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Review Article

Recent Developments in Instantons in Noncommutative $\mathbb{R}^4$

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We review recent developments in noncommutative deformations of instantons in $\mathbb{R}^4$. In the operator formalism, we study how to make noncommutative instantons by using the ADHM method, and we review the relation between topological charges and noncommutativity. In the ADHM methods, there exist instantons whose commutative limits are singular. We review smooth noncommutative deformations of instantons, spinor zero-modes, the Green's functions, and the ADHM constructions from commutative ones that have no singularities. It is found that the instanton charges of these noncommutative instanton solutions coincide with the instanton charges of commutative instantons before noncommutative deformation. These smooth deformations are the latest developments in noncommutative gauge theories, and we can extend the procedure to other types of solitons. As an example, vortex deformations are studied.

1. Introduction

Instantons in commutative space are one of the most important objects for nonperturbative analysis. We can overview them for example in [1] from the physicist's view points or in [2] from mathematical view points. See for example [3] for recent developments of them. Noncommutative (NC for short) instantons were discovered by Nekrasov and Schwarz [4]. After [4], NC instantons have been investigated by many physicists and mathematicians. However, many enigmas are left until now. Let us focus into instantons of $U(N)$ gauge theories in NC $\mathbb{R}^4$ and understand what is clarified and what is unknown.

Instanton connections in the 4-dim Yang-Mills theory are defined by

$$F^+ = \frac{1}{2}(1 + *)F = 0, \tag{1.1}$$

where F is a curvature 2-form and $*$ is the Hodge star operator. This condition says that curvature is anti-self-dual. In this paper, we call anti-self-dual connections instantons. The choice of anti-self-dual connection or self-dual connection to define instantons is not important to mathematics but just a habit.

NC instanton solutions were discovered by Nekrasov and Schwartz by using the ADHM method [4]. (See also [5] for the original ADHM method.) The ADHM construction which generates the instanton $U(N)$ gauge field requires a pair of the two complex vector spaces $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$. Here $-k$ is an integer called instanton number. Introduce $B_1, B_2 \in \mathrm{Hom}(V,V)$, $I \in \mathrm{Hom}(W,V)$, and $J \in \mathrm{Hom}(V,W)$ which are called ADHM data that satisfy the ADHM equations that we will see soon. In other words, B_1 and B_2 are complex-valued $k \times k$ matrices, and I and $J^\dagger$ are complex-valued $k \times N$ matrices that satisfy (2.13) and (2.14) in Section 2.2. Using these ADHM data, we can construct instanton [6–17]. We call it NC ADHM instanton in the following. The NC ADHM construction is a strong method. A lot of instanton solutions are constructed by using the NC ADHM construction [6–17]. The NC ADHM method also clarifies some important features, for example, topological charge, index theorems, Green's functions, and so on. As a characteristic feature of NC ADHM construction, the NC ADHM instantons can be instantons that have singularities in the commutative limit. On the other hand, we can study NC instantons from a point of view of deformation quantization. Recently, NC instanton that is smoothly deformed from commutative instanton is constructed [18]. The method in [18] makes success in analysis for topological charges, index theorems, and the method derives the ADHM equations from NC instanton and proves a one-to-one correspondence between the ADHM data and NC instantons [19]. We review them in this article.

This paper is organized as follows. In Section 2, we review the NC ADHM instanton and their natures (For example, we investigate topological charges of instantons. We distinguish the terms "instanton number" from "instanton charge". In this article, we define the instanton number by the dimension of some vector space V; on the other hand, the instanton charge is defined by integral of the 2nd Chern class. We will soon see more details.) . In Section 3, we construct an NC instanton solution which is a smooth deformation of the commutative instanton [18]. We study the NC instanton charge, an index theorem, and the correspondence relation with the ADHM construction for the smooth NC deformations of instantons [19]. In Section 4, we apply the method in Section 3 to a gauge theory in $\mathbb{R}^2$, and we make NC vortex solutions which are smooth deformations of commutative vortex solutions [20, 21].

2. Noncommutative ADHM Instantons

In this section, we review the NC ADHM instanton that may have singularities in commutative limit. An NC $U(1)$ instanton is a typical example that has a singularity in commutative limit.

2.1. Notations for the Fock Space Formalism

Let us consider coordinate operators x^μ ($\mu = 1,2,3,4$) satisfying $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where θ is a skew symmetric real valued matrix and we call $\theta^{\mu\nu}$ NC parameter. We set the noncommutativity of the space to the self-dual case of $\theta^{12} = -\zeta_1$, $\theta^{34} = -\zeta_2$, and the other

$\theta^{\mu\nu} = 0$ for convenience. By transformations of coordinates x^μ, the NC parameters are possible to be put in this form in general. Here we introduce complex coordinate operators

$$z_1 = \frac{1}{\sqrt{2}}\left(x^1 + ix^2\right), \qquad z_2 = \frac{1}{\sqrt{2}}\left(x^3 + ix^4\right). \tag{2.1}$$

Then the commutation relations become

$$[z_1, \overline{z}_1] = -\zeta_1, \quad [z_2, \overline{z}_2] = -\zeta_2, \quad \text{others are zero.} \tag{2.2}$$

We define creation and annihilation operators by

$$c_\alpha^\dagger = \frac{z_\alpha}{\sqrt{\zeta_\alpha}}, \quad c_\alpha = \frac{\overline{z}_\alpha}{\sqrt{\zeta_\alpha}}, \quad (\alpha = 1, 2); \tag{2.3}$$

then they satisfy

$$\left[c_\alpha, c_\alpha^\dagger\right] = 1, \quad [c_\alpha, c_\beta] = \left[c_\alpha^\dagger, c_\beta^\dagger\right] = 0 \quad (\alpha, \beta = 1, 2). \tag{2.4}$$

The Fock space $\mathscr{H}$ on which the creation and annihilation operators (2.4) act is spanned by the Fock state

$$|n_1, n_2\rangle = \frac{\left(c_1^\dagger\right)^{n_1}\left(c_2^\dagger\right)^{n_2}}{\sqrt{n_1! n_2!}}|0, 0\rangle, \tag{2.5}$$

with

$$\begin{aligned} c_1|n_1, n_2\rangle &= \sqrt{n_1}|n_1 - 1, n_2\rangle, \qquad & c_1^\dagger|n_1, n_2\rangle &= \sqrt{n_1 + 1}|n_1 + 1, n_2\rangle, \\ c_2|n_1, n_2\rangle &= \sqrt{n_2}|n_1, n_2 - 1\rangle, \qquad & c_2^\dagger|n_1, n_2\rangle &= \sqrt{n_2 + 1}|n_1, n_2 + 1\rangle, \end{aligned} \tag{2.6}$$

where n_1 and n_2 are the occupation number. The number operators are also defined by

$$\widehat{n}_\alpha = c_\alpha^\dagger c_\alpha, \qquad \widehat{N} = \widehat{n}_1 + \widehat{n}_2, \tag{2.7}$$

which act on the Fock states as

$$\widehat{n}_\alpha|n_1, n_2\rangle = n_\alpha|n_1, n_2\rangle, \qquad \widehat{N}|n_1, n_2\rangle = (n_1 + n_2)|n_1, n_2\rangle. \tag{2.8}$$

In the operator representation, derivatives of a function f are defined by

$$\partial_\alpha f(z) = \left[\widehat{\partial}_\alpha, f(z)\right], \qquad \partial_{\overline{\alpha}} f(z) = \left[\widehat{\partial}_{\overline{\alpha}}, f(z)\right], \tag{2.9}$$

where $\hat{\partial}_\alpha = \overline{z}_\alpha/\zeta_\alpha$ and $\hat{\partial}_{\overline{\alpha}} = -z_\alpha/\zeta_\alpha$ which satisfy $[\hat{\partial}_\alpha, \hat{\partial}_{\overline{\alpha}}] = -1/\zeta_\alpha$. The integral on NC $\mathbb{R}^4$ is defined by the standard trace in the operator representation,

$$\int d^4x = \int d^4z = (2\pi)^2 \zeta_1\zeta_2 \mathrm{Tr}_{\mathscr{H}}. \tag{2.10}$$

Note that $\mathrm{Tr}_{\mathscr{H}}$ represents the trace over the Fock space whereas the trace over the gauge group is denoted by $\mathrm{tr}_{U(N)}$.

2.2. Noncommutative ADHM Instantons

Let us consider the $U(N)$ Yang-Mills theory on NC $\mathbb{R}^4$. Let M be a projective module over the algebra that is generated by the operator x_μ.

In the NC space, the Yang-Mills connection is defined by $D_\mu\psi = -\psi\hat{\partial}_\mu + \hat{D}_\mu\psi$, where ψ is a matter field in fundamental representation type and $\hat{D}_\mu \in \mathrm{End}(M)$ are anti-Hermitian gauge fields [22–24]. The relation between $\hat{D}_\mu$ and usual gauge connection A_μ is $\hat{D}_\mu = -i\theta_{\mu\nu}x^\nu + A_\mu$, where $\theta_{\mu\nu}$ is an inverse matrix of $\theta^{\mu\nu}$. In our notation of the complex coordinates (2.1) and (2.2), the curvature is given as

$$F_{\alpha\overline{\alpha}} = \frac{1}{\zeta_\alpha} + \left[\hat{D}_\alpha, \hat{D}_{\overline{\alpha}}\right], \quad F_{\alpha\overline{\beta}} = \left[\hat{D}_\alpha, \hat{D}_{\overline{\beta}}\right] \quad (\alpha \neq \beta). \tag{2.11}$$

Note that there is a constant term originated with the noncommutativity in $F_{\alpha\overline{\alpha}}$. Instanton solutions satisfy the antiself-duality condition $F = -*F$. These conditions are rewritten in the complex coordinates as

$$F_{1\overline{1}} = -F_{2\overline{2}}, \qquad F_{12} = F_{\overline{1}\,\overline{2}} = 0. \tag{2.12}$$

In the commutative spaces, instantons are classified by the topological charge $Q = (1/8\pi^2)\int \mathrm{tr}_{U(N)} F \wedge F$, which is always integer $-k$ and coincide with the opposite sign of dimension of the vector space V in the ADHM methods, and $-k$ is called instanton number. In the NC spaces, the same statement is conjectured, and some partial proofs are given. (See Section 2.4 and see also [18, 25–30].)

