographies are given. Rogovchenko [4] followed the ideas of the theory of impulsive differential equations which treats the changes of the state of the evolution process due to a short-term perturbations whose duration can be negligible in comparison with the duration of the process as an instant impulses. In 2001, Lakshmikantham and McRae [5] studied basic results for fuzzy impulsive differential equations. Park et al. [6] studied the existence and uniqueness of fuzzy solutions and controllability for the impulsive semilinear fuzzy integrodifferential equations in one-dimensional fuzzy vector space E_N^1. Rodríguez-López [7] studied periodic boundary value problems for impulsive fuzzy differential equations. Fuzzy integrodifferential equations are a field of interest, due to their applicability to the analysis of phenomena with memory where imprecision is inherent. Balasubramaniam and Muralisankar [8] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. They considered the semilinear one-dimensional heat equation on a connected domain $(0,1)$ for material with memory. In one-dimensional fuzzy vector space E_N^1, Park et al. [9] proved the existence and uniqueness of fuzzy solutions and presented the sufficient condition of nonlocal controllability for the following semilinear fuzzy integrodifferential equation with nonlocal initial condition.

In [10], Kwun et al. proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equations by using successive iteration. In [11], Kwun et al. investigated the continuously initial observability for the semilinear fuzzy integrodifferential equations. Bede and Gal [12] studied almost periodic fuzzy-number-valued functions. Gal and N'Guerekata [13] studied almost automorphic fuzzy-number-valued functions. More recently, Kwun et al. [14] studied the existence and uniqueness of solutions and nonlocal controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space.

In this paper, we study the existence and uniqueness of solutions and nonlocal controllability for the following impulsive semilinear nonlocal fuzzy integrodifferential equations in n-dimensional fuzzy vector space by using short-term perturbations techniques and Banach fixed point theorem:

$$\begin{gathered}
\frac{dx_i(t)}{dt} = A_i\left[x_i(t) + \int_0^t G(t-s)x_i(s)ds\right] + f_i\left(t, x_i(t), \int_0^t q_i(t,s,x_i(s))ds\right) + u_i(t) \quad \text{on } E_N^i, \\
x_i(0) + g_i(x_i) = x_{0_i} \in E_N^i, \\
\Delta x_i(t_k) = I_k(x_i(t_k)), \quad t \neq t_k,\ k = 1,2,\dots,m,\ i = 1,2,\dots,n,
\end{gathered} \tag{1.1}$$

where $A_i : [0,T] \to E_N^i$ is fuzzy coefficient, E_N^i is the set of all upper semicontinuously convex fuzzy numbers on R with $E_N^i \neq E_N^j$ $(i \neq j)$, $f_i : [0,T] \times E_N^i \times E_N^i \to E_N^i$ and $q_i : [0,T] \times [0,T] \times E_N^i$ are nonlinear regular fuzzy functions, $g_i : E_N^i \to E_N^i$ is a nonlinear continuous function, $G(t)$ is an $n \times n$ continuous matrix such that $dG(t)x_i/dt$ is continuous for $x_i \in E_N^i$ and $t \in [0,T]$ with $\|G(t)\| \leq k$, $k > 0$, $u_i : [0,T] \to E_N^i$ is a control function, $x_{0_i} \in E_N^i$ is an initial value and $I_k \in C(E_N^i, E_N^i)$ are bounded functions, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, where $x_i(t_k^-)$ and $x_i(t_k^+)$ represent the left and right limits of $x_i(t)$ at $t = t_k$, respectively.

2. Preliminaries

A fuzzy set u of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$ its α-level set.

Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$.

Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_o) = 1$;

(2) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \leq \lambda \leq 1$;

(3) $u(x)$ is upper semicontinuous, that is, $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n (k = 0, 1, 2, \ldots)$, $x_k \longrightarrow x_0$;

(4) $[u]^0$ is compact.

We call $u \in E^n$ an n-dimension fuzzy number.

Wang et al. [15] defined n-dimensional fuzzy vector space and investigated its properties.

For any $u_i \in E$, $i = 1, 2, \ldots, n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, \ldots, u_n$ (i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, \ldots, u_n$) an n-dimension fuzzy vector, denote it as $(u_1, u_2, \ldots, u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}$) n-dimensional fuzzy vector space, and denote it as $(E)^n$.

Definition 2.1 (see [15]). If $u \in E^n$, and $[u]^\alpha$ is a hyperrectangle, that is, $[u]^\alpha$ can be represented by $\prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$, that is, $[u_{1l}^\alpha, u_{1r}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha]$ for every $\alpha \in [0,1]$, where $u_{il}^\alpha, u_{ir}^\alpha \in R$ with $u_{il}^\alpha \leq u_{ir}^\alpha$ when $\alpha \in (0,1]$, $i = 1, 2, \ldots, n$, then we call u a fuzzy n-cell number. We denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

Theorem 2.2 (see [15]). *For any $u \in L(E^n)$ with $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha] (\alpha \in [0,1])$, there exists a unique $(u_1, u_2, \ldots, u_n) \in (E)^n$ such that $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \ldots, n$ and $\alpha \in [0,1])$. Conversely, for any $(u_1, u_2, \ldots, u_n) \in (E)^n$ with $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \ldots, n$ and $\alpha \in [0,1])$, there exists a unique $u \in L(E^n)$ such that $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha] (\alpha \in [0,1])$.*

Note 1 (see [15]). Theorem 2.2 indicates that fuzzy n-cell numbers and n-dimension fuzzy vectors can represent each other, so $L(E^n)$ and $(E)^n$ may be regarded as identity. If $(u_1, u_2, \ldots, u_n) \in (E)^n$ is the unique n-dimension fuzzy vector determined by $u \in L(E^n)$, then we denote $u = (u_1, u_2, \ldots, u_n)$.

Let $(E_N^i)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n$, where E_N^i $(i = 1, 2, \ldots, n)$ is a fuzzy subset of R. Then $(E_N^i)^n \subseteq (E)^n$.

Definition 2.3 (see [15]). The complete metric D_L on $(E_N^i)^n$ is defined by

$$D_L(u, v) = \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) = \sup_{0<\alpha\leq 1} \max_{1\leq i\leq n} \{ |u_{il}^\alpha - v_{il}^\alpha|, |u_{ir}^\alpha - v_{ir}^\alpha| \} \tag{2.1}$$

for any $u, v \in (E_N^i)^n$, which satisfies $d_L(u + w, v + w) = d_L(u, v)$.

Definition 2.4. Let $u, v \in C([0,T] : (E_N^i)^n)$,

$$H_1(u, v) = \sup_{0\leq t\leq T} D_L(u(t), v(t)). \tag{2.2}$$

Definition 2.5 (see [15]). The derivative $x'(t)$ of a fuzzy process $x \in (E_N^i)^n$ is defined by

$$[x'(t)]^\alpha = \prod_{i=1}^{n} \left[(x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t)\right] \tag{2.3}$$

provided that equation defines a fuzzy $x'(t) \in (E_N^i)^n$.

Definition 2.6 (see [15]). The fuzzy integral $\int_b^a x(t)dt$, $a, b \in [0, T]$ is defined by

$$\left[\int_b^a x(t)dt\right]^\alpha = \prod_{i=1}^{n} \left[\int_b^a x_{il}^\alpha(t)dt, \int_b^a x_{ir}^\alpha(t)dt\right] \tag{2.4}$$

provided that the Lebesgue integrals on the right-hand side exist.

