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Research Article

Nonlocal Symmetries of Systems of Evolution Equations

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We prove that any potential symmetry of a system of evolution equations reduces to a Lie symmetry through a nonlocal transformation of variables. This fact is in the core of our approach to computation of potential and more general nonlocal symmetries of systems of evolution equations having nontrivial Lie symmetry. Several examples are considered.

1. Introduction

The Lie symmetries and their various generalizations have become an inseparable part of the modern physical description of wide range of phenomena of nature from quantum physics to hydrodynamics. Such success of a purely mathematical theory of continuous groups developed by Lie and Engel in 19th century [1] is explained by the remarkable fact that the overwhelming majority of mathematical models of physical, chemical, and biological processes possess nontrivial Lie symmetry.

One can even argue that this very property, invariance under Lie symmetries, distinguishes the popular models of mathematical and theoretical physics from a continuum of possible models in the form of differential or integral equations (see, e.g., [2, 3]). Based on this observation is the symmetry selection principle stating that if an equation describing some physical process contains arbitrary elements, then the latter should be so chosen that the resulting model possesses the highest possible symmetry. In this sense, the Lie theory effectively predicts which equation is the best candidate to serve as a mathematical model of a specific physical, chemical, or biological process.

The procedure of selecting partial differential equations (PDEs) enjoying the highest Lie symmetry from a prescribed class of PDEs is called group classification. In the case when non-Lie symmetries are involved, the more general term, symmetry classification, is used.

In this paper we study symmetries of systems of evolution equations in one spatial variable

$$\mathbf{u}_t = \mathbf{f}(t, x, \mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n), \tag{1.1}$$

where $\mathbf{u} = \{u^1(t,x), u^2(t,x), \ldots, u^m(t,x)\}$, $\mathbf{u}_{i+1} = \partial \mathbf{u}_i/\partial x$, $n \geq 2$, $m \geq 2$. Note that we use the boldface font to denote a multicomponent variable.

The problem of the Lie group classification of PDEs of the form (1.1) has been extensively studied (see, e.g. [4–7] and the references therein). The centerpiece of any approach used in this respect is the classical infinitesimal Lie method. The latter enables to reduce the problem of description of transformation groups admitted by (1.1) to integrating some linear system of PDEs (further details can be found in [8–10]).

However, with all its importance and power, the traditional Lie approach does not provide all the answers to mounting challenges of the modern nonlinear physics. By this very reason there were numerous attempts of generalization of Lie symmetries so that the generalized symmetries retain the most important features of Lie symmetries and allow for a broader scope of applicability. A natural move in this direction would be letting the coefficients of infinitesimal generators of the Lie symmetries to contain not only independent and dependent variables and their derivatives but also integrals of dependent variables, as well. In this way, the so-called nonlocal symmetries have been introduced into mathematical physics.

The concept of nonlocal symmetry of linear PDEs is relatively well understood (see, e.g., [11]). This is not the case for nonlinear differential equations. The problem of developing regular methods for constructing nonlocal symmetries of nonlinear PDEs is still waiting for its Sophus Lie. Still, there are a number of results on nonlocal symmetries for specific equations.

One of the possible approaches to construction of nonlocal symmetries has been suggested by Bluman et al. [12, 13]. They put forward the concept of potential symmetry, which is a special case of nonlocal symmetry. The basic idea of the method for constructing potential symmetries of PDEs can be formulated in the following way. Consider an evolution equation

$$u_t = f(t, x, u, u_1, \ldots, u_n). \tag{1.2}$$

Suppose that it can be rewritten in the form of a conservation law

$$\frac{\partial}{\partial t}(G(t,x,u)) = \frac{\partial}{\partial x}(F(t,x,u,u_1,\ldots,u_{n-1})). \tag{1.3}$$

By force of (1.3), we can introduce the new dependent variable $v = v(t,x)$ and rewrite (1.1) as follows:

$$v_x = G(t,x,u), \qquad v_t = F(t,x,u,u_1,\ldots,u_{n-1}). \tag{1.4}$$

If the system of two equations (1.4) admits a Lie symmetry such that at least one of the coefficients of its infinitesimal operator depends on $v = \partial_x^{-1} G(t,x,u)$, then this symmetry is the nonlocal symmetry for the initial evolution equation (1.2). Here ∂_x^{-1} is the inverse of ∂_x, that is, $\partial_x \partial_x^{-1} \equiv \partial_x^{-1}\partial_x \equiv 1$. This nonlocal symmetry is also called potential symmetry of (1.2).