In the commutative spaces, the ADHM construction is proposed by Atiyah et al. [5] to construct instantons. Nekrasov and Schwarz first extended this method to NC cases [4]. Here we review briefly on the ADHM construction of $U(N)$ instantons [22, 23].

The first step of ADHM construction on NC $\mathbb{R}^4$ is looking for $B_1, B_2 \in \mathrm{End}(\mathbb{C}^k)$, $I \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$, and $J \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ which satisfy the deformed ADHM equations

$$\left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = \zeta_1 + \zeta_2, \tag{2.13}$$

$$[B_1, B_2] + IJ = 0. \tag{2.14}$$

We call $-k$ "instanton number" in this article. In the previous section, we denote V as the vector space $\mathbb{C}^k$. Note that the right-hand side of (2.13) is caused by the noncommutativity of the space $\mathbb{R}^4$. The set of B_1, B_2, I, and J satisfying (2.13) and (2.14) is called ADHM data. Using this ADHM data, we define operator $\mathfrak{D} : \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^n \rightarrow \mathbb{C}^k \oplus \mathbb{C}^k$ by

$$\mathfrak{D}^\dagger = \begin{pmatrix} \tau \\ \sigma^\dagger \end{pmatrix},$$
$$\tau = (B_2 - z_2, B_1 - z_1, I) = \left(B_2 - \sqrt{\zeta_2} c_2^\dagger, B_1 - \sqrt{\zeta_1} c_1^\dagger, I\right), \tag{2.15}$$
$$\sigma^\dagger = \left(-B_1^\dagger + \overline{z}_1, B_2^\dagger - \overline{z}_2, J^\dagger\right) = \left(-B_1^\dagger + \sqrt{\zeta_1} c_1, B_2^\dagger - \sqrt{\zeta_2} c_2, J^\dagger\right).$$

The ADHM equations (2.13) and (2.14) are replaced by

$$\tau\tau^\dagger = \sigma^\dagger\sigma \equiv \Box, \qquad \tau\sigma = 0. \tag{2.16}$$

Let us denote by $\Psi : \mathbb{C}^n \rightarrow \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^n$ the solution to the following equation:

$$\mathfrak{D}^\dagger \Psi^a = 0 \quad (a = 1, \ldots, n), \quad \Psi^{\dagger a} \Psi^b = \delta^{ab}. \tag{2.17}$$

Theorem 2.1. *Let Ψ^a be orthonormal zero-modes defined in* (2.17)*. Then NC $U(N)$ instanton A_μ with instanton number $-k$ is obtained by*

$$A_\mu = \Psi^\dagger \partial_\mu \Psi = -i\Psi^\dagger \theta_{\mu\nu} [x^\nu, \Psi]. \tag{2.18}$$

Here $\theta_{\mu\nu}$ is inverse of $\theta^{\mu\nu}$, that is, $\theta_{\mu\nu}\theta^{\nu\rho} = \delta_\mu^\rho$.

Proof. The curvature two-form determined by this connection is given as follows.

$$\begin{aligned} F &= dA + A \wedge A \\ &= d\left(\Psi^\dagger d\Psi\right) + \left(\Psi^\dagger d\Psi\right) \wedge \left(\Psi^\dagger d\Psi\right) \\ &= d\Psi^\dagger \wedge d\Psi - (d\Psi)\Psi\Psi^\dagger \wedge d\Psi \\ &= d\Psi^\dagger \left(1 - \Psi\Psi^\dagger\right) \wedge d\Psi. \end{aligned} \tag{2.19}$$

Here we use $d\Psi^\dagger\Psi + \Psi^\dagger d\Psi = 0$ that follows from the differentiating of (2.17). Note that

$$\Psi\Psi^\dagger = I - \mathfrak{D}\frac{1}{\mathfrak{D}^\dagger\mathfrak{D}}\mathfrak{D}^\dagger, \tag{2.20}$$

since

$$
\begin{aligned}
I &= (\mathfrak{D}\ \ \Psi)(\mathfrak{D}\ \ \Psi)^{-1}\left((\mathfrak{D}\ \ \Psi)^{\dagger}\right)^{-1}(\mathfrak{D}\ \ \Psi)^{\dagger} \\
&= (\mathfrak{D}\ \ \Psi)\begin{pmatrix} \mathfrak{D}^{\dagger}\mathfrak{D} & 0 \\ 0 & 1 \end{pmatrix}^{-1}(\mathfrak{D}\ \ \Psi)^{\dagger} = \mathfrak{D}\frac{1}{\mathfrak{D}^{\dagger}\mathfrak{D}}\mathfrak{D}^{\dagger} + \Psi\Psi^{\dagger}.
\end{aligned} \tag{2.21}
$$

From (2.19) and (2.20),

$$
F = d\Psi^{\dagger}\left(\mathfrak{D}\frac{1}{\mathfrak{D}^{\dagger}\mathfrak{D}}\mathfrak{D}^{\dagger}\right)\wedge d\Psi = \Psi^{\dagger}(d\mathfrak{D})\wedge\frac{1}{\mathfrak{D}^{\dagger}\mathfrak{D}}\left(d\mathfrak{D}^{\dagger}\right)\Psi, \tag{2.22}
$$

where we use $(d\mathfrak{D}^{\dagger})\Psi + \mathfrak{D}^{\dagger}d\Psi = 0$ that follows from differentiating $\mathfrak{D}^{\dagger}\Psi = 0$. If the coordinates (x_1, x_2, x_3, x_4) are renamed (x_2, x_1, x_4, x_3) for convenience, we obtain

$$
\partial_{\mu}\mathfrak{D}^{\dagger} = \frac{1}{\sqrt{2}}(-\overline{\sigma}_{\mu}\ \ 0), \qquad \partial_{\mu}\mathfrak{D} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma_{\mu} \\ 0 \end{pmatrix}. \tag{2.23}
$$

Here, we define σ_{μ} and $\overline{\sigma}_{\mu}$ by

$$
\begin{aligned}
(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &:= (-i\tau_1, -i\tau_2, -i\tau_3, 1_{2\times 2}), \\
(\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3, \overline{\sigma}_4) &:= (i\tau_1, i\tau_2, i\tau_3, 1_{2\times 2}),
\end{aligned} \tag{2.24}
$$

where τ_i are the Pauli matrices and $1_{2\times 2}$ is an identity matrix of degree 2. Note that $\mathfrak{D}^{\dagger}\mathfrak{D} = \left(\begin{smallmatrix} \square & 0 \\ 0 & \square \end{smallmatrix}\right)$ owing to (2.16), and $\mathfrak{D}^{\dagger}\mathfrak{D}$ and its inverse commute with σ_{μ}. Then we find (2.22) is in proportion to

$$
\sigma_{\mu}\overline{\sigma}_{\nu}dx^{\mu}\wedge dx^{\nu}. \tag{2.25}
$$

$\sigma_{\mu}\overline{\sigma}_{\nu} - \sigma_{\nu}\overline{\sigma}_{\mu}$ is a component of anti-self-dual two-form, that is easily checked by direct calculations. This fact and (2.22) show that the curvature F is anti-self-dual and the connections given by (2.18) are instantons. $\square$

With the complex coordinate z_{α}, NC instanton connections are given by

$$
\widehat{D}_{\alpha} = \frac{1}{\zeta_{\alpha}}\Psi^{\dagger}\overline{z}_{\alpha}\Psi, \qquad \widehat{D}_{\overline{\alpha}} = -\frac{1}{\zeta_{\alpha}}\Psi^{\dagger}z_{\alpha}\Psi. \tag{2.26}
$$

One of the most important feature to understand the origin of the instanton charges is existence of zero-modes of $\Psi\Psi^{\dagger}$.

Theorem 2.2 (Zero-mode of $\Psi\Psi^{\dagger}$). *Suppose that Ψ and $\Psi^{\dagger}$ are given as above. The vector $|v\rangle \in (\mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^n)\otimes\mathscr{H}$ satisfying*

$$
\Psi\Psi^{\dagger}|v\rangle = \langle v|\Psi\Psi^{\dagger} = 0, \qquad |v\rangle \neq 0 \tag{2.27}
$$

is said to be a zero-mode of $\Psi\Psi^\dagger$. The zero-modes are given by following three types:

$$|v_1\rangle = \begin{pmatrix} \left(-B_1 + \sqrt{\zeta_1}c_1\right)|u\rangle \\ \left(B_2 - \sqrt{\zeta_2}c_2\right)|u\rangle \\ J|u\rangle \end{pmatrix}, \qquad |v_2\rangle = \begin{pmatrix} \left(B_2^\dagger - \sqrt{\zeta_2}c_2^\dagger\right)|u'\rangle \\ \left(B_1^\dagger - \sqrt{\zeta_1}c_1^\dagger\right)|u'\rangle \\ I^\dagger|u'\rangle \end{pmatrix},$$
$$|v_0\rangle = \begin{pmatrix} \left(\exp \sum_\alpha B_\alpha^\dagger c_\alpha^\dagger\right)|0,0\rangle v_0^i \\ \left(\exp \sum_\alpha B_\alpha^\dagger c_\alpha^\dagger\right)|0,0\rangle v_0^i \\ 0 \end{pmatrix}. \tag{2.28}$$

Here $|u\rangle$ ($|u'\rangle$) is some element of $\mathbb{C}^k \otimes \mathcal{H}$ (i.e., $|u\rangle$ is expressed with the coefficients $u_i^{nm} \in \mathbb{C}$ as $|u\rangle = \sum_i \sum_{n,m} u_i^{nm}|n,m\rangle e_i$, where e_i is a base of k-dim vector space). v_0^i is a element of k-dim vector.