3. Existence and Uniqueness

In this section we consider the existence and uniqueness of the fuzzy solution for (1.1) ($u \equiv 0$).

We define

$$\begin{aligned}
A &= (A_1, A_2, \ldots, A_n),\\
x &= (x_1, x_2, \ldots, x_n),\\
f &= (f_1, f_2, \ldots, f_n),\\
q &= (q_1, q_2, \ldots, q_n),\\
u &= (u_1, u_2, \ldots, u_n),\\
g &= (g_1, g_2, \ldots, g_n),
\end{aligned} \tag{3.1}$$

$$x_0 = (x_{0_1}, x_{0_2}, \ldots, x_{0_n}). \tag{3.2}$$

Then

$$A,\ x,\ f,\ q,\ x_0,\ u,\ g \in \left(E_N^i\right)^n. \tag{3.3}$$

Instead of (1.1), we consider the following fuzzy integrodifferential equations in $(E_N^i)^n$

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] + f\left(t, x(t), \int_0^t q(t,s,x(s))ds\right) + u(t) \quad \text{on } \left(E_N^i\right)^n, \tag{3.4}$$

$$x(0) + g(x) = x_0 \in \left(E_N^i\right)^n, \tag{3.5}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad t \neq t_k,\ k = 1, 2, \ldots, m,\ i = 1, 2, \ldots, n, \tag{3.6}$$

with fuzzy coefficient $A : [0,T] \to (E_N^i)^n$, initial value $x_0 \in (E_N^i)^n$, and $u : [0,T] \to (E_N^i)^n$ being a control function. Given nonlinear regular fuzzy functions $f : [0,T] \times (E_N^i)^n \times (E_N^i)^n \to (E_N^i)^n$ and $q : [0,T] \times [0,T] \times (E_N^i)^n \to (E_N^i)^n$ satisfy global Lipschitz conditions, that is, there exist finite constants $k_1, k_2, M > 0$ such that

$$\begin{aligned} &d_L\left(\left[f\left(s, \xi_1(s), \eta_1(s)\right)\right]^\alpha, \left[f\left(s, \xi_2(s), \eta_2(s)\right)\right]^\alpha\right) \\ &\qquad \le k_1 d_L\left([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha\right) + k_2 d_L\left([\eta_1(s)]^\alpha, [\eta_2(s)]^\alpha\right), \end{aligned} \tag{3.7}$$

$$d_L\left(\left[q\left(t, s, \varphi_1(s)\right)\right]^\alpha, \left[q\left(t, s, \varphi_2(s)\right)\right]^\alpha\right) \le M d_L\left([\varphi_1(s)]^\alpha, [\varphi_2(s)]^\alpha\right) \tag{3.8}$$

for all $\xi_j(s), \eta_j(s), \varphi_j(s) \in (E_N^i)^n (j = 1, 2)$, the nonlinear function $g : (E_N^i)^n \to (E_N^i)^n$ is continuous and satisfies the Lipschitz condition

$$d_L\left([g(x(\cdot))]^\alpha, [g(y(\cdot))]^\alpha\right) \le h d_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) \tag{3.9}$$

for all $x(\cdot), y(\cdot) \in (E_N^i)^n$, h is a finite positive constant.

Definition 3.1. The fuzzy process $x : I = [0,T] \to (E_N^i)^n$ with α-level set $[x(t)]^\alpha = \Pi_{i=1}^n [x_i]^\alpha = \Pi_{i=1}^n [x_{il}^\alpha, x_{ir}^\alpha]$ is a fuzzy solution of (3.4) and (3.5) without nonhomogeneous term if and only if

$$\begin{aligned} &\left(x_{il}^\alpha\right)'(t) = \min\left\{ A_{ij}^\alpha(t)\left[x_{ik}^\alpha(t) + \int_0^t G(t-s) x_{ik}^\alpha(s) ds\right] : j,k = l,r \right\}, \\ &\left(x_{ir}^\alpha\right)'(t) = \max\left\{ A_{ij}^\alpha(t)\left[x_{ik}^\alpha(t) + \int_0^t G(t-s) x_{ik}^\alpha(s) ds\right] : j,k = l,r \right\}, \\ &x_{il}^\alpha(0) + g_{il}^\alpha\left(x_{il}^\alpha\right) = x_{0_{il}}^\alpha, \quad x_{ir}^\alpha(0) + g_{ir}^\alpha\left(x_{ir}^\alpha\right) = x_{0_{ir}}^\alpha, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.10}$$

For the sequel, we need the following assumption:

(H1) $S(t)$ is a fuzzy number satisfying, for $y \in (E_N^i)^n$, $(d/dt)S(t)y \in C^1(I : (E_N^i)^n) \bigcap C(I : (E_N^i)^n)$, the equation

$$\frac{d}{dt} S(t)y = A\left[S(t)y + \int_0^t G(t-s) S(s) y \, ds\right], \quad t \in I, \tag{3.11}$$

where

$$[S(t)]^\alpha = \prod_{i=1}^n [S_i(t)]^\alpha = \prod_{i=1}^n \left[S_{il}^\alpha(t), S_{ir}^\alpha(t)\right], \quad S(0) = I \tag{3.12}$$

and $S_{ij}^\alpha(t) (j = l, r)$ is continuous with $|S_{ij}^\alpha(t)| \le c$, $c > 0$, for all $t \in I = [0,T]$.

In order to define the solution of (3.4)–(3.6), we will consider the space $\Omega_i = \{x_i : J \to E^i_N : (x_i)_k \in C(J_k, E^i_N), J_k = (t_k, t_{k+1}], k = 0, 1, \ldots, m$, and there exist $(x_i)(t_k^-)$ and $(x_i)(t_k^+)$ $(k = 1, 2, \ldots, m)$, with $(x_i)(t_k^-) = (x_i)(t_k)\}$, $i = 1, 2, \ldots, n$.

Let $\Omega' = \Pi_{i=1}^n \Omega'_i$, $\Omega'_i = \Omega_i \bigcap C([0,T] : E^i_N)$, $i = 1, 2, \ldots, n$.

Lemma 3.2. *If x is an integral solution of* (3.4)–(3.6) $(u \equiv 0)$, *then x is given by*

$$\begin{aligned} x(t) &= S(t)\left(x_0 - g(x)\right) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau\right) ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k) I_k\left(x\left(t_k^-\right)\right), \quad \textit{for } t \in J. \end{aligned} \tag{3.13}$$

Proof. Let x be a solution of (3.4)–(3.6). Define $\omega(s) = S(t-s)x(s)$. Then we have that

$$\begin{aligned} \frac{d\omega(s)}{ds} &= -\frac{dS(t-s)}{ds} x(s) + S(t-s)\frac{dx(s)}{ds} \\ &= -A\left[S(t-s)x(s) + \int_0^t G(t-s)S(s)x(s)ds\right] + S(t-s)\frac{dx(s)}{ds} \\ &= S(t-s) f\left(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau\right). \end{aligned} \tag{3.14}$$

Consider $t_k < t, k = 1, 2, \ldots, m$. Then integrating the previous equation, we have

$$\int_0^t \frac{d\omega(s)}{ds} ds = \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau\right) ds. \tag{3.15}$$

For $k = 1$,

$$\omega(t) - \omega(0) = \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau\right) ds \tag{3.16}$$

or

$$x(t) = S(t)\left(x_0 - g(x)\right) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau\right) ds. \tag{3.17}$$