Pucci and Saccomandi [14] and Saccomandi [15] proved that potential symmetries can be derived using nonclassical symmetries of PDE (1.2). Recently, we established much stronger assertion by associating potential symmetries with classical contact symmetries [16, 17]. More precisely, we proved that any potential symmetry of evolution equation (1.2) can be reduced to contact symmetry by a suitable nonlocal transformation of dependent and independent variables. As a consequence, one can obtain exhaustive description of potential symmetries of (1.2) through classification of contact symmetries of PDEs of the form (1.2).

Some applications of potential symmetries to specific subclasses of the class of PDEs (1.2) can be found in [18–23].

In the present paper, we generalize the results of [16] for system of evolution equations (1.1) and prove that any potential symmetry of the system in question reduces to classical Lie symmetry under a suitable nonlocal transformation of dependent and independent variables (Sections 2 and 3). Next, we suggest in Section 4 a more general approach to constructing nonlocal symmetries that goes far beyond the concept of potential symmetries. It enables generating systems of evolution equations associated with a given system of the system (1.1), provided the latter admits a nontrivial Lie symmetry. Some applications of the approach in question are presented in Section 4.

2. Conservation Law Representation and Classical Symmetries

Definition 2.1. One says that system (1.1) admits complete conservation law representation (CLR) if it can be written in the form

$$\frac{\partial}{\partial t}(\mathbf{G}(t,x,\mathbf{u})) = \frac{\partial}{\partial x}(\mathbf{F}(t,x,\mathbf{u},\mathbf{u}_1,\dots,\mathbf{u}_{n-1})). \tag{2.1}$$

Here $\mathbf{u},\mathbf{F}$, and $\mathbf{G}$ are m-component vectors.

Definition 2.2. One says that system (1.1) admits partial CLR if it can be written in the form

$$\begin{gathered}\frac{\partial}{\partial t}(\mathbf{F}(t,x,\mathbf{u},\mathbf{w})) = \frac{\partial}{\partial x}(\mathbf{G}(t,x,\mathbf{u},\mathbf{u}_1,\dots,\mathbf{u}_{n-1},\mathbf{w},\mathbf{w}_1,\dots,\mathbf{w}_{n-1})),\\ \mathbf{w}_t = \mathbf{H}(t,x,\mathbf{u},\mathbf{u}_1,\dots,\mathbf{u}_n,\mathbf{w},\mathbf{w}_1,\dots,\mathbf{w}_n).\end{gathered} \tag{2.2}$$

Here $\mathbf{u},\mathbf{F},\mathbf{G}$ and $\mathbf{w},\mathbf{H}$ are r-component and m-r-component vectors, respectively.

Below we present theorems that provide exhaustive characterization of conservation law representability in terms of classical Lie symmetries. We give the detailed proof of the assertion regarding complete CLR; the case of partial CLR is handled in a similar way.

Theorem 2.3. *System* (1.1) *admits complete CLR if and only if it is invariant under m-dimensional commutative Lie algebra* $\mathcal{L}_m = \langle e_1, \ldots, e_m \rangle$, *where*

$$e_i = \xi_i(t, x, \mathbf{u})\partial_x + \sum_{j=1}^{m} \eta_i^j(t, x, \mathbf{u})\partial_{u^j}, \tag{2.3}$$

and besides

$$\operatorname{rank}\begin{pmatrix} \xi_1 & \eta_1^1 & \cdots & \eta_1^m \\ \vdots & \vdots & \vdots & \vdots \\ \xi_m & \eta_m^1 & \cdots & \eta_m^m \end{pmatrix} = m. \tag{2.4}$$