The proof is given in [25]. We will see the fact that zero-modes $|v_0\rangle$ play an essential role, in the following subsections.

2.3. $U(1)$ *N.C. ADHM Multi-Instanton*

One of the most characteristic features of NC instantons is found in regularizations of the singularities. In commutative $\mathbb{R}^4$, we cannot construct a nonsingular $U(1)$ instanton. On the other hand, there exist in NC $\mathbb{R}^4$. Let us see how to construct them as typical NC ADHM instantons.

At the beginning, we review the methods in [23]. Let B_1, B_2, I, J be constant matrices satisfying (2.13) and (2.14). We consider $\zeta = \zeta_1 + \zeta_2 > 0$; then we can put $J = 0$ in general by using a symmetry. B_1 and B_2 are $k \times k$ matrices, and I is $k \times 1$ matrices:

$$B_1 = \begin{array}{c} \\ 1 \\ \vdots \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & \cdots & k \end{array} \\ \begin{pmatrix} B_{11}^1 & \cdots & B_{1k}^1 \\ \vdots & \ddots & \vdots \\ B_{k1}^1 & \cdots & B_{kk}^1 \end{pmatrix} \end{array}, \qquad B_2 = \begin{array}{c} \\ 1 \\ \vdots \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & \cdots & k \end{array} \\ \begin{pmatrix} B_{11}^2 & \cdots & B_{1k}^2 \\ \vdots & \ddots & \vdots \\ B_{k1}^2 & \cdots & B_{kk}^2 \end{pmatrix} \end{array},$$
$$I = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \end{array}\!\!\begin{array}{c} 1 \\ \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_k \end{pmatrix} \end{array}, \qquad J = 0. \tag{2.29}$$

We define β_α, $c_\alpha^\dagger$ and c_α by

$$B_\alpha = \sqrt{\zeta_\alpha}\beta_\alpha \quad (\alpha = 1, 2). \tag{2.30}$$

We introduce $\widehat{\Delta}$ as

$$\widehat{\Delta} = \sum_\alpha \zeta_\alpha \left(\beta_\alpha - c_\alpha^\dagger\right)\left(\beta_\alpha^\dagger - c_\alpha\right), \tag{2.31}$$

and we define a projection operator P as a projection onto 0-eigenstates of $\widehat{\Delta}$ by

$$P = I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}|0,0\rangle G^{-1}\langle 0,0|e^{\sum_\alpha \beta_\alpha c_\alpha} I, \tag{2.32}$$

where

$$G = \langle 0,0|e^{\sum_\alpha \beta_\alpha c_\alpha} I I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}|0,0\rangle. \tag{2.33}$$

We define shift operators S and $S^\dagger$ and a operator Λ by

$$\begin{gathered} SS^\dagger = 1, \qquad S^\dagger S = 1 - P, \\ \Lambda = 1 + I^\dagger \frac{1}{\widehat{\Delta}} I. \end{gathered} \tag{2.34}$$

Theorem 2.3 (Nekrasov). *$U(1)$ instantons are given by*

$$D_\alpha = -\sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{-1/2} c_\alpha \Lambda^{1/2} S^\dagger, \qquad D_{\overline{\alpha}} = \sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{1/2} c_\alpha^\dagger \Lambda^{-1/2} S^\dagger. \tag{2.35}$$

Proof. At first, we check that the inverse of $\widehat{\Delta}$ in (2.34) is well defined. $\widehat{\Delta}$ has k zero-modes:

$$e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}|0,0\rangle \otimes e_i \quad (i = 1, \ldots, k) \tag{2.36}$$

which satisfy $\widehat{\Delta} e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}|0,0\rangle \otimes e_i = 0$. Here $e_i = (\delta_{1i}, \delta_{2,i}, \ldots, \delta_{ki})^t$ is a base of V. Note that $S \cdots S^\dagger = S(1-P)\cdots S^\dagger$. This implies that S removes the zero-modes, and Hilbert spaces $\mathscr{H}$ is projected on to a space that does not include the zero-modes. Therefore, the inverse of Λ exists if it is sandwiched between S and $S^\dagger$ and (2.35) is well defined.

Next, we check that (2.35) is an instanton. Let us see how the equation $\mathfrak{D}^\dagger \Psi = 0$ is solved under orthonormalization condition $\Psi^\dagger \Psi = 1$. $\psi_\pm$ and ξ are introduced as

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix}, \quad \psi_\pm \in V \otimes \mathscr{H}, \ \xi \in \mathscr{H}. \tag{2.37}$$

The orthonormalization condition is expressed as

$$\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \xi^\dagger \xi = 1. \tag{2.38}$$

We put anzats for the solution by

$$\psi_+ = -\sqrt{\zeta_2}\left(\beta_2^\dagger - c_2\right)v, \qquad \psi_- = \sqrt{\zeta_1}\left(\beta_1^\dagger - c_1\right)v, \tag{2.39}$$

and substitute them into $\mathfrak{D}^\dagger \Psi = 0$; then we get

$$\widehat{\Delta} v + I\xi = 0. \tag{2.40}$$

The orthonormalization condition is rewritten as

$$v^\dagger \widehat{\Delta} v + \xi^\dagger \xi = 1. \tag{2.41}$$

If there exist the inverse of $\widehat{\Delta}$,

$$v = -\frac{1}{\widehat{\Delta}} I\xi. \tag{2.42}$$

0-eigenstates of $\widehat{\Delta}$ are (2.36) and we define the projection operator to project out the 0-eigenstates by

$$P = \mathrm{I}^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} |0,0\rangle G^{-1} \langle 0,0| e^{\sum_\alpha \beta_\alpha c_\alpha} I. \tag{2.43}$$

Shift operators $S, S^\dagger$ satisfying

$$SS^\dagger = 1, \qquad S^\dagger S = 1 - P \tag{2.44}$$

are determined by the definition of P. Then the inverse of $\widehat{\Delta}$ is well defined at the left side of $S^\dagger$ or the right side of S.

Using the orthonormalization condition, we obtain

$$\xi = \Lambda^{-1/2} S^\dagger, \qquad \Lambda = 1 + I^\dagger \frac{1}{\widehat{\Delta}} I. \tag{2.45}$$

Through these processes, Ψ is determined by the ADHM data, and after substituting this Ψ into (2.26) we obtain the instantons.

$$\begin{aligned} D_\alpha &= -\frac{1}{\xi_\alpha}\psi^\dagger \widehat{\overline{z}}_\alpha \psi = -\frac{1}{\xi_\alpha} S\Lambda^{-1/2}\left(I^\dagger \frac{1}{\widehat{\Delta}}\widehat{\Delta}\widehat{\overline{z}}_\alpha \frac{1}{\widehat{\Delta}} I - \widehat{\overline{z}}_\alpha\right)\Lambda^{-1/2}S^\dagger \\ &= -\frac{1}{\sqrt{\xi_\alpha}} S\Lambda^{-1/2} c_\alpha \Lambda^{1/2} S^\dagger. \end{aligned} \tag{2.46}$$

$D_{\overline{\alpha}}$ is given similarly. □

This expression (2.35) is useful, but there exist other issues to get concrete expression of instantons. For example, it is not easy to obtain the explicit expression of $\widehat{\Delta}^{-1}$.

As an example, let us construct an NC $U(1)$ multi-instanton having concrete expressions with the instanton number $-k$ [31, 32], which is made from the ADHM data:

$$\begin{gathered} B_1 = \sum_{l=1}^{k-1}\sqrt{l\zeta}e_l e_{l+1}^\dagger = \sqrt{\zeta}\begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \sqrt{k-2} & 0 \\ 0 & \cdots & & \cdots & 0 & \sqrt{k-1} \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}, \qquad B_2 = 0, \\ I = \sqrt{k\zeta}e_k = \sqrt{\zeta}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{k} \end{pmatrix}, \qquad J = 0, \end{gathered} \tag{2.47}$$

where $\zeta = \zeta_1 + \zeta_2$. It is easy to check that this data satisfies the ADHM equations (2.13) and (2.14), and substituting them into definition of P derives

$$P = \sum_{n_1=0}^{k-1} |n_1, 0\rangle\langle n_1, 0|. \tag{2.48}$$

To construct an instanton, it is necessary to obtain $\widehat{\Delta}$ or Λ. By definition,

$$\begin{gathered} \widehat{\Delta}(k) = \zeta_1\widehat{n}_1 + \zeta_2\widehat{n}_2 + \zeta\sum_{i=1}^{k-1} i e_i e_i^\dagger - \sqrt{\zeta_1\zeta}\sum_{i=1}^{k-1}\sqrt{i}\left\{c_1 e_i e_{i+1}^\dagger + c_1^\dagger e_{i+1}e_i^\dagger\right\}, \\ \Lambda(k) = 1 + \zeta k \widehat{\Delta}_{kk}^{-1}(k). \end{gathered} \tag{2.49}$$