Now for $k = 2, \ldots, m$, we have that

$$\int_0^{t_1} \frac{d\omega(s)}{ds} ds + \int_{t_1}^{t_2} \frac{d\omega(s)}{ds} ds + \cdots + \int_{t_k}^{t} \frac{d\omega(s)}{ds} ds = \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right) ds. \tag{3.18}$$

Then

$$\omega(t_1^-) - \omega(0) + \omega(t_2^-) - \omega(t_1^+) + \cdots - \omega(t_k^+) + \omega(t) = \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right) ds \tag{3.19}$$

if and only if

$$\omega(t) = \omega(0) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right) ds + \sum_{0<t_k<t} \left[\omega(t_k^+) - \omega(t_k^-)\right]. \tag{3.20}$$

Hence

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right) ds + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)), \tag{3.21}$$

which proves the lemma. □

Assume the following:

(H2) there exists $d > 0$ such that

$$d_L\left(\left[I_k(x(t_k^-))\right]^\alpha, \left[I_k(y(t_k^-))\right]^\alpha\right) \le d\, d_L\left([x(t)]^\alpha, [y(t)]^\alpha\right), \tag{3.22}$$

where $x(t), y(t) \in \Omega'$;

(H3)

$$c\left[T\left\{h(1+c) + cd + k_1\left(1 + \frac{cT}{2}\right) + k_2 MT\left(12 + \frac{cT}{3}\right)\right\} + (h+d)\right] < 1. \tag{3.23}$$

Theorem 3.3. *Let $T > 0$. If hypotheses (H1)–(H3) are hold, then, for every $x_0 \in (E_N^i)^n$,* (3.13) *has a unique fuzzy solution $x \in \Omega'$.*

Proof. For each $x(t) \in \Omega'$ and $t \in [0,T]$, define $(G_0x)(t) \in \Omega'$ by

$$\begin{aligned}(G_0x)(t) = {}& S(t)\big(x_0 - g(x)\big) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right)ds \\ &+ \sum_{0<t_k<t} S(t-t_k) I_k\big(x(t_k^-)\big).\end{aligned} \tag{3.24}$$

Thus, $G_0x : [0,T] \to \Omega'$ is continuous, so G_0 is a mapping from Ω' into itself. By Definitions 2.3 and 2.4, some properties of d_L and inequalities (3.7), (3.8), and (3.9), we have the following inequalities. For $x, y \in \Omega'$,

$$\begin{aligned}
&d_L\big([(G_0x)(t)]^\alpha, [(G_0y)(t)]^\alpha\big) \\
&\quad\le d_L\Bigg(\left[S(t)\big(x_0 - g(x)\big) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right)ds\right]^\alpha, \\
&\qquad\qquad \left[S(t)\big(x_0 - g(y)\big) + \int_0^t S(t-s) f\left(s, y(s), \int_0^s q(s,\tau,y(\tau))d\tau\right)ds\right]^\alpha\Bigg) \\
&\qquad + d_L\left(\left[\sum_{0<t_k<t} S(t-t_k) I_k\big(x(t_k^-)\big)\right]^\alpha, \left[\sum_{0<t_k<t} S(t-t_k) I_k\big(y(t_k^-)\big)\right]^\alpha\right) \\
&\quad\le d_L\big([S(t)g(x)]^\alpha, [S(t)g(y)]^\alpha\big) \\
&\qquad + \int_0^t d_L\Bigg(\left[S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right)\right]^\alpha, \\
&\qquad\qquad \left[S(t-s) f\left(s, y(s), \int_0^s q(s,\tau,y(\tau))d\tau\right)\right]^\alpha\Bigg)ds \\
&\qquad + d_L\left(\left[\sum_{0<t_k<t} S(t-t_k) I_k\big(x(t_k^-)\big)\right]^\alpha, \left[\sum_{0<t_k<t} S(t-t_k) I_k\big(y(t_k^-)\big)\right]^\alpha\right) \\
&\quad\le chd_L\big([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\big) \\
&\qquad + c\int_0^t \left(k_1 d_L\big([x(s)]^\alpha, [y(s)]^\alpha\big) + k_2 M \int_0^s d_L\big([x(\tau)]^\alpha, [y(\tau)]^\alpha\big)d\tau\right)ds \\
&\qquad + cdd_L\big([x(t)]^\alpha, [y(t)]^\alpha\big).
\end{aligned} \tag{3.25}$$

Therefore

$$
\begin{aligned}
&D_L\big((G_0x)(t),(G_0y)(t)\big)\\
&\quad=\sup_{0<\alpha\le 1} d_L\big([(G_0x)(t)]^\alpha,[(G_0y)(t)]^\alpha\big)\\
&\quad\le ch\sup_{0<\alpha\le 1} d_L\big([x(\cdot)]^\alpha,[y(\cdot)]^\alpha\big)\\
&\qquad+c\int_0^t\left(k_1\sup_{0<\alpha\le 1} d_L\big([x(s)]^\alpha,[y(s)]^\alpha\big)+k_2M\int_0^s\sup_{0<\alpha\le 1} d_L\big([x(\tau)]^\alpha,[y(\tau)]^\alpha\big)d\tau\right)ds\\
&\qquad+cd\sup_{0<\alpha\le 1} d_L\big([x(t)]^\alpha,[y(t)]^\alpha\big)\\
&\quad\le chD_L\big(x(\cdot),y(\cdot)\big)\\
&\qquad+c\int_0^t\left(k_1D_L\big(x(s),y(s)\big)+k_2M\int_0^s D_L\big(x(\tau),y(\tau)\big)d\tau\right)ds+cdD_L\big(x(t),y(t)\big).
\end{aligned}
\tag{3.26}
$$

Hence

$$
\begin{aligned}
H_1(G_0x,G_0y)&=\sup_{0\le t\le T} D_L\big((G_0x)(t),(G_0y)(t)\big)\\
&\le ch\sup_{0\le t\le T} D_L\big(x(\cdot),y(\cdot)\big)\\
&\quad+c\sup_{0\le t\le T}\int_0^t\left(k_1D_L\big(x(s),y(s)\big)+k_2M\int_0^s D_L\big(x(\tau),y(\tau)\big)d\tau\right)ds\\
&\quad+cd\sup_{0\le t\le T} D_L\big(x(t),y(t)\big)\\
&\le c\left(h+d+\left(k_1+k_2M\frac{T}{2}\right)T\right)H_1(x,y).
\end{aligned}
\tag{3.27}
$$

By hypothesis (H3), G_0 is a contraction mapping. Using the Banach fixed point theorem, (3.13) has a unique fixed point $x\in\Omega'$. □

4. Nonlocal Controllability

In this section, we show the nonlocal controllability for the control system (1.1).

The control system (1.1) is related to the following fuzzy integral system:

$$\begin{aligned} x(t) = {} & S(t)\left(x_0 - g(x)\right) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right) ds \\ & + \int_0^t S(t-s)u(s)ds + \sum_{0<t_k<t} S(t-t_k) I_k\left(x\left(t_k^-\right)\right). \end{aligned} \tag{4.1}$$

Definition 4.1. Equations (1.1)–(3) are nonlocal controllable. Then there exists $u(t)$ such that the fuzzy solution $x(t)$ for (4.1) as $x(T) = x^1 - g(x)$(i.e., $[x(T)]^\alpha = [x^1 - g(x)]^\alpha$), where $x^1 \in (E_N^i)^n$, is target set.