Proof. Suppose that system (1.1) admits CLR (2.1). Introducing new m-component function

$$\mathbf{v}_x = \mathbf{G}(t, x, \mathbf{u}) \tag{2.5}$$

and eliminating $\mathbf{u}$ from (2.1), we get

$$\mathbf{v}_{xt} = \frac{\partial}{\partial x}\left(\widetilde{\mathbf{f}}(t, x, \mathbf{v}_1, \ldots, \mathbf{v}_n)\right). \tag{2.6}$$

Integrating the obtained system of PDEs with respect to x yields

$$\mathbf{v}_t = \widetilde{\mathbf{f}}(t, x, \mathbf{v}_1, \ldots, \mathbf{v}_n). \tag{2.7}$$

Note that the integration constant $\mathbf{w}(t)$ is absorbed into the function $\mathbf{v}$. Evidently, system (2.7) is invariant under the commutative m-dimensional Lie algebra $\mathcal{L}_m = \langle \partial_{v^1}, \ldots, \partial_{v^m} \rangle$. What is more is that the coefficients of the basis elements of the algebra $\mathcal{L}_m$ satisfy condition (2.4).

Let us prove now that the inverse assertion is also true. Suppose that (1.1) admits Lie algebra $\mathcal{L}_m = \langle e_1, \ldots, e_m \rangle$, whose basis elements have the form (2.3) and satisfy (2.4). Then, there is a change of variables (see, e.g., [8])

$$\bar{t} = t, \qquad \bar{x} = X(t, x, \mathbf{u}), \qquad \bar{\mathbf{u}} = \mathbf{U}(t, x, \mathbf{u}), \tag{2.8}$$

reducing basis elements of $\mathcal{L}_m$ to the form $e_i = \partial_{\bar{u}^i}$, $i = 1, \ldots, m$. In what follows, we drop the bars.

Now, (1.1) necessarily takes the form

$$\mathbf{u}_t = \widetilde{\mathbf{f}}(t, x, \mathbf{u}_1, \ldots, \mathbf{u}_n). \tag{2.9}$$

Differentiating (2.9) with respect to x and making the (nonlocal) change of dependent variables $\mathbf{v}_x = \mathbf{u}$, we finally get

$$\mathbf{v}_t = \frac{\partial}{\partial x}\widetilde{\mathbf{f}}(t, x, \mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}), \tag{2.10}$$

which completes the proof. □

Note 1. The fact that symmetry operators $e_1, \dots, e_m$ are of specific form (2.3) is crucial for the whole procedure of reducing a system of evolution equations to "conserved" form (2.1). If a symmetry group generated by some operator e_i does not preserve the temporal variable t (which means that the coefficient of ∂_t in e_i is nonzero for some i), then this operator cannot be reduced to the canonical form ∂_{v^i}, and the reduction routine does not work.

Theorem 2.4. *System* (1.1) *admits partial CLR if and only if it is invariant under r-dimensional commutative Lie algebra $\mathcal{L}_r = \langle e_1, \dots, e_r\rangle$, where*

$$\begin{aligned} e_i = {} & \xi_i(t, x, \mathbf{u}, \mathbf{w})\partial_x + \sum_{j=1}^{m}\eta_i^j(t, x, \mathbf{u}, \mathbf{w})\partial_{u^j} \\ & + \sum_{j=1}^{m}\zeta_i^j(t, x, \mathbf{u}, \mathbf{w})\partial_{w^j}, \quad i = 1, \dots, r \end{aligned} \tag{2.11}$$

and besides

$$\operatorname{rank}\begin{pmatrix} \xi_1 & \eta_1^1 & \dots & \eta_1^m & \zeta_1^1 & \dots & \zeta_1^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_r & \eta_r^1 & \dots & \eta_r^m & \zeta_r^1 & \dots & \zeta_r^m \end{pmatrix} = r. \tag{2.12}$$

3. Potential Symmetries

Potential symmetries of system of evolution equations (1.1) appear in the same way as they do for a single evolution equation. For simplicity, we consider the case of complete CLR. By force of (2.1), we can introduce the new dependent variable $\mathbf{v}$, so that

$$\mathbf{v}_t = \mathbf{F}(t, x, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}), \qquad \mathbf{v}_x = \mathbf{G}(t, x, \mathbf{u}). \tag{3.1}$$

Note that $\mathbf{v}$ is nonlocal variable since $\mathbf{v} = \partial_x^{-1}\mathbf{G}(t, x, \mathbf{u})$.