$\widehat{\Delta}$ and Λ depend on k, so we denote them $\widehat{\Delta}(k)$ and $\Lambda(k)$, respectively. $\widehat{\Delta}^{-1}_{kk}(k)$ is (k,k) entry of matrix $\widehat{\Delta}^{-1}(k)$. To obtain $\widehat{\Delta}^{-1}_{kk}(k)$, it is enough to calculate the kth row vector of $\widehat{\Delta}^{-1}(k)$. The kth row vector of $\widehat{\Delta}^{-1}(k)$ is determined by $\widehat{\Delta}^{-1}\widehat{\Delta} = 1$. We denote the kth row vector of $\widehat{\Delta}^{-1}(k)$ by $(u_1, \ldots, u_k)$, that is, $\widehat{\Delta}^{-1}_{ki}(k) = u_i$. Then, we obtain the following recurrence equation from the kth row of $\widehat{\Delta}^{-1}\widehat{\Delta} = 1$

$$\begin{gathered}
u_2 c_1^{\dagger} - \frac{1}{\sqrt{\widetilde{\theta}}} u_1 \left(1 + \widetilde{\theta}\widehat{n}_1 + \left(1 - \widetilde{\theta}\right)\widehat{n}_2\right) = 0, \\
\sqrt{i} u_{i+1} c_1^{\dagger} - \frac{1}{\sqrt{\widetilde{\theta}}} u_i \left(i + \widetilde{\theta}\widehat{n}_1 + \left(1 - \widetilde{\theta}\right)\widehat{n}_2\right) + \sqrt{i-1}\, u_{i-1} c_1 = 0 \quad (1 \leq i \leq k-2),
\end{gathered} \tag{2.50}$$

where $\widetilde{\theta} = \zeta_1/(\zeta_1 + \zeta_2)$. We change variables as

$$u_i = w_{i-1} \frac{c_1^{\dagger(k-i)}}{\sqrt{(i-1)!}}; \tag{2.51}$$

then we can rewrite the above recurrence relation by w_i as

$$\begin{aligned}
& w_1 - \frac{1}{\sqrt{\widetilde{\theta}}} \left(1 + \widetilde{\theta}(\widehat{n}_1 - k + 1) - \left(1 - \widetilde{\theta}\right)\widehat{n}_2\right) w_0 = 0, \\
& w_{i+1} - \left\{ i \left(\frac{1}{\sqrt{\widetilde{\theta}}} + \sqrt{\widetilde{\theta}} \right) + \frac{1}{\sqrt{\widetilde{\theta}}} \left(\widetilde{\theta}(\widehat{n}_1 - k) + \left(1 - \widetilde{\theta}\right)\widehat{n}_2 \right) + \left(\frac{1}{\sqrt{\widetilde{\theta}}} + \sqrt{\widetilde{\theta}} \right) \right\} w_i \\
& \quad + i((i-1)\widehat{n}_1 - k + 2) w_{i-1} = 0, \quad (2 \leq i \leq k-1).
\end{aligned} \tag{2.52}$$

Note that $\widehat{n}_1$ and $\widehat{n}_2$ are commutative to each other, so we can treat them like C-numbers in the following. We introduce an anzats for the generating function $F(t;k)$ by

$$\begin{aligned}
F(t;k) &= e^{f(t)}(1 - at)^{\alpha} = \sum_{i=0}^{\infty} \frac{w_i}{i!} t^i, \\
f(t) &= \int dt \frac{ct}{(1 - at)(1 - bt)} \\
&= \frac{c}{2ab} \left\{ \ln\left(1 - (a+b)t + abt^2\right) + \frac{a+b}{\sqrt{D}} \ln\left| \frac{2abt - (a+b) - \sqrt{D}}{2abt - (a+b) + \sqrt{D}} \right| \right\},
\end{aligned} \tag{2.53}$$

where a, b, c, and α are real parameter determined by the request that w_i satisfy (2.52), and $D = (a-b)^2$. From the differentiation of $F(t;k)$, we obtain

$$(1 - at)(1 - bt) \sum_{i=1}^{\infty} \frac{w_1}{(i-1)!} t^{i-1} = \{-a\alpha + (c + ab\alpha)t\} \sum_{i=0}^{\infty} \frac{w_i}{i!} t^i, \tag{2.54}$$

and we find that w_i satisfy the following relation:

$$w_{i+1} - (i(a+b) - a\alpha)w_i + i(ab(i-1) - c - ab\alpha)w_{i-1} = 0. \tag{2.55}$$

From (2.52) and (2.55), we obtain

$$\begin{aligned} a &= \sqrt{\tilde{\theta}} \quad \text{or} \quad \frac{1}{\sqrt{\tilde{\theta}}}, \qquad b = \frac{1}{a}, \\ \alpha &= -\frac{h\left(\tilde{\theta}, n_1, n_2\right)}{a}, \qquad c = -n_1 + k - 2 + \frac{h\left(\tilde{\theta}, n_1, n_2\right)}{a}, \end{aligned} \tag{2.56}$$

where

$$h\left(\tilde{\theta}, n_1, n_2\right) = \frac{1}{\sqrt{\tilde{\theta}}}\left\{\tilde{\theta}(n_1 - k) + \left(1 - \tilde{\theta}\right)n_2\right\} + \tilde{\theta} + \frac{1}{\sqrt{\tilde{\theta}}}. \tag{2.57}$$

Thus the generating function $F(t;k)$ is determined as an elementary function for each instanton number $-k$. Using this $F(t;k)$, we obtain w_i, and $\Delta_{kk}^{-1}(k)$ is determined as

$$\Delta_{kk}^{-1}(k) = u_k = \left(1 + \sqrt{\zeta_1\zeta}\frac{\sqrt{k-1}}{\sqrt{(k-2)!}}w_{k-2}\hat{n}_1\right)(\zeta_1\hat{n}_1 + \zeta_2\hat{n}_2)^{-1}. \tag{2.58}$$

Using them, G, P, S, and Λ are determined as

$$\begin{aligned} G &= \zeta k! \sum_{i=1}^{k}\left\{i!(k-i)!\tilde{\theta}^{k-i}\right\}^{-1} e_i e_i^{\dagger}, \\ P &= I^{\dagger} e^{\sum_\alpha \beta_\alpha^{\dagger} c_\alpha^{\dagger}} |0,0\rangle G^{-1} \langle 0,0| e^{\sum_\alpha \beta_\alpha c_\alpha} I = \sum_{n_1=0}^{k-1} |n_1, 0\rangle\langle n_1, 0|, \\ S^{\dagger} &= \sum_{n_1=0}^{\infty} |n_1 + k, 0\rangle\langle n_1, 0| + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} |n_1, n_2\rangle\langle n_1, n_2|, \\ \Lambda &= 1 + \zeta k u_k. \end{aligned} \tag{2.59}$$

Finally we obtain instanton gauge fields with instanton number $-k$ as

$$\begin{aligned} D_1 &= \sqrt{\frac{1}{\zeta_1}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |n_1, n_2\rangle\langle n_1 + 1, n_2| d_1(n_1, n_2; k), \\ D_2 &= \sqrt{\frac{1}{\zeta_2}} \left\{\sum_{n_1=0}^{\infty} |n_1, 0\rangle\langle n_1 + k, 1| d_2(n_1, 0; k) + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} |n_1, n_2\rangle\langle n_1, n_2 + 1| d_2(n_1, n_2; k)\right\}, \end{aligned} \tag{2.60}$$

where

$$
d_1(n_1,n_2;k)=\begin{cases}\sqrt{n_1+k+1}\left[\dfrac{\Lambda(n_1+k+1,0)}{\Lambda(n_1+k,0)}\right]^{1/2}, & (n_2=0),\\ \sqrt{n_1+1}\left[\dfrac{\Lambda(n_1+1,n_2)}{\Lambda(n_1,n_2)}\right]^{1/2}, & (n_2\neq 0),\end{cases}
$$

$$
d_2(n_1,n_2;k)=\begin{cases}\left[\dfrac{\Lambda(n_1+k,1)}{\Lambda(n_1+k,0)}\right]^{1/2}, & (n_2=0),\\ \sqrt{n_2+1}\left[\dfrac{\Lambda(n_1,n_2+1)}{\Lambda(n_1,n_2)}\right]^{1/2}, & (n_2\neq 0).\end{cases}
\tag{2.61}
$$

Therefore, we obtain NC multi-instanton solutions expressed completely by elementary functions. This solution is one of the examples of the many kinds of the NC multi-instantons discovered until now [6–17].

2.4. Some Aspects

In this section, we overview some facts and important aspects of NC instantons without detailed derivations.

2.4.1. Instanton Charges and Instanton Numbers

Let us see a rough sketch of how to define instanton charges by using characteristic classes. The instanton charge in commutative space is determined $(1/8\pi^2)\int \mathrm{tr}\, F\wedge\star F$ and coincides with the instanton number defined by the dimension of the vector space V in the ADHM construction. A naive definition of the instanton charges in NC $\mathbb{R}^4$ is given by replacement of $\int d^4x$ by $(2\pi)^2\zeta_1\zeta_2\mathrm{Tr}_{\mathscr{H}}$, but it is conditionally convergent in general. In [25, 26], we introduce cut-off N_C for the Fock space and make the instanton charge be a converge series. The region of the initial and final state of the Fock space with the boundary is

$$
|n_1,n_2\rangle \quad (n_1=0,\ldots,N_1(n_2),\ n_2=0,\ldots,N_2(n_1)),
\tag{2.62}
$$

where $N_1(n_2)$ $(N_2(n_1))$ is a function of n_2 (n_1) and we suppose that the length of the boundary is order $N_C\gg k$, that is, $N_1(n_2)\approx N_2(n_1)\approx N_C\gg k$.

Using this cut-off (boundary), we define the instanton charge by

$$
\begin{aligned}
Q&=\lim_{N_C\to\infty}Q_{N_C},\\
Q_{N_C}&=\zeta^2\sum_{n_1=0}\sum_{n_2=0}^{N_1(n_1)}\langle n_1,n_2|\left(F_{1\bar{1}}F_{2\bar{2}}-F_{1\bar{2}}F_{2\bar{1}}\right)|n_1,n_2\rangle.
\end{aligned}
\tag{2.63}
$$

As described in [25, 26], the regions for summations of intermediate states are shifted. This phenomenon is caused by the existence of the $\Psi\Psi^\dagger$ zero-mode $\langle v_0|$.