Define the fuzzy mapping $\widetilde{\beta} : \widetilde{P}(R^n) \to (E_N^i)^n$ by

$$\widetilde{\beta}^\alpha(v) = \begin{cases} \displaystyle\int_0^T S^\alpha(T-s)v(s)ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases} \tag{4.2}$$

where $\overline{\Gamma}_u$ is closed support of u. Then there exists

$$\widetilde{\beta}_i : \widetilde{P}(R) \longrightarrow E_N^i \qquad (i = 1, 2, \ldots, n) \tag{4.3}$$

such that

$$\widetilde{\beta}_i^\alpha(v_i) = \begin{cases} \displaystyle\int_0^T S_i^\alpha(T-s)v_i(s)ds, & v_i(s) \subset \overline{\Gamma}_{u_i} \\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

Then there exists $\widetilde{\beta}_{ij}^\alpha (j = l, r)$ such that

$$\begin{aligned} \widetilde{\beta}_{il}^\alpha(v_{il}) &= \int_0^T S_{il}^\alpha(T-s)v_{il}(s)ds, \quad v_{il}(s) \in \left[u_{il}^\alpha(s), u_i^1\right], \\ \widetilde{\beta}_{ir}^\alpha(v_{ir}) &= \int_0^T S_{ir}^\alpha(T-s)v_{ir}(s)ds, \quad v_{ir}(s) \in \left[u_i^1, u_{ir}^\alpha(s)\right]. \end{aligned} \tag{4.5}$$

We assume that $\widetilde{\beta}_{il}^\alpha, \widetilde{\beta}_{ir}^\alpha$ are bijective mappings.

We can introduce α-level set of $u(s)$ of (4.1):

$$
\begin{aligned}
[u(s)]^\alpha &= \prod_{i=1}^{n}[u_i(s)]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha(s), u_{ir}^\alpha(s)] \\
&= \prod_{i=1}^{n}\Bigg[\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}\Bigg(\left(\left(x^1\right)_{il}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) - S_{il}^\alpha(T)\left(x_{0_{il}}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) \\
&\qquad - \int_0^T S_{il}^\alpha(T-s) f_{il}^\alpha\left(s, x_{il}^\alpha(s), \int_0^s q_{il}^\alpha(s,\tau,x_{il}^\alpha(\tau))d\tau\right)ds \\
&\qquad - \sum_{0<t_k<T} S_{il}^\alpha(T-t_k) I_{k_{il}}^\alpha(x_{il}^\alpha(t_k^-))\Bigg), \\
&\quad \left(\widetilde{\beta}_{ir}^\alpha\right)^{-1}\Bigg(\left(\left(x^1\right)_{ir}^\alpha - g_{ir}^\alpha(x_{ir}^\alpha)\right) - S_{ir}^\alpha(T)\left(x_{0_{ir}}^\alpha - g_{ir}^\alpha(x_{ir}^\alpha)\right) \\
&\qquad - \int_0^T S_{ir}^\alpha(T-s) f_{ir}^\alpha\left(s, x_{ir}^\alpha(s), \int_0^s q_{ir}^\alpha(s,\tau,x_{ir}^\alpha(\tau))d\tau\right)\Bigg)ds \\
&\qquad - \sum_{0<t_k<T} S_{ir}^\alpha(T-t_k) I_{k_{ir}}^\alpha(x_{ir}^\alpha(t_k^-))\Bigg)\Bigg].
\end{aligned} \tag{4.6}
$$

Then substituting this expression into (4.1) yields α-level of $x(T)$.

For each $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
[x_i(T)]^\alpha = \Bigg[& S_{il}^\alpha(T)\left(x_{0_{il}}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) + \int_0^T S_{il}^\alpha(T-s) \\
& \times f_{il}^\alpha\left(s, x_{il}^\alpha(s), \int_0^s q_{il}^\alpha(s,\tau,x_{il}^\alpha(\tau))d\tau\right)ds + \sum_{0<t_k<T} S_{il}^\alpha(T-t_k) I_{k_{il}}^\alpha(x_{il}^\alpha(t_k^-)) \\
& + \int_0^T S_{il}^\alpha(T-s)\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}\Bigg(\left(\left(x^1\right)_{il}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) - S_{il}^\alpha(T)\left(x_{0_{il}}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) \\
& \qquad - \int_0^T S_{il}^\alpha(T-s) f_{il}^\alpha\left(s, x_{il}^\alpha(s), \int_0^s q_{il}^\alpha(s,\tau,x_{il}^\alpha(\tau))d\tau\right)ds \\
& \qquad - \sum_{0<t_k<T} S_{il}^\alpha(T-t_k) I_{k_{il}}^\alpha(x_{il}^\alpha(t_k^-))\Bigg)ds,
\end{aligned}
$$

$$\begin{aligned}
&S_{ir}^{\alpha}(T)\left(x_{0_{ir}}^{\alpha}-g_{ir}^{\alpha}\left(x_{ir}^{\alpha}\right)\right)+\int_{0}^{T}S_{ir}^{\alpha}(T-s)\times f_{ir}^{\alpha}\left(s,x_{ir}^{\alpha}(s),\int_{0}^{s}q_{il}^{\alpha}\left(s,\tau,x_{ir}^{\alpha}(\tau)\right)d\tau\right)ds\\
&\quad+\sum_{0<t_k<T}S_{ir}^{\alpha}(T-t_k)I_{k_{ir}}^{\alpha}\left(x_{ir}^{\alpha}\left(t_k^{-}\right)\right)\\
&\quad+\int_{0}^{T}S_{ir}^{\alpha}(T-s)\left(\widetilde{\beta}_{ir}^{\alpha}\right)^{-1}\\
&\qquad\times\left(\left(\left(\left(x^{1}\right)_{ir}^{\alpha}-g_{ir}^{\alpha}\left(x_{ir}^{\alpha}\right)\right)\right)-S_{ir}^{\alpha}(T)\left(x_{0_{ir}}^{\alpha}-g_{ir}^{\alpha}\left(x_{ir}^{\alpha}\right)\right)\right.\\
&\qquad\quad-\int_{0}^{T}S_{ir}^{\alpha}(T-s)f_{ir}^{\alpha}\left(s,x_{ir}^{\alpha}(s),\int_{0}^{s}q_{il}^{\alpha}\left(s,\tau,x_{ir}^{\alpha}(\tau)\right)d\tau\right)ds\\
&\qquad\quad\left.-\sum_{0<t_k<T}S_{ir}^{\alpha}(T-t_k)I_{k_{ir}}^{\alpha}\left(x_{ir}^{\alpha}\left(t_k^{-}\right)\right)\right)ds\Bigg]\\
&=\left[\left(x^{1}-g(x)\right)_{il}^{\alpha},\left(x^{1}-g(x)\right)_{ir}^{\alpha}\right]=\left[\left(x^{1}-g(x)\right)_{i}\right]^{\alpha}.
\end{aligned}\tag{4.7}$$

Therefore

$$[x(T)]^{\alpha}=\prod_{i=1}^{n}\left[x_i(T)\right]^{\alpha}=\prod_{i=1}^{n}\left[\left(x^{1}-g(x)\right)_{i}\right]^{\alpha}=\left[x^{1}-g(x)\right]^{\alpha}.\tag{4.8}$$

We now set

$$\begin{aligned}
\Phi x(t)&=S(t)\left(x_0-g(x)\right)+\int_{0}^{t}S(t-s)f\left(s,x(s),\int_{0}^{t}q(s,\tau,x(\tau))d\tau\right)ds\\
&\quad+\sum_{0<t_k<t}S(t-t_k)I_k\left(x\left(t_k^{-}\right)\right)\\
&\quad+\int_{0}^{t}S(t-s)\widetilde{\beta}^{-1}\left(x^{1}-g(x)-S(T)\left(x_0-g(x)\right)\right.\\
&\qquad\quad-\int_{0}^{T}S(T-s)f\left(s,x(s),\int_{0}^{s}q(s,\tau,x(\tau))d\tau\right)ds\\
&\qquad\quad\left.-\sum_{0<t_k<T}S(T-t_k)I_k\left(x\left(t_k^{-}\right)\right)\right)ds,
\end{aligned}\tag{4.9}$$

where the fuzzy mapping $\widetilde{\beta}^{-1}$ satisfies the previous statements.