Suppose now that system (3.1) admits the Lie symmetry

$$\begin{aligned} t' &= T(t, x, \mathbf{u}, \mathbf{v}, \theta), & x' &= X(t, x, \mathbf{u}, \mathbf{v}, \theta), \\ \mathbf{u}' &= \mathbf{U}(t, x, \mathbf{u}, \mathbf{v}, \theta), & \mathbf{v}' &= \mathbf{V}(t, x, \mathbf{u}, \mathbf{v}, \theta), \end{aligned} \tag{3.2}$$

such that one of the derivatives

$$\frac{\partial T}{\partial v^i}, \quad \frac{\partial T}{\partial v^i}, \quad \frac{\partial \mathbf{U}}{\partial v^i}, \quad \frac{\partial \mathbf{V}}{\partial v^i}, \quad i = 1, \ldots, m, \tag{3.3}$$

does not vanish identically. Rewriting group (3.2) in terms of variables t, x, and $\mathbf{u}$ and taking into account that $\mathbf{v} = \partial_x^{-1}\mathbf{u}$ yield the nonlocal symmetry of the initial system of evolution equations (1.1). This means, in particular, that the symmetry in question cannot be obtained within the Lie infinitesimal approach. What we are going to prove is that this symmetry can be derived by regular Lie approach if the later is combined with the nonlocal transformation of the dependent variables.

Indeed, let system (1.1) admit complete CLR (2.1). In addition, we suppose that (1.1) possesses potential symmetry. Making the nonlocal change of dependant variables, $\mathbf{u} \to \mathbf{v}$,

$$\mathbf{v}_x = \mathbf{G}(t, x, \mathbf{u}), \qquad \mathbf{u} = \widetilde{\mathbf{G}}(t, x, \mathbf{v}_x), \qquad G\Big(t, x, \widetilde{\mathbf{G}}(t, x, \mathbf{v}_x)\Big) \equiv \mathbf{v}_x, \tag{3.4}$$

we rewrite (2.1) in the form (2.6). As initial system (1.1) admits a potential symmetry, system (3.1) is invariant under the Lie transformation group of the form (3.2).

Integrating (2.6) with respect to x yields system of evolution equations

$$\mathbf{v}_t = \widetilde{\mathbf{f}}(t, x, \mathbf{v}_1, \ldots, \mathbf{v}_n). \tag{3.5}$$

Next, we rewrite the Lie symmetry (3.2) by eliminating $\mathbf{u}$ according to (3.4) which yields

$$\begin{gathered} t' = T\Big(t, x, \widetilde{\mathbf{G}}(t, x, \mathbf{v}_x), \mathbf{v}, \theta\Big), \qquad x' = X\Big(t, x, \widetilde{\mathbf{G}}(t, x, \mathbf{v}_x), \mathbf{v}, \theta\Big), \\ \mathbf{v}' = \mathbf{V}\Big(t, x, \widetilde{\mathbf{G}}(t, x, \mathbf{v}_x), \mathbf{v}, \theta\Big). \end{gathered} \tag{3.6}$$

By construction, Lie transformation group (3.6) maps the set of solutions of (3.5) into itself. Consequently, (3.6) is the Lie group of contact symmetries of system of evolution equations (3.5).