The following terms appear in the instanton charge Q_{N_C}:

$$-\mathrm{tr}_{U(N)}\mathrm{Tr}_{N_C}\left(\frac{1}{2}\left[\Psi^\dagger c_2^\dagger\Psi, \Psi^\dagger c_2\Psi\right] + \frac{1}{2}\left[\Psi^\dagger c_1^\dagger\Psi, \Psi^\dagger c_1\Psi\right]\right). \tag{2.64}$$

We denote Tr_{N_C} as trace over some finite domain of Fock space characterized by N_C which is the length of the Fock space boundary. Using the Stokes' like theorem in [25], only trace over the boundary is left, then $\mathrm{Tr}_{N_C}[\Psi^\dagger c_2^\dagger\Psi, \Psi^\dagger c_2\Psi]$ becomes

$$-\mathrm{tr}_{U(N)}\sum_{\text{boundary}}(N_2(n_1)+1) = -\mathrm{tr}_{U(N)}\mathrm{Tr}_{N_C}1 - k. \tag{2.65}$$

The same value is obtained from $\mathrm{Tr}_N[\Psi^\dagger c_1^\dagger\Psi, \Psi^\dagger c_1\Psi]$, too. The first term in (2.65) and the term from the constant curvature in (2.11) cancel out. The second term $-k$ is occurred by zero-modes $|v_0\rangle$. Finally the second term of (2.65) is understood as the source of the instanton charge. The origin of the instanton charge is shift of intermediate states caused by k zero-modes $|v_0\rangle$. After all, we get

$$Q_N = -k + O\left(N^{-1/2}\right), \qquad Q = \lim_{N\to\infty} Q_N = -k. \tag{2.66}$$

Theorem 2.4 (Instanton number). *Consider U(N) gauge theory on NC $\mathbb{R}^4$ with self-dual $\theta^{\mu\nu}$. The instanton charge Q is possible to be defined by limit of converge series and it is identified with the dimension k that appears in the ADHM construction and is called "instanton number".*

The strict proof is given in [25].

Note that the proof of the equivalence between the topological charge defined as the integral of the second Chern class and the instanton number given by the dimension of the vector space in the ADHM construction is not completed in NC space. In [27], Furuuchi shows how to appear zero-modes in the NC ADHM construction, and he shows that zero-modes project out some states in Fock space. In [28, 29], the geometrical origin of the instanton number for NC $U(1)$ gauge theory is clarified. In [25], the identification between the topological charge and the dimension of the vector space in the ADHM construction is shown for a $U(1)$ gauge theory. In [26], this identification is shown when the NC parameter is self-dual for a $U(N)$ gauge theory. In [30], the equivalence between the instanton numbers and instanton charges is shown with no restrictions on the NC parameters, but an NC version of the Osborn's identity (Corrigan's identity) is assumed. Until now, the relation between the instanton numbers and the topological charges in NC spaces had not been clarified completely. Moreover, the calculation in [25, 26] shows that the origin of the instanton number is deeply related to the noncommutativity. These results make us feel anomalous, because the instanton number of course exists for the instanton in the commutative space but $|v_0\rangle$ zero-modes or some counterparts of them do not exist in the commutative space. From these observations, we might wonder if there is a deep disconnection between commutative instantons and NC instantons. To clarify the connection between the NC instantons and commutative instantons, let us consider the smooth NC deformation from the commutative instanton in the next section.

Propagators and the Index Theorems

The zero-modes of the Dirac operator in the ADHM instanton background are studied in [33]. They show that the Atiyah-Singer index of the Dirac operator is equal to the instanton number. In [34], Green functions are constructed for a field in an arbitrary representation of gauge group propagating in NC ADHM instanton backgrounds.

Other Kinds of Solutions

We have reviewed the ADHM method. There are some other methods to construct NC instantons.

In [35], Lechtenfeld and Popov study the NC generalization of 't Hooft's multi-instanton configurations for the $U(2)$ gauge group. They solve the problem in the naive application of Nekrasov and Schwarz method to the 't Hooft instanton solution. The problem originates from the appearance of a source term in the equation in the Corrigan-Fairlie-'t Hooft-Wilczek ansatz. They generalize the method of [36] to naive NC multi-instantons.

In [37], Horváth et al. use the method of dressing transformations, an iterative procedure for generating solutions from a given solution, and they generalize Belavin and Zakharov method to the NC case.

In [38], Hamanaka and Terashima construct NC instantons by using the solution generating technique introduced by Harvey et al. [39].

More details and an embracive list including other kinds of NC space and other kinds of BPS states are found in [40] for example.

Another approach that is smooth deformation of commutative instanton is given in the last few years. We will see it in the next section.

3. Smooth NC Deformation of Instantons

In this section, we construct NC instantons deformed smoothly from commutative instantons, and we study their natures.

We define NC deformations by formal expansions in a deformation parameter $\hbar$. So, let us pay attention to the mathematical meaning of the formal expansion. We introduce our star products by using formal expansions in $\hbar$, as we will see soon. Such products are not closed in the set of all smooth functions in general, so one of the simple ways to define the star products is using formal expansion. The star product is defined by putting some conditions on each order of $\hbar$ expansion to be a smooth bounded function or a square integrable function and so on. Therefore, we have to check their conditions for all quantities represented by using the star product. Someone might wonder how can we manage such difficulties when the Fock space formalism is used. The Fock space formalism itself is regarded as a formal expansion by complex coordinates of $\mathbb{C}^2 \cong \mathbb{R}^4$. For example, an integrable condition of a function in the star product formulation is replaced by a convergence of the corresponding series. Space integrations are replaced by the trace operations (2.10). When we estimate topological charges like instanton charges by mathematically rigorous calculation, we have to use the Stokes' like theorem in the Fock space, as mentioned in Section 2.4. Therefore, the complexities of calculations are essentially same as the ones in star product formalism. One of the merits of using the star product formalism is that it does not require some specific representation. In calculations in the operator formalism, we have to introduce some basis like the Fock basis, but in the star product formalism, we can obtain physical values without introducing any representation.

3.1. Smooth NC Deformations

In this section, to easy understand that NC instantons smoothly connect into commutative instantons, we use a star product formulation. In the previous section, we use an operator formalism. Formally, there is a one-to-one correspondence between the operator formalism and the star product formalism, and the Weyl-transformation connects them with each other. Commutation relations of coordinates are given by

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad \mu,\nu = 1,2,\ldots,4, \tag{3.1}$$

where $(\theta^{\mu\nu})$ are a real, x-independent, skew-symmetric matrix entries, called the NC parameters. $\star$ is known as the Moyal product [41]. The Moyal product (or star product) is defined on functions by

$$f(x) \star g(x) := f(x) \exp\left(\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x). \tag{3.2}$$

Here $\overleftarrow{\partial}_\mu$ and $\overrightarrow{\partial}_\nu$ are partial derivatives with respect to x^μ for $f(x)$ and to x^ν for $g(x)$, respectively.

The curvature two form F is defined by $F := (1/2)F_{\mu\nu} dx^\mu \wedge_\star dx^\nu = dA + A \wedge_\star A$, where $\wedge_\star$ is defined by $A \wedge_\star A := (1/2)(A_\mu \star A_\nu) dx^\mu \wedge dx^\nu$.

To consider smooth NC deformations, we introduce a parameter $\hbar$ and a fixed constant $\theta_0^{\mu\nu} < \infty$ with $\theta^{\mu\nu} = \hbar\theta_0^{\mu\nu}$. We define the commutative limit by letting $\hbar \to 0$.

Formally we expand the connection as

$$A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)} \hbar^l. \tag{3.3}$$

Then,

$$\begin{gathered} A_\mu \star A_\nu = \sum_{l,m,n=0}^{\infty} \hbar^{l+m+n} \frac{1}{l!} A_\mu^{(m)} \left(\overline{\Delta}\right)^l A_\mu^{(n)}, \\ \overline{\Delta} \equiv \frac{i}{2} \overleftarrow{\partial}_\mu \theta_0^{\mu\nu} \overrightarrow{\partial}_v. \end{gathered} \tag{3.4}$$

We introduce the self-dual projection operator P by

$$P := \frac{1+*}{2}; \qquad P_{\mu\nu,\rho\tau} = \frac{1}{4}\left(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\nu\rho}\delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}\right). \tag{3.5}$$

Then the instanton equation is given as

$$P_{\mu\nu,\rho\tau}F^{\rho\tau} = 0. \tag{3.6}$$

In the NC case, the lth order equation of (3.6) is given by

$$\begin{aligned}
&P^{\mu\nu,\rho\tau}\left(\partial_\rho A_\tau^{(l)} - \partial_\tau A_\rho^{(l)} + i\left[A_\rho^{(l)}, A_\tau^{(0)}\right] + i\left[A_\rho^{(0)}, A_\tau^{(l)}\right] + C_{\rho\tau}^{(l)}\right) = 0,\\
&C_{\rho\tau}^{(l)} := \sum_{(p;m,n)\in I(l)} \hbar^{p+m+n}\frac{1}{p!}\left(A_\rho^{(m)}\left(\overleftarrow{\Delta}\right)^p A_\tau^{(n)} - A_\tau^{(m)}\left(\overleftarrow{\Delta}\right)^p A_\rho^{(n)}\right),\\
&I(l) \equiv \left\{(p;m,n)\in\mathbb{Z}^3 \mid p+m+n=l,\ p,m,n\geq 0,\ m\neq l,\ n\neq l\right\}.
\end{aligned} \tag{3.7}$$

Note that the 0th order is the commutative instanton equation with solution $A_\mu^{(0)}$ being a commutative instanton. The asymptotic behavior of commutative instanton $A_\mu^{(0)}$ is given by

$$A_\mu^{(0)} = gdg^{-1} + O\left(|x|^{-2}\right), \quad gdg^{-1} = O\left(|x|^{-1}\right), \tag{3.8}$$

where $g \in G$ and G is a gauge group. (See, e.g., [2].) We introduce covariant derivatives associated to the commutative instanton connection by

$$D_\mu^{(0)} f := \partial_\mu f + i\left[A_\mu^{(0)}, f\right], \qquad D_{A^{(0)}} f := df + A^{(0)} \wedge f. \tag{3.9}$$

Using this, (3.7) is given by

$$P^{\mu\nu,\rho\tau}\left(D_\rho^{(0)} A_\tau^{(l)} - D_\tau^{(0)} A_\rho^{(l)} + C_{\rho\tau}^{(l)}\right) = 0. \tag{3.10}$$