Notice that $\Phi x(T)=x^{1}-g(x)$, which means that the control $u(t)$ steers (4.9) from the origin to $x^{1}-g(x)$ in time T provided we can obtain a fixed point of the operator Φ.

(H4) Assume that the linear system of (4.9) ($f\equiv 0$) is controllable.

Theorem 4.2. *Suppose that hypotheses (H1)–(H4) are satisfied. Then* (4.9) *is nonlocal controllable.*

Proof. We can easily check that Φ is continuous function from Ω' to itself. By Definitions 2.3 and 2.4, some properties of d_L, and inequalities (3.7), (3.8), and (3.9), we have following inequalities. For any $x, y \in \Omega'$,

$$
\begin{aligned}
&d_L\left([\Phi x(t)]^\alpha, [\Phi y(t)]^\alpha\right) \\
&= d_L\Bigg(\Bigg[S(t)\ (x_0 - g(x)) + \int_0^t S(t-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right)ds \\
&\qquad + \sum_{0<t_k<t} S(t-t_k) I_k\left(x\left(t_k^-\right)\right) \\
&\qquad + \int_0^t S(t-s)\widetilde{\beta}^{-1}\Bigg(x^1 - g(x) - S(T)(x_0 - g(x)) \\
&\qquad\qquad - \int_0^T S(T-s) f\left(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau\right)ds \\
&\qquad\qquad - \sum_{0<t_k<T} S(T-t_k) I_k\left(x\left(t_k^-\right)\right)\Bigg)ds\Bigg]^\alpha, \\
&\qquad \Bigg[S(t)(x_0 - g(y)) + \int_0^t S(t-s) f\left(s, y(s), \int_0^s q(s,\tau,y(\tau))d\tau\right)ds \\
&\qquad + \sum_{0<t_k<t} S(t-t_k) I_k\left(y\left(t_k^-\right)\right) \\
&\qquad + \int_0^t S(t-s)\widetilde{\beta}^{-1}\Bigg(x^1 - g(y) - S(T)(x_0 - g(y)) \\
&\qquad\qquad - \int_0^T S(T-s) f\left(s, y(s), \int_0^s q(s,\tau,y(\tau))d\tau\right)ds \\
&\qquad\qquad - \sum_{0<t_k<T} S(T-t_k) I_k\left(y\left(t_k^-\right)\right)\Bigg)ds\Bigg]^\alpha\Bigg) \\
&\le chd_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) \\
&\quad + c\int_0^t \left(k_1 d_L\left([x(s)]^\alpha, [y(s)]^\alpha\right) + k_2 M \int_0^s d_L\left([x(\tau)]^\alpha, [y(\tau)]^\alpha\right)d\tau\right)ds \\
&\quad + cdd_L\left([x(t)]^\alpha, [y(t)]^\alpha\right) \\
&\quad + c\int_0^t \Bigg\{hd_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) + chd_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) \\
&\qquad + c\int_0^T \left(k_1 d_L\left([x(s)]^\alpha, [y(s)]^\alpha\right) + k_2 M \int_0^s d_L\left([x(\tau)]^\alpha, [y(\tau)]^\alpha\right)d\tau\right)ds \\
&\qquad + cdd_L\left([x(t)]^\alpha, [y(t)]^\alpha\right)\Bigg\}ds.
\end{aligned}
\tag{4.10}
$$

Therefore

$$
\begin{aligned}
D_L(\Phi x(t), \Phi y(t)) = {} & \sup_{0<\alpha\le 1} d_L\left([\Phi x(t)]^\alpha, [\Phi y(t)]^\alpha\right) \\
& \le ch \sup_{0<\alpha\le 1} d_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) + cd \sup_{0<\alpha\le 1} d_L\left([x(t)]^\alpha, [y(t)]^\alpha\right) \\
& \quad + c\int_0^t \Bigg(k_1 \sup_{0<\alpha\le 1} d_L\left([x(s)]^\alpha, [y(s)]^\alpha\right) \\
& \qquad + k_2 M \int_0^s \sup_{0<\alpha\le 1} d_L\left([x(\tau)]^\alpha, [y(\tau)]^\alpha\right) d\tau \Bigg) ds \\
& \quad + c\int_0^t \Bigg\{ h(1+c) \sup_{0<\alpha\le 1} d_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) \\
& \qquad + c\int_0^T \Bigg(k_1 \sup_{0<\alpha\le 1} d_L\left([x(s)]^\alpha, [y(s)]^\alpha\right) \\
& \qquad\quad + k_2 M \int_0^s \sup_{0<\alpha\le 1} d_L\left([x(\tau)]^\alpha, [y(\tau)]^\alpha\right) d\tau \Bigg) ds \\
& \qquad + cd \sup_{0<\alpha\le 1} d_L\left([x(s)]^\alpha, [y(s)]^\alpha\right) \Bigg\} ds \\
& \le chD_L(x(\cdot), y(\cdot)) + cdD_L(x(t), y(t)) \\
& \quad + c\int_0^t \left(k_1 D_L(x(s), y(s)) + k_2 M \int_0^s D_L(x(\tau), y(\tau)) d\tau \right) ds \\
& \quad + c\int_0^t \Bigg\{ h(1+c) D_L(x(\cdot), y(\cdot)) \\
& \qquad + c\int_0^T \left(k_1 D_L(x(s), y(s)) + k_2 M \int_0^s D_L(x(\tau), y(\tau)) d\tau \right) ds \\
& \qquad + cdD_L(x(s), y(s)) \Bigg\} ds.
\end{aligned}
\tag{4.11}
$$