It is a common knowledge that any contact symmetry of a system of PDEs boils down to the first prolongation of a classical symmetry [24]. Consequently, the derivatives of T, X, and $\mathbf{V}$ with respect to the third argument vanish identically and we get

$$t' = T(t, x, \mathbf{v}, \theta), \qquad x' = X(t, x, \mathbf{v}, \theta), \qquad \mathbf{v}' = \mathbf{V}(t, x, \mathbf{v}, \theta). \tag{3.7}$$

This group is nothing else than the standard Lie symmetry group of system (3.5).

The same assertion holds true for the case of partial CLR.

Theorem 3.1. *Let system of evolution equations* (1.1) *admit complete or partial CLR and be invariant under a potential symmetry. Then, there exists a (nonlocal) change of variables mapping* (1.1) *into another system of the form* (1.1) *so that the potential symmetry of* (1.1) *becomes the standard Lie symmetry of the transformed system.*

This assertion is, in fact, the no-go theorem for potential symmetries of systems of evolution equations. It states that the concept of potential symmetry does not produce essentially new symmetries. The system admitting potential symmetry is equivalent to the one admitting the standard Lie symmetry, which is the image of the potential symmetry in question.

However, there is more to it. Theorem 3.1 implies the regular algorithm for group classification system of nonlinear evolution equations admitting nonlocal symmetries. Again, for the sake of simplicity, we consider the case of complete CLR.

Indeed, let system of evolution equations (1.1) be invariant under $(m+1)$-dimensional Lie algebra $\mathcal{L}_{m+1} = \langle e_1, \ldots, e_{m+1} \rangle$. Here $e_1, \ldots, e_m$ are commuting operators of the form (2.3) and their coefficients satisfy constraint (2.4). Basis operator e_{m+1} is of the generic form

$$e_{m+1} = \tau(t, x, \mathbf{u})\partial_t + \xi_i(t, x, \mathbf{u})\partial_x + \sum_{j=1}^{m} \eta_i^j(t, x, \mathbf{u})\partial_{u^j}. \tag{3.8}$$

Making an appropriate change of variables, we can reduce the operators $e_1, \ldots, e_m$ to the canonical forms, namely, $e_i = \partial_{u^i}$, $i = 1, \ldots, m$. Then, system (1.1) necessarily takes the form (3.5).

Let (3.7) be the Lie transformation group generated by the symmetry operator e_{m+1}. Calculating the first prolongation of formulas (3.7) we get the transformation rule for the first derivatives of $\mathbf{v}$:

$$\mathbf{v}'_x = \mathbf{W}(t, x, \mathbf{v}, \mathbf{v}_x, \theta). \tag{3.9}$$

Now, we differentiate (2.6) with respect to x and make the following change of dependent variables:

$$\mathbf{w} = \mathbf{v}_x, \tag{3.10}$$

which yields

$$\mathbf{w}_t = \frac{\partial}{\partial x}\Big(\tilde{\mathbf{f}}(t, x, \mathbf{w}, \ldots, \mathbf{w}_{n-1})\Big). \tag{3.11}$$

Formulas (3.7), (3.9) provide the image of the transformation group (3.7) under the mapping (3.9), so that

$$t' = T(t, x, \mathbf{v}, \theta), \qquad x' = X(t, x, \mathbf{v}, \theta), \qquad \mathbf{w}'_x = \mathbf{W}(t, x, \mathbf{v}, \mathbf{w}, \theta). \tag{3.12}$$

Here $\mathbf{v} = \partial_x^{-1}\mathbf{w}$.

Consequently, if one of the derivatives, $\partial T/\partial v^i$, $\partial X/\partial v^i$, $\partial \mathbf{W}/\partial v^i$, does not vanish identically, then (3.12) is the nonlocal symmetry group of system of evolution equations (3.11).

The same line of reasoning applies to the case when system (1.1) admits partial CLR.

We summarize the above speculations in the form of the procedure for computation of nonlocal symmetries of systems of evolution equations associated with a given system of the form (1.1).

Let system of evolution equations (1.1) be invariant under N-dimensional Lie symmetry algebra $\mathcal{L}_N$. For simplicity, we consider the case of complete CLR.