In the following, we fix a commutative instanton connection $A^{(0)}$. We impose the following gauge fixing condition for $A^{(l)}$ $(l \geq 1)$ [18, 42]

$$A - A^{(0)} = D_{A^{(0)}}^* B, \quad B \in \Omega_+^2, \tag{3.11}$$

where $D_{A^{(0)}}^*$ is defined by

$$\begin{aligned}
\left(D_{A^{(0)}}^*\right)_\rho^{\mu\nu} B_{\mu\nu} &= \delta_\rho^\nu \partial^\mu B_{\mu\nu} - \delta_\rho^\mu \partial^\nu B_{\mu\nu} + i\delta_\rho^\nu\left[A^\mu, B_{\mu\nu}\right] - \delta_\rho^\mu\left[A^\nu, B_{\mu\nu}\right]\\
&= \delta_\rho^\nu D^{(0)\mu} B_{\mu\nu} - \delta_\rho^\mu D^{(0)\nu} B_{\mu\nu}.
\end{aligned} \tag{3.12}$$

We expand B in $\hbar$ as we did with A. Then $A^{(l)} = D_{A^{(0)}}^* B^{(l)}$. In this gauge, using the fact that the $A^{(0)}$ is an anti-self-dual connection, (3.10) simplified to

$$2D_{(0)}^2 B^{(l)\mu\nu} + P^{\mu\nu,\rho\tau} C_{\rho\tau}^{(l)} = 0, \tag{3.13}$$

where

$$D_{(0)}^2 \equiv D^{\rho}_{A^{(0)}} D_{A^{(0)}\rho}. \tag{3.14}$$

We consider the Green's function for $D_{(0)}^2$:

$$D_{(0)}^2 G_0(x,y) = \delta(x-y), \tag{3.15}$$

where $\delta(x-y)$ is a four-dimensional delta function. $G_0(x,y)$ has been constructed in [43] (see also [44, 45]). Using the Green's function, we solve (3.13) as

$$B^{(l)\mu\nu} = -\frac{1}{2}\int_{\mathbb{R}^4} G_0(x,y) P^{\mu\nu,\rho\tau} C^{(l)}_{\rho\tau}(y) d^4 y \tag{3.16}$$

and the NC instanton $A = \sum A^{(l)}\hbar^l$ is given by

$$A^{(l)} = D^{*}_{A^{(0)}} B^{(l)}. \tag{3.17}$$

In the following, we call NC instantons smoothly deformed from commutative instantons SNCD instantons. The asymptotic behavior of Green's function of $D_{(0)}^2$ is important, which is given by

$$G_0(x,y) = O\left(|x-y|^{-2}\right). \tag{3.18}$$

We introduce the notation $O'(|x|^{-m})$ as in [2]. If s is a function of $\mathbb{R}^4$ which is $O(|x|^{-m})$ as $|x| \to \infty$ and $|D^k_{(0)} s| = O(|x|^{-m-k})$, then we denote this natural growth condition by $s = O'(|x|^{-m})$.

Theorem 3.1. *If $C^{(l)} = O'(|x|^{-4})$, then $B^{(k)} = O'(|x|^{-2})$.*

We gave a proof of this theorem in [18].

In our case, $C^{(1)}_{\rho\tau} = O'(x^{-4})$ by (3.8), and so $B^{(1)} = O'(|x|^{-2})$, $A^{(1)} = O'(|x|^{-3})$ as $A^{(l)} = D^{*}_{A^{(0)}} B^{(l)}$. Repeating the argument l times, we get

$$\left|A^{(l)}\right| < O'\left(|x|^{-3+\epsilon}\right), \quad \forall \epsilon > 0. \tag{3.19}$$

3.2. Instanton Charge

The instanton charge is defined by

$$Q_\hbar := \frac{1}{8\pi^2}\int \mathrm{tr}_{U(N)} F \wedge \star F. \tag{3.20}$$

We rewrite (3.20) as

$$\frac{1}{8\pi^2}\int \mathrm{tr}_{U(N)} d\left(A \wedge \star dA + \frac{2}{3} A \wedge \star A \wedge \star A\right) + \frac{1}{8\pi^2}\int \mathrm{tr}_{U(N)} P_\star, \tag{3.21}$$

where

$$P_\star = \frac{1}{3}\{F \wedge \star A \wedge \star A + 2A \wedge \star F \wedge \star A + A \wedge \star A \wedge \star F + A \wedge \star A \wedge \star A \wedge \star A\}. \tag{3.22}$$

$\int \mathrm{tr}_{U(N)} P_\star$ is 0 in the commutative limit, but it does not vanish in NC space, because the cyclic symmetry of trace operation is broken by the NC deformation.

The terms in $\int \mathrm{tr}_{U(N)} P_\star$ are typically written as

$$\int_{\mathbb{R}^d} \mathrm{tr}_{U(N)}\left(P \wedge \star R - (-1)^{n(4-n)} R \wedge \star P\right), \tag{3.23}$$

where P and R are some n-form and $(4-n)$-form $(n = 0, \ldots, 4)$, respectively, and let $P \wedge R$ be $O(\hbar^k)$. The lowest order term in $\hbar$ vanishes because of the cyclic symmetry of the trace, that is, $\int \mathrm{tr}_{U(N)}(P \wedge R - (-1)^{n(4-n)} R \wedge P) = 0$. The term of order $\hbar$ is given by

$$\begin{aligned} &\frac{i}{2}\int_{\mathbb{R}^4} \mathrm{tr}_{U(N)}\left\{\hbar\theta_0^{\mu\nu}(\partial_\mu P \wedge \partial_\nu R)\right\} \\ &\quad = \frac{i}{2}\int_{\mathbb{R}^4} (n!(4-n)!)\epsilon^{\mu_1\mu_2\mu_3\mu_4} \mathrm{tr}_{U(N)} d\left\{(*\theta) \wedge \left(P_{\mu_1\cdots\mu_n} dR_{\mu_{n+1}\cdots\mu_4}\right)\right\}, \end{aligned} \tag{3.24}$$

where $*\theta = \epsilon_{\mu\nu\rho\tau}\theta^{\rho\tau} dx^\mu \wedge dx^\nu/4$. These integrals are zero if $P_{\mu_1\cdots\mu_n} dR_{\mu_{n+1}\cdots\mu_4}$ is $O'(|x|^{-(4-1+\epsilon)})$ $(\epsilon > 0)$ and this condition is satisfied for SNCD instantons. Similarly, higher-order terms in $\hbar$ in (3.23) can be written as total divergences and hence vanish under the decay hypothesis. This fact and (3.19) imply that $\int \mathrm{tr}_{U(N)} P_\star = 0$.

Because of the similar estimation, we found the other terms of $\int \mathrm{tr}_{U(N)} F \wedge \star F - \int \mathrm{tr}_{U(N)} F^{(0)} \wedge F^{(0)}$ vanish, where $F^{(0)}$ is the curvature two form associated to $A^{(0)}$.

Summarizing the above discussions, we get following theorems [18].

Theorem 3.2. *Let $A_\mu^{(0)}$ be a commutative instanton solution in $\mathbb{R}^4$. There exists a formal NC instanton solution $A_\mu = \sum_{l=0}^\infty A_\mu^{(l)}\hbar^l$ (SNCD instanton) such that the instanton number $Q_\hbar$ defined by* (3.20) *is independent of the NC parameter $\hbar$:*

$$\frac{1}{8\pi^2}\int \mathrm{tr}_{U(N)} F \wedge \star F = \frac{1}{8\pi^2}\int \mathrm{tr}_{U(N)} F^{(0)} \wedge F^{(0)}. \tag{3.25}$$

3.3. Index of the Dirac Operator and Green's Function

Dirac(-Weyl) operators $\mathfrak{D}_A : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ and $\overline{\mathfrak{D}}_A : \Gamma(S^- \otimes E) \to \Gamma(S^+ \otimes E)$ are defined as

$$\mathfrak{D}_A := \sigma^\mu D_\mu, \qquad \overline{\mathfrak{D}}_A := \overline{\sigma}^\mu D_\mu^\dagger. \tag{3.26}$$

Here, σ_μ and $\overline{\sigma}_\mu$ are defined by (2.24). Consider $\hbar$ expansion of $\psi \in \Gamma(S^+ \otimes E)$ and $\overline{\psi} \in \Gamma(S^- \otimes E)$ as

$$\psi = \sum_{n=0}^{\infty} \hbar^n \psi^{(n)}, \qquad \overline{\psi} = \sum_{n=0}^{\infty} \hbar^n \overline{\psi}^{(n)}. \tag{3.27}$$

In [19], the zero-modes of $\mathfrak{D}_A$ and $\overline{\mathfrak{D}}_A$, which are defined by

$$\mathfrak{D}_A \star \psi = 0, \qquad \overline{\mathfrak{D}}_A \star \overline{\psi} = 0, \tag{3.28}$$

are investigated, and the following theorem is obtained.

Theorem 3.3. *Let $\mathfrak{D}_A$ and $\overline{\mathfrak{D}}_A$ be the Dirac(-Weyl) operators for an SNCD instanton background with its instanton number $-k$. There is no zero-mode for $\mathfrak{D}_A \star \psi = 0$, and there are k zero-modes for $\overline{\mathfrak{D}}_A \star \overline{\psi}_i = 0 \quad (i = 1, \ldots, k)$ that are given as*

$$\overline{\psi}_i = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} a_{n,i}^{j} \eta_j \right) \hbar^n + O'\left(|x|^{-5+\epsilon}\right), \quad \eta_j = O'\left(|x|^{-3}\right), \tag{3.29}$$

where $a_{n,i}^{j}$ is a constant matrix and η_j is a base of the zero mode of $\overline{\mathfrak{D}}_A^{(0)}$.