Hence

$$
\begin{aligned}
H_1(\Phi x, \Phi y) = {} & \sup_{0\le t\le T} D_L(\Phi x(t), \Phi y(t)) \\
& \le ch \sup_{0\le t\le T} D_L(x(\cdot), y(\cdot)) + cd \sup_{0\le t\le T} D_L(x(t), y(t)) \\
& \quad + c \sup_{0\le t\le T} \int_0^t \left(k_1 D_L(x(\cdot), y(\cdot)) + k_2 M \int_0^s D_L(x(\tau), y(\tau)) d\tau \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+ c \sup_{0\le t\le T} \int_0^t \Bigg\{ h(1+c) D_L\big(x(\cdot), y(\cdot)\big) \\
&\qquad + c\int_0^T \left(k_1 D_L\big(x(s), y(s)\big) + k_2 M \int_0^s D_L\big(x(\tau), y(\tau)\big) d\tau \right) ds \\
&\qquad + c d D_L\big(x(s), y(s)\big) \Bigg\} ds \\
&\le c h H_1(x,y) + c d H_1(x,y) + c\int_0^T \left(k_1 H_1(x,y) + k_2 M \int_0^s H_1(x,y) d\tau \right) ds \\
&+ c\int_0^T \Bigg\{ h(1+c) H_1(x,y) \\
&\qquad + c\int_0^T \left(k_1 H_1(x,y) + k_2 M \int_0^s H_1(x,y) d\tau \right) ds + c d H_1(x,y) \Bigg\} ds \\
&= c\left[T\left\{ h(1+c) + cd + k_1\left(1 + \frac{cT}{2}\right) + k_2 M T\left(\frac{1}{2} + \frac{cT}{3}\right) \right\} + (h+d) \right] H_1(x,y).
\end{aligned}
\tag{4.12}
$$

By hypothesis (H3), Φ is a contraction mapping. Using the Banach fixed point theorem, (4.9) has a unique fixed point $x \in \Omega'$. □

5. Example

Consider the two semilinear one-dimensional heat equations on a connected domain $(0,1)$ for material with memory on E_N^i, $i = 1,2$, boundary condition

$$x_i(t,0) = x_i(t,1) = 0, \quad i = 1,2 \tag{5.1}$$

and with initial conditions

$$x_i(0,z_i) + \sum_{k=1}^{p} (c_k)_i x_i(t_k, z_i) = x_{0_i}(z_i), \tag{5.2}$$

where $x_{0_i}(z_i) \in E_N^i$,

$$\sum_{k=1}^{p} (c_k)_i x_i(t_k, z_i) = g_i(x_i), \quad i = 1,2. \tag{5.3}$$

Let $x_i(t,z_i)$, $i = 1,2$ be the internal energy and

$$f_i(t, x_i(t,z_i)), \int_0^t q_i(t,s,x_i(s,z_i)) ds = \tilde{2} t x_i(t,z_i)^2 + \int_0^t (t-s) x_i(s,z_i) ds, \quad i = 1,2 \tag{5.4}$$

be the external heat with memory.

$$\Delta x_i(t_k, z_i) = x_i\left(t_k^+, z_i\right) - x_i\left(t_k^-, z_i\right) \tag{5.5}$$

is impulsive effect at $t = t_k (k = 1, 2, \ldots, m)$.

Let

$$
\begin{aligned}
A &= (A_1, A_2) = \left(\widetilde{2}\frac{\partial^2}{\partial z_1^2}, \widetilde{2}\frac{\partial^2}{\partial z_2^2}\right), \\
f&\left(t, x(t), \int_0^t q(t, s, x(s))ds\right) \\
&= \left(f_1\left(t, x_1(t), \int_0^t q_1(t, s, x_1(s))ds\right), f_2\left(t, x_2(t), \int_0^t q_2(t, s, x_2(s))ds\right)\right) \\
&= \left(\widetilde{2}tx_1(t, z_1)^2 + \int_0^t (t-s)x_1(s, z_1)ds, \widetilde{2}tx_2(t, z_2)^2 + \int_0^t (t-s)x_2(s, z_2)ds\right), \\
g(x) &= (g_1(x_1), g_2(x_2)) = \left(\sum_{k=1}^{p}(c_k)_1 x_1(t_k, z_1), \sum_{k=1}^{p}(c_k)_2 x_2(t_k, z_2)\right), \\
x(0) + g(x) &= (x_1(0) + g_1(x), x_2(0) + g_2(x)), \quad x_0 = (x_{0_1}, x_{0_2}) = \left(\widetilde{0}, \widetilde{0}\right), \\
\Delta x(t_k) &= (\Delta x_1(t_k), \Delta x_2(t_k)), \quad t \neq t_k,\ k = 1, 2, \ldots, m, \\
I_k(x(t_k)) &= (I_k(x_1(t_k)), I_k(x_2(t_k))) \\
&= \left(x_1\left(t_k^+\right) - x_1\left(t_k^-\right), x_2\left(t_k^+\right) - x_2\left(t_k^-\right)\right) \\
&= \left(x_1\left(t_k^+, z_1\right) - x_1\left(t_k^-, z_1\right), x_2\left(t_k^+, z_2\right) - x_2\left(t_k^-, z_2\right)\right), \\
&\qquad t \neq t_k,\ k = 1, 2, \ldots, m, \\
G(t-s) &= \left(e^{-(t-s)}, e^{-(t-s)}\right)
\end{aligned}
\tag{5.6}
$$

then the balance equations become

$$
\begin{aligned}
\frac{dx(t)}{dt} &= A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \\
&\quad + f\left(t, x(t), \int_0^t q(t, s, x(s))ds\right) + u(t) \quad \text{on } \left(E_N^i\right)^2, \\
x(0) + g(x) &= x_0 \in \left(E_N^i\right)^2, \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t \neq t_k,\ k = 1, 2, \ldots, m,\ i = 1, 2.
\end{aligned}
\tag{5.7}
$$

The α-level sets of fuzzy numbers are the following

$[\widetilde{0}]^\alpha = [\alpha-1, 1-\alpha]$, $[\widetilde{2}]^\alpha = [\alpha+1, 3-\alpha]$ for all $\alpha \in [0,1]$. Then α-level set of $f(t, x(t), \int_0^t q(t,s,x(s))ds)$ is

$$\begin{aligned}
&\left[f\left(t, x(t), \int_0^t q(t,s,x(s))ds\right)\right]^\alpha \\
&\quad = \left[\widetilde{2}tx_1(t)^2 + \int_0^t (t-s)x_1(s)ds\right]^\alpha \times \left[\widetilde{2}tx_2(t)^2 + \int_0^t (t-s)x_2(s)ds\right]^\alpha \\
&\quad = \left(\left[\widetilde{2}\right]^\alpha \cdot t\left[x_1(t)^2\right]^\alpha + \int_0^t (t-s)[x_1(s)]^\alpha ds\right) \\
&\qquad \times \left(\left[\widetilde{2}\right]^\alpha \cdot t\left[x_2(t)^2\right]^\alpha + \int_0^t (t-s)[x_2(s)]^\alpha ds\right) \\
&\quad = \left([\alpha+1, 3-\alpha] \cdot t\left[\left(x_{1l}^\alpha(t)\right)^2, \left(x_{1r}^\alpha(t)\right)^2\right] + \int_0^t (t-s)\left[\left(x_{1l}^\alpha(s)\right), \left(x_{1r}^\alpha(s)\right)\right]ds\right) \\
&\qquad \times \left([\alpha+1, 3-\alpha] \cdot t\left[\left(x_{2l}^\alpha(t)\right)^2, \left(x_{2r}^\alpha(t)\right)^2\right] + \int_0^t (t-s)\left[\left(x_{2l}^\alpha(s)\right), \left(x_{2r}^\alpha(s)\right)\right]ds\right) \\
&\quad = \left[(\alpha+1)t\left(x_{1l}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{1l}^\alpha(s)ds, (3-\alpha)t\left(x_{1r}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{1r}^\alpha(s)ds\right] \\
&\qquad \times \left[(\alpha+1)t\left(x_{2l}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{2l}^\alpha(s)ds, (3-\alpha)t\left(x_{2r}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{2r}^\alpha(s)ds\right].
\end{aligned} \tag{5.8}$$