Procedure 1. Classification of Potential Symmetries of (1.1)

(1) Calculate inequivalent subalgebras $\mathcal{M}$ of the algebra $\mathcal{L}_N$.

(2) Select those subalgebras $\mathcal{M}$, which contain commutative subalgebras $\mathcal{M}_m$ of operators of the form (2.3).

(3) For each commutative subalgebra $\mathcal{M}_m$ perform change of variables reducing its basis elements to the canonical forms $\partial_{v^1}, \dots, \partial_{v^m}$ and transform the initial system (1.1) accordingly.

(4) Perform nonlocal transformation (3.10).

(5) Eliminate "old" dependent variables $\mathbf{v}$ from (3.7) in order to derive symmetry group (3.12) of the transformed system of evolution equations (3.11).

(6) Verify that there is, at least, one derivative from the list $\partial T/\partial v^i$, $\partial X/\partial v^i$, $\partial \mathbf{W}/\partial v^i$ that does not vanish identically. If this is the case, then (3.12) is the nonlocal (potential) symmetry of (3.11).

The steps needed to implement the above procedure for the case of system of evolution equations admitting partial CLR are the same, the only difference is that intermediate formulas (3.7)–(3.12) are more cumbersome, since we need to distinguish between two sets of dependent variables $\mathbf{u}$ and $\mathbf{w}$ (see, (2.2)).

Note that by force of Theorems 2.3 and 2.4, any potential symmetry of equations of the form (1.1) can be obtained in the above-described manner.

As an example, we consider the Galilei-invariant nonlinear Schrödinger equation introduced in [25]

$$\begin{aligned} i\psi_t = \psi_{xx} + 2(x+i\alpha)^{-1}\psi_x - \left(\frac{i}{2}\right)(x+i\alpha) \\ + F\big(2i\alpha(x+i\alpha)\psi_x - (x-i\alpha)(\psi-\psi^*)\big), \end{aligned} \tag{3.13}$$

where $\psi = \phi(t,x) + i\varphi(t,x)$, $\psi^* = \phi(t,x) - i\varphi(t,x)$, $\alpha \neq 0$ is an arbitrary real constant, and F is an arbitrary complex-valued function. Equation (3.13) admits the Lie algebra of the Galilei group having the following basis operators [25]:

$$\begin{aligned} e_1 &= \partial_t, \\ e_2 &= \partial_\psi + \partial_{\psi^*}, \\ e_3 &= (x+i\alpha)^{-1}\partial_\psi + (x-i\alpha)^{-1}\partial_{\psi^*}, \\ e_4 &= \partial_x - \left(t + (x+i\alpha)^{-1}\psi\right)\partial_\psi - \left(t + (x-i\alpha)^{-1}\psi^*\right)\partial_{\psi^*}. \end{aligned} \tag{3.14}$$

Operators e_2, e_3 commute and the rank of the matrix of coefficients of operators ∂_t, ∂_x, ∂_ψ, and ∂_{ψ^*} is equal to 2. Consequently, there is a change of variables that reduces e_2, e_3 to canonical forms ∂_u, ∂_v. Indeed, making the change of variables

$$u(t,x) = \left(\frac{1}{2}\right)(\psi + \psi^*), \qquad v(t,x) = (2i\alpha)^{-1}\left(x^2 + \alpha^2\right)(\psi - \psi^*), \tag{3.15}$$

transforms e_1, e_2 to become $e_1 = \partial_u$, $e_2 = \partial_v$. So we can apply Procedure 1 to (3.13) transformed according to (3.15). As a result, the transformed operator e_3 becomes the potential symmetry of the transformed nonlinear system of two evolution equations.

4. Some Generalizations

Denote the class of partial differential equations of the form (1.1) as $\mathfrak{E}_n$. Then any system of the form

$$\mathbf{u}_t = f(t, x, \mathbf{u}_1, \ldots, \mathbf{u}_n) \tag{4.1}$$

(i) belongs to $\mathfrak{E}_n$, and (ii) its image under nonlocal transformation $\mathbf{u} = \mathbf{v}_x$ also belongs