Note that it is a well-known fact as an index theorem in commutative space that the dimension of $\ker \overline{\mathfrak{D}}_A^{(0)}$ is equal to k the instanton number (of opposite sign), and there exists k zero-mode η_i $(i = 1, 2, \ldots, k)$. Theorem 3.3 says that zero-modes deformed from the ones in commutative space are obtained, but there is no new zero-mode appearing. Then we get the following theorem [19].

Theorem 3.4. *If* $\mathrm{Ind}\,\not{D}^0 := \dim\ker \mathfrak{D}_A^{(0)} - \dim\ker \overline{\mathfrak{D}}_A^{(0)} = -k$, *then* $\mathrm{Ind}\,\not{D} := \dim\ker \mathfrak{D}_A - \dim\ker \overline{\mathfrak{D}}_A = -k$.

Next, we construct the Green's function of $\Delta_A \equiv D_\mu \star D^\mu$,

$$\Delta_A \star G_A(x, y) = \delta(x - y). \tag{3.30}$$

We expand (3.18) by $\hbar$, for $n > 0$, then $\hbar^n$ order equation is given as

$$\Delta_A^{(0)} G_A^{(n)}(x,y) + \left[\Delta_A \sum_{0\le k<n} \hbar^k G_A^{(k)}(x,y)\right]^{(n)} = 0, \tag{3.31}$$

where $G_A^{(n)}$ is defined by $G_A(x,y) = \sum_{k=0}^{\infty} G_A^{(k)} \hbar^k$. We solve them recursively

$$G_A^{(n)}(x,y) = \int d^4 w G_A^{(0)}(x,w) \left[\Delta_A \sum_{0\le k<n} \hbar^k G_A^{(k)}(w,y)\right]^{(n)}. \tag{3.32}$$

Note that $G_A^{(0)}(x,w)$ was constructed in [43–45]. Using property of $G_A^{(0)}(x,w)$ and $A^{(n)}$, we obtain the following decay condition in [19]:

$$G_A^{(n)}(x,y) = O'\left(|x|^{-3}\right). \tag{3.33}$$

3.4. From an Instanton to the ADHM Equations

Let us see how to derive the ADHM equations from an SNCD instanton.

Let $\overline{\psi}_i$ $(i = 1,\dots,k)$ be orthonormal zero-modes of $\overline{\mathfrak{D}}_A$ and $\overline{\psi} = (\overline{\psi}_i)$, which are introduced in Section 3.3.

At first we define T^μ by

$$T^\mu := \int_{\mathbb{R}^4} d^4 x \frac{1}{2}\left(x^\mu \star \overline{\psi}^\dagger \star \overline{\psi} + \overline{\psi}^\dagger \star \overline{\psi} \star x^\mu\right). \tag{3.34}$$

Next we introduce an asymptotically parallel section $g^{-1}S$ of $S^+ \otimes E$ by

$$\tilde{\psi} = -\frac{g^{-1} S x^\dagger}{|x|^4} + O'\left(|x|^{-4}\right), \tag{3.35}$$

where $x^\dagger := \overline{\sigma}_\mu x^\mu$ and $\tilde{\psi} := {}^t\overline{\psi}\sigma_2$. This t means transposing spinor suffixes.

Using various properties and decay conditions of $A^{(n)}, G_A^{(n)}, \overline{\psi}^{(n)}$, and theorems in the previous subsections, we finally obtain the following theorem.

Theorem 3.5. *Let A^μ be an SNCD instanton and $\overline{\psi}$ the zero-mode of $\overline{\mathfrak{D}}_A$ determined by A^μ as in Section 3.3. Let T^μ, S be constant matrices defined by* (3.34) *and* (3.35)*, respectively. Then, they satisfy the ADHM equations:*

$$[T^\mu, T^\nu]^+ = \frac{1}{2}\operatorname{tr}\left(S^\dagger S \overline{\sigma}^{\mu\nu}\right) - i\theta^{\mu\nu+} 1_{k\times k}. \tag{3.36}$$

Here $\overline{\sigma}_{\mu\nu} := (1/4)(\overline{\sigma}_\mu \sigma_\nu - \overline{\sigma}_\nu \sigma_\mu)$ and $1_{k\times k}$ is an identity matrix.

Rough sketch of the proof

Let us see the essence of the proof. Let us introduce $\star_x$ as $\star$ associated with variable x. The completeness of $\overline{\varphi}(x)$ is written as

$$\star_x \overline{\varphi}(x)\overline{\varphi}^\dagger(y)\star_y = \star_x \delta(x-y)\star_y - \star_x \mathfrak{D}_A \star_x G_A(x,y)\star_y \overleftarrow{\overline{\mathfrak{D}}}_A \star_y. \tag{3.37}$$

From the definition of the T^μ,

$$T^\mu T^\nu = \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \left(x^\mu \star_x \overline{\varphi}^\dagger(x)\star_x \overline{\varphi}(x)\right)\left(\overline{\varphi}^\dagger(y)\star_y \overline{\varphi}(y)\star_y y^\nu\right). \tag{3.38}$$

Using Theorem 3.3, (3.37), and integration by parts, (3.38) becomes

$$\begin{aligned} T^\mu T^\nu = &\int_{\mathbb{R}^4} d^4x x^\mu \star \overline{\varphi}^\dagger \star \overline{\varphi} \star x^\nu \\ &+ \int_{S^3} dS_x^\rho \int_{\mathbb{R}^4} d^4y \left(x^\mu \star_x \overline{\varphi}^\dagger(x)\sigma_\rho\right)\star_x G_A(x,y)\star_y \overleftarrow{\overline{\mathfrak{D}}}_A \star_y \left(\overline{\varphi}(y)\star_y y^\nu\right) \\ &- \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \left(\overline{\varphi}^\dagger(x)\sigma^\mu\right)\star_x G_A(x,y)\star_y \overleftarrow{\overline{\mathfrak{D}}}_A \star_y \left(\overline{\varphi}(y)\star_y y^\nu\right), \end{aligned} \tag{3.39}$$

where $dS_x^\mu = |x|^2 x^\mu d\Omega$ and $d\Omega$ is the solid angle. The first term is deformed as follows.

$$\begin{aligned} &\int_{\mathbb{R}^4} d^4x x^\mu \star \overline{\varphi}^\dagger \star \overline{\varphi} \star x^\nu \\ &\quad = \int_{\mathbb{R}^4} d^4x \left(\overline{\varphi}^\dagger \star \overline{\varphi} \star x^\nu \star x^\mu + \left[x^\mu, \overline{\varphi}^\dagger \star \overline{\varphi}\right]_\star \star x^\nu + \overline{\varphi}^\dagger \star \overline{\varphi} \star [x^\mu, x^\nu]_\star\right) \\ &\quad = \int_{\mathbb{R}^4} d^4x \left(\overline{\varphi}^\dagger \star \overline{\varphi} \star x^\nu \star x^\mu + i\theta^{\mu\rho}\partial_\rho\left(\overline{\varphi}^\dagger \star \overline{\varphi}\right) \star x^\nu + i\theta^{\mu\nu}\overline{\varphi}^\dagger \star \overline{\varphi}\right) \\ &\quad = \int_{\mathbb{R}^4} d^4x \overline{\varphi}^\dagger \star \overline{\varphi} \star x^\nu \star x^\mu. \end{aligned} \tag{3.40}$$

Here $\overline{\psi} = O'(|x|^{-3})$ given in Theorem 3.3 is used in the third equality. By integration by parts again, we get

$$T^\mu T^\nu = \int_{\mathbb{R}^4} d^4x \overline{\psi}^\dagger \star \overline{\psi} \star x^\nu \star x^\mu \tag{3.41}$$

$$+ \int_{S^3} dS_x^\rho \int_{S^3} dS_y^\tau \left(x^\mu \star_x \overline{\psi}^\dagger(x) \sigma_\rho \right) \star_x G_A(x,y) \star_y (\overline{\sigma}_\tau \overline{\psi}(y) \star_y y^\nu) \tag{3.42}$$

$$- \int_{S^3} dS_x^\rho \int_{\mathbb{R}^4} d^4y \left(x^\mu \star_x \overline{\psi}^\dagger(x) \sigma_\rho \right) \star_x G_A(x,y) \star_y (\overline{\sigma}^\nu \overline{\psi}(y)) \tag{3.43}$$

$$- \int_{\mathbb{R}^4} d^4x \int_{S^3} dS_y^\tau \left(\overline{\psi}^\dagger(x) \sigma^\mu \right) \star_x G_A(x,y) \star_y (\overline{\sigma}^\tau \overline{\psi}(y) \star_y y^\nu) \tag{3.44}$$

$$+ \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \left(\overline{\psi}^\dagger(x) \sigma^\mu \right) \star_x G_A(x,y) \star_y (\overline{\sigma}^\nu \overline{\psi}(y)). \tag{3.45}$$

Equations (3.42) and (3.44) vanish when $R_y \to \infty$, where R_y is a radius of S_y^3. Equation (3.45) will vanish on the self-dual projection $[T^\mu, T^\nu]^+ := P^{\mu\nu,\rho\tau}[T_\rho, T_\tau]$, because $\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu$ is anti-self-dual with respect to the μ, ν. Thus only (3.41) and (3.43) remain. By the asymptotic behaviors of $\overline{\psi}$ and some calculations, we can prove that (3.43) becomes

$$\frac{1}{8} \operatorname{tr}\left(S^\dagger S \overline{\sigma}^\mu \sigma^\nu \right), \tag{3.46}$$

where the trace tr is taken with respect to the spinor indices. In the $[T^\mu, T^\nu]^+$ combination, (3.41) becomes $-i\theta^{\mu\nu+} = -iP^{\mu\nu,\rho\tau}\theta^{\rho\tau}$. Therefore, we get (3.36). The complete proof is given in [19].

These ADHM equations (3.36) are coincident with the ones provided by Nekrasov and Schwarz [4]. After identification of

$$S^\dagger = \begin{pmatrix} I \\ J^\dagger \end{pmatrix}, \qquad T^\mu \overline{\sigma}_\mu = \begin{pmatrix} -B_2 & -B_1 \\ B_1^\dagger & -B_2^\dagger \end{pmatrix}, \tag{3.47}$$

and setting the NC parameter as in (2.2), we find that (3.36) is identified with (2.13) and (2.14).