Further, we have

$$\begin{aligned}
&d_L\left(\left[f\left(t, x(t), \int_0^t q(t,s,x(s))ds\right)\right]^\alpha, \left[f\left(t, y(t), \int_0^t q(t,s,y(s))ds\right)\right]^\alpha\right) \\
&\quad = d_L\left(\left[(\alpha+1)t\left(x_{il}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{il}^\alpha(s)ds, (3-\alpha)t\left(x_{ir}^\alpha(t)\right)^2 + \int_0^t (t-s)x_{ir}^\alpha(s)ds\right],\right. \\
&\qquad\qquad \left.\left[(\alpha+1)t\left(y_{il}^\alpha(t)\right)^2 + \int_0^t (t-s)y_{il}^\alpha(s)ds, (3-\alpha)t\left(y_{ir}^\alpha(t)\right)^2 + \int_0^t (t-s)y_{ir}^\alpha(s)ds\right]\right) \\
&\quad = t\max_{1\le i\le 2}\left\{(\alpha+1)\left|\left(x_{il}^\alpha(t)\right)^2 - \left(y_{il}^\alpha(t)\right)^2\right|, (3-\alpha)\left|\left(x_{ir}^\alpha(t)\right)^2 - \left(y_{ir}^\alpha(t)\right)^2\right|\right\} \\
&\qquad + \int_0^t (t-s)\max_{1\le i\le 2}\left\{\left|x_{il}^\alpha(s) - y_{il}^\alpha(s)\right|, \left|x_{ir}^\alpha(s) - y_{ir}^\alpha(s)\right|\right\}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq T(3-\alpha)\max_{1\leq i\leq 2}\left\{\left|x_{il}^{\alpha}(t)-y_{il}^{\alpha}(t)\right|\left|x_{il}^{\alpha}(t)+y_{il}^{\alpha}(t)\right|,\left|x_{ir}^{\alpha}(t)-y_{ir}^{\alpha}(t)\right|\left|x_{ir}^{\alpha}(t)+y_{ir}^{\alpha}(t)\right|\right\}\\
&\quad+\frac{T^2}{2}\max_{1\leq i\leq 2}\left\{\left|x_{il}^{\alpha}(t)-y_{il}^{\alpha}(t)\right|,\left|x_{ir}^{\alpha}(t)-y_{ir}^{\alpha}(t)\right|\right\}\\
&\leq 3T\left|x_{ir}^{\alpha}(t)+y_{ir}^{\alpha}(t)\right|d_L\left([x(t)]^{\alpha},[y(t)]^{\alpha}\right)+\frac{T^2}{2}d_L\left([x(t)]^{\alpha},[y(t)]^{\alpha}\right)\\
&=k_1d_L\left([x(t)]^{\alpha},[y(t)]^{\alpha}\right)+k_2d_L\left([x(t)]^{\alpha},[y(t)]^{\alpha}\right),\\
d_L\left([g(x(\cdot))]^{\alpha},[g(y(\cdot))]^{\alpha}\right)&=d_L\left(\left[\sum_{k=1}^{p}c_k(x(t_k))\right]^{\alpha},\left[\sum_{k=1}^{p}c_k(y(t_k))\right]^{\alpha}\right)\\
&=\max_{1\leq i\leq 2}\left\{\left|\sum_{k=1}^{p}(c_k)_i\left(x_{il}^{\alpha}(t_k)\right)-\sum_{k=1}^{p}(c_k)_i\left(y_{il}^{\alpha}(t_k)\right)\right|,\left|\sum_{k=1}^{p}(c_k)_i\left(x_{ir}^{\alpha}(t_k)\right)-\sum_{k=1}^{p}(c_k)_i\left(y_{ir}^{\alpha}(t_k)\right)\right|\right\}\\
&\leq\left|\sum_{k=1}^{p}c_k\right|\max_{1\leq i\leq 2}\left\{\left|x_{il}^{\alpha}(t_k)-y_{il}^{\alpha}(t_k)\right|,\left|x_{ir}^{\alpha}(t_k)-y_{ir}^{\alpha}(t_k)\right|\right\}=\left|\sum_{k=1}^{p}c_k\right|d_L\left([x(t_k)]^{\alpha},[y(t_k)]^{\alpha}\right)\\
&\leq\left|\sum_{k=1}^{p}c_k\right|\max_{k}d_L\left([x(t_k)]^{\alpha},[y(t_k)]^{\alpha}\right)=hd_L\left([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}\right),
\end{aligned}
\tag{5.9}
$$

where k_1, k_2, and h satisfy inequalities (3.7), (3.8), and (3.9), respectively. Choose T such that $T<(1-c(h+d))/c$. Then all conditions stated in Theorem 3.3 are satisfied, so the problem (5.7) has a unique fuzzy solution.

Let target set be $x^1=(x_1^1,x_2^1)=(\widetilde{2},\widetilde{3})$. The α-level set of fuzzy numbersis $\widetilde{3}[\widetilde{3}]^{\alpha}=[\alpha+2,4-\alpha]$.

From the definition of fuzzy solution,

$$
\begin{aligned}
x_{il}^{\alpha}(t)&=S_{il}^{\alpha}(t)\left((x_0)_{il}^{\alpha}-\sum_{k=1}^{p}(c_k)_i\left(x_{il}^{\alpha}(t_k)\right)\right)\\
&\quad+\int_0^t S_{il}^{\alpha}(t-s)\left((\alpha+1)s\left(x_{il}^{\alpha}(s)\right)^2+\int_0^s(s-\tau)x_{il}^{\alpha}d\tau\right)ds\\
&\quad+\int_0^t S_{il}^{\alpha}(t-s)u_{il}^{\alpha}(s)ds+\sum_{0<t_k<t}S_{il}^{\alpha}(t-t_k)I_{kil}^{\alpha}\left(x(t_k^-)\right),\\
x_{ir}^{\alpha}(t)&=S_{ir}^{\alpha}(t)\left((x_0)_{ir}^{\alpha}-\sum_{k=1}^{p}(c_k)_i\left(x_{ir}^{\alpha}(t_k)\right)\right)\\
&\quad+\int_0^t S_{ir}^{\alpha}(t-s)\left((3-\alpha)s\left(x_{ir}^{\alpha}(s)\right)^2+\int_0^s x_{ir}^{\alpha}d\tau\right)ds\\
&\quad+\int_0^t S_{ir}^{\alpha}(t-s)u_{ir}^{\alpha}(s)ds+\sum_{0<t_k<t}S_{ir}^{\alpha}(t-t_k)I_{kir}^{\alpha}\left(x(t_k^-)\right),
\end{aligned}
\tag{5.10}
$$

where $i=1,2$.

Thus the α-levels of $u(s)$

$$
\begin{aligned}
u_{1l}^{\alpha}(s) = \left(\widetilde{\beta}_{1l}^{\alpha}\right)^{-1}\Bigg(&(\alpha+1) - \sum_{k=1}^{p}(c_k)_1\left(x_{il}^{\alpha}(t_k)\right) \\
&- \Bigg[S_{1l}^{\alpha}(T)\Bigg((\alpha-1) - \sum_{k=1}^{p}(c_k)_1\left(x_{1l}^{\alpha}(t_k)\right)\Bigg) \\
&\quad - \int_0^T (\alpha+1) S_{1l}^{\alpha}(T-s)\Bigg(s\left(x_{1l}^{\alpha}(s)\right)^2 + \int_0^s (s-\tau)x_{1l}^{\alpha}(\tau)d\tau\Bigg)ds\Bigg] \\
&- \sum_{0<t_k<T} S_{1l}^{\alpha}(T-t_k) I_{k1l}^{\alpha}\left(x_{1l}^{\alpha}(t_k^-)\right)\Bigg), \\
u_{1r}^{\alpha}(s) = \left(\widetilde{\beta}_{1r}^{\alpha}\right)^{-1}\Bigg(&(3-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{ir}^{\alpha}(t_k)\right) \\
&- \Bigg[S_{1r}^{\alpha}(T)\Bigg((1-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^{\alpha}(t_k)\right)\Bigg) \\
&\quad - \int_0^T (3-\alpha) S_{1r}^{\alpha}(T-s)\Bigg(s\left(x_{1r}^{\alpha}(s)\right)^2 + \int_0^s (s-\tau)x_{1r}^{\alpha}(\tau)d\tau\Bigg)ds\Bigg] \\
&- \sum_{0<t_k<T} S_{1r}^{\alpha}(T-t_k) I_{k1r}^{\alpha}\left(x_{1r}^{\alpha}(t_k^-)\right)\Bigg), \\
u_{2l}^{\alpha}(s) = \left(\widetilde{\beta}_{2l}^{\alpha}\right)^{-1}\Bigg(&(\alpha+2) - \sum_{k=1}^{p}(c_k)_2\left(x_{il}^{\alpha}(t_k)\right) \\
&- \Bigg[S_{2l}^{\alpha}(T)\Bigg((\alpha-1) - \sum_{k=1}^{p}(c_k)_2\left(x_{2l}^{\alpha}(t_k)\right)\Bigg) \\
&\quad - \int_0^T (\alpha+1) S_{2l}^{\alpha}(T-s)\Bigg(s\left(x_{2l}^{\alpha}s\right)^2 + \int_0^s (s-\tau)x_{2l}^{\alpha}(\tau)d\tau\Bigg)ds\Bigg] \\
&- \sum_{0<t_k<T} S_{2l}^{\alpha}(T-t_k) I_{k2l}^{\alpha}\left(x_{2l}^{\alpha}(t_k^-)\right)\Bigg), \\
u_{2r}^{\alpha}(s) = \left(\widetilde{\beta}_{2r}^{\alpha}\right)^{-1}\Bigg(&(4-\alpha) - \sum_{k=1}^{p}(c_k)_2\left(x_{ir}^{\alpha}(t_k)\right) \\
&- \Bigg[S_{2r}^{\alpha}(T)\Bigg((1-\alpha) - \sum_{k=1}^{p}(c_k)_2\left(x_{2r}^{\alpha}(t_k)\right)\Bigg) \\
&\quad - \int_0^T (3-\alpha) S_{2r}^{\alpha}(T-s)\Bigg(s\left(x_{2r}^{\alpha}(s)\right)^2 + \int_0^s (s-\tau)x_{2r}^{\alpha}(\tau)d\tau\Bigg)ds\Bigg] \\
&- \sum_{0<t_k<T} S_{2r}^{\alpha}(T-t_k) I_{k2r}^{\alpha}\left(x_{2r}^{\alpha}(t_k^-)\right)\Bigg).
\end{aligned} \tag{5.11}
$$

Then α-level of $x(T) = (x_1(T), x_2(T))$ is

$$
\begin{aligned}
[x_1(T)]^\alpha &= \left[x_{1l}^\alpha(T), x_{1r}^\alpha(T)\right] \\
&= \Bigg[S_{1l}^\alpha(T)\Bigg((\alpha-1) - \sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right)\Bigg) \\
&\quad + \int_0^T(\alpha+1)S_{1l}^\alpha(T-s)\left(s\left(x_{1l}^\alpha(s)\right)^2 + \int_0^s(s-\tau)x_{1l}^\alpha(\tau)d\tau\right)ds \\
&\quad + \widetilde{\beta}_{1l}^\alpha\left(\widetilde{\beta}_{1l}^\alpha\right)^{-1}\Bigg((\alpha+1) - \sum_{k=1}^{p}(c_k)_1\left(x_{il}^\alpha(t_k)\right) \\
&\qquad - \Bigg\{S_{1l}^\alpha(T)\Bigg((\alpha-1) - \sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right)\Bigg) \\
&\qquad + \int_0^T(\alpha+1)S_{1l}^\alpha(T-s)\left(s\left(x_{1l}^\alpha(s)\right)^2 + \int_0^s(s-\tau)x_{1l}^\alpha(\tau)d\tau\right)ds \\
&\qquad + \sum_{0<t_k<T}S_{1l}^\alpha(T-t_k)I_{k1l}^\alpha\left(x_{1l}^\alpha(t_k^-)\right)\Bigg\}\Bigg) \\
&\quad + \sum_{0<t_k<T}S_{1l}^\alpha(T-t_k)I_{k1l}^\alpha\left(x_{1l}^\alpha(t_k^-)\right), S_{1r}^\alpha(T)\Bigg((1-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\Bigg) \\
&\quad + \int_0^T(3-\alpha)S_{1r}^\alpha(T-s)\left(s\left(x_{1r}^\alpha(s)\right)^2 + \int_0^s(s-\tau)x_{1r}^\alpha(\tau)d\tau\right)ds \\
&\quad + \widetilde{\beta}_{1r}^\alpha\left(\widetilde{\beta}_{1r}^\alpha\right)^{-1}\Bigg((3-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right) \\
&\qquad - \Bigg\{S_{1r}^\alpha(T)\Bigg((1-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\Bigg) \\
&\qquad + \int_0^T(3-\alpha)S_{1r}^\alpha(T-s)\left(s\left(x_{1r}^\alpha(s)\right)^2 + \int_0^s(s-\tau)x_{1r}^\alpha(\tau)d\tau\right)ds \\
&\qquad + \sum_{0<t_k<T}S_{1r}^\alpha(T-t_k)I_{k1r}^\alpha\left(x_{1r}^\alpha(t_k^-)\right)\Bigg\}\Bigg) \\
&\quad + \sum_{0<t_k<T}S_{1r}^\alpha(T-t_k)I_{k1r}^\alpha\left(x_{1r}^\alpha(t_k^-)\right)\Bigg] \\
&= \left[(\alpha+1) - \sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right), (3-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\right] \\
&= \left[\widetilde{2} - \sum_{k=1}^{p}(c_k)_1(x_1(t_k))\right]^\alpha.
\end{aligned}
\tag{5.12}
$$

Similarly,

$$[x_2(T)]^\alpha = \left[x_{2l}^\alpha(T), x_{2r}^\alpha(T)\right] = \left[\widetilde{3} - \sum_{k=1}^{p}(c_k)_2(x_2(t_k))\right]^\alpha. \tag{5.13}$$

Hence

$$x(T) = (x_1(T), x_2(T)) = \left(\widetilde{2} - \sum_{k=1}^{p}(c_k)_1(x_1(t_k)), \widetilde{3} - \sum_{k=1}^{p}(c_k)_2(x_2(t_k))\right) = x^1 - g(x). \tag{5.14}$$

Then all the conditions stated in Theorem 4.2 are satisfied, so the system (5.7) is nonlocal controllable on $[0, T]$.

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