Similar to the commutative case, we obtain the following theorem.

Theorem 3.6. *There is a one-to-one correspondence between ADHM data satisfying* (3.36) *and SNCD instantons in NC* $\mathbb{R}^4$.

The proof is given in [19].

4. Smooth NC Deformation of Vortexes

In the previous section, we investigate the smooth deformation of instantons. This method is applicable to gauge theories in other dimensions. In this section we study NC deformation of the vortex solutions [46, 47]. We consider the Abelian-Higgs model in commutative $\mathbb{R}^2$ and deform the Taubes' vortex solutions into NC vortexes [48].

Let coordinates of NC Euclidean space $\mathbb{R}^2$ be $x^\mu,\ \mu = 1,2$, with commutation relations

$$[x^\mu, x^\nu]_\star = i\hbar\epsilon^{\mu\nu} \quad (\mu,\nu = 1,2), \tag{4.1}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \quad (\epsilon^{12} = 1)$ is an antisymmetric tensor.

The curvature components of the connection A are given by

$$\begin{gathered} F_{zz} = F_{\overline{z}\,\overline{z}} = 0, \\ F_{z\overline{z}} = iF_{12} = \partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z - i[A_z, A_{\overline{z}}]_\star =: iB. \end{gathered} \tag{4.2}$$

Using these complex coordinates, the covariant derivatives of the Higgs fields are

$$\begin{aligned} D\star\phi &= (\partial - iA)\star\phi, &\qquad \overline{D}\star\phi &= \left(\overline{\partial} - i\overline{A}\right)\star\phi, \\ D\star\overline{\phi} &= \partial\overline{\phi} + i\overline{\phi}\star A, &\qquad \overline{D}\star\overline{\phi} &= \overline{\partial}\,\overline{\phi} + i\overline{\phi}\star\overline{A}. \end{aligned} \tag{4.3}$$

The vortex equations are defined by

$$\overline{D}\star\phi = \left(\overline{\partial} - i\overline{A}\right)\star\phi = 0, \qquad B + \phi\star\overline{\phi} - 1 = 0. \tag{4.4}$$

We call solutions of these equations NC vortexes.

The formal expansions of the fields are

$$\phi = \sum_{n=0}^{\infty}\hbar^n\phi_n(z,\overline{z}), \qquad A = \sum_{n=0}^{\infty}\hbar^n A_n(z,\overline{z}). \tag{4.5}$$

The kth order equations for (4.4) are

$$-i\left(\partial\overline{A}_k + \overline{\partial}A_k\right) + \phi_k\overline{\phi}_0 + \phi_0\overline{\phi}_k - \delta_{k0} + C_k(z,\overline{z}) = 0, \tag{4.6}$$

$$\overline{\partial}\phi_k - i\overline{A}_k\phi_0 - i\overline{A}_0\phi_k + D_k(z,\overline{z}) = 0. \tag{4.7}$$

Here $C_k(z,\overline{z})$ is the coefficient of $\hbar^k$ in $-[A,\overline{A}]_\star + \phi\star\overline{\phi} - (\phi_k\overline{\phi}_0 + \phi_0\overline{\phi}_k)$, so $C_k(z,\overline{z})$ is a function of $\{A_i, \overline{A}_j, \phi_m, \overline{\phi}_n \mid 0 \le i,j,m,n \le k-1\}$. Similarly, $D_k(z,\overline{z})$ is the coefficient of $\hbar^k$ in $-i\overline{A}\star\phi - (-i\overline{A}_k\phi_0 - i\overline{A}_0\phi_k)$ and a function of $\{A_i, \overline{A}_j, \phi_m, \overline{\phi}_n \mid 0 \le i,j,m,n \le k-1\}$.

In the case of $k = 0$, (4.6) and (4.7) coincide with the commutative $U(1)$ vortex equations $\overline{D}\phi_0 = (\overline{\partial} - i\overline{A}_0)\phi_0 = 0$ and $B_0 + \phi_0\overline{\phi}_0 - 1 = 0$, where $B_0 = -i(\partial\overline{A}_0 - \overline{\partial}A_0)$. In the following, we consider the case that $A_0, \overline{A}_0$, and ϕ_0 are smooth finite vortex solutions. We call it Taubes' vortex solution.

In the region $\phi_0 \neq 0$, substituting (4.7) into (4.6) for A_k and $\overline{A}_k$, we get

$$\begin{aligned}&\left\{\frac{\partial\phi_0}{\phi_0^2}\left(\overline{\partial}\phi_k - i\overline{A}_0\phi_k + D_k\right) - \frac{1}{\phi_0}\left(\Delta\phi_k - i\partial\overline{A}_0\phi_k - i\overline{A}_0\partial\phi_k + \partial D_k\right)\right\} + \{\text{c.c.}\}\\ &\qquad + \phi_k\overline{\phi}_0 + \phi_0\overline{\phi}_0 - \delta_{k0} + C_k = 0.\end{aligned} \tag{4.8}$$

Here {c.c.} is the complex conjugate of preceding terms and $\Delta = \partial\overline{\partial}$.

Setting

$$\varphi_k = \frac{\phi_k}{\phi_0} + \frac{\overline{\phi}_k}{\overline{\phi}_0} = 2\,\mathrm{Re}\left(\frac{\phi_k}{\phi_0}\right), \qquad d_k = \frac{D_k}{\phi_0}. \tag{4.9}$$

Equation (4.8) is simplified to

$$\left(-\Delta + |\phi_0|^2\right)\varphi_k = E_k, \tag{4.10}$$

where

$$E_k := -C_k + \partial d_k - \overline{\partial}\,\overline{d}_k. \tag{4.11}$$

To show that there exists a unique NC vortex solution deformed from the Taubes' vortex solution, we consider the stationary Schrödinger equation

$$(-\Delta + V(x))u(x) = f(x) \tag{4.12}$$

in $\mathbb{R}^2$, where $V(x)$ is a real-valued C^∞ function. We impose the following assumptions for $V(x)$.

($a1$) $V(x) \geq 0$, for all $x \subset \mathbb{R}^2$.

($a2$) There exist $K \subset \mathbb{R}^2$ and $\exists c > 0$ such that K is a compact set and for $x \in \mathbb{R}^2 \setminus K$, $V(x) \geq c$.

($a3$) There exist $x_1, \ldots, x_N \in \mathbb{R}^2$ such that $V(x_i) = 0$ and $V(x) > 0$ for $x \notin \{x_1, \ldots, x_N\}$.

($a4$) For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, There exists a positive constant c such that $|\partial_x^\alpha(V - c)| \leq c$ for any $x \in \mathbb{R}^2$.

Note that the system (4.10) satisfies the assumptions ($a1$)–($a4$). We set

$$H_l(n) := \left\{ f \mid \|f\| := \sup_{x\in\mathbb{R}^2}(1 + |x|^n)\left|\partial_x^\alpha f(x)\right| < \infty \text{ for any } |\alpha| \leq l \right] \tag{4.13}$$

for $n \in \mathbb{Z}_+$. Then we obtain the following theorem.

Theorem 4.1. *Under the assumptions* $(a1)$*–*$(a4)$*, there exists a unique solution* $u \in H_l(n)$ *of* (4.12) *for any* $f \in H_l(n)$*.*

This theorem's proof is given by using standard techniques of Green's function [20].

Equation (4.10) is a particular example of (4.12). Theorem 4.1 and some asymptotic analysis derive the following theorem.

Theorem 4.2. *Let* A_0 *and* ϕ_0 *be a Taubes' vortex solution, in other words,* (A_0, ϕ_0) *is a finite and smooth solution of the commutative vortex equations. Then there exists a unique solution* (A, ϕ) *of the NC vortex equations* (4.4) *with* $A|_{\hbar=0} = A_0$*,* $\phi|_{\hbar=0} = \phi_0$*, and its vortex number is preserved*

$$\frac{1}{2\pi}\int d^2xB = \frac{1}{2\pi}\int d^2xB_0. \tag{4.14}$$

The proof is given in [20].

5. Conclusions

We have reviewed developments for the last dozen years in NC instantons in $\mathbb{R}^4$. The ADHM methods made great progress and broke ground to make strict solutions of the NC soliton equations. A lot of kinds of NC instanton solutions have been made by the ADHM method. Using the solutions and ADHM data, many aspects have been investigated. For example, topological charges, Dirac zero-modes, index theorems, and Green's functions in the NC ADHM instanton backgrounds. However, we could not understand the relation with commutative instantons and how instantons deform from commutative ones rigorously. In recent few years, the smooth NC deformation method has been investigated. For the smooth NC deformed instantons, many features are clarified. For example, the instanton charge, the number of the spinor zero-modes, and the index of the Dirac operator in the NC deformed instanton backgrounds coincide with the ones in commutative instanton backgrounds. The ADHM equations are derived from the NC deformed instantons and we find the ADHM equations coincide with the ones by Nekrasov and Schwarz. A one-to-one correspondence between smooth NC deformed instantons and the ADHM data are also obtained. Thus, about instantons in NC $\mathbb{R}^4$, a lot of features have been investigated. The smooth NC deformation method is useful for other dimensional gauge theories. As an example, smooth deformations of vortexes are studied similarly. Their vortex numbers also coincide with the ones in commutative $\mathbb{R}^2$.

We have considered gauge theories in $\mathbb{R}^n$. One of the essences to prove some theorems of NC instantons or NC vortexes is in infinity of size of the space. So, some of the theorems are changed when we consider finite size spaces. For example, topological charges are deformed under NC deformations of the spaces and they depend on the NC parameters in general [18]. The generic investigations of such changes from the point of view of smooth NC deformations are left for future subjects. Most NC instantons or some other solitons in gauge theories are still in deep mist.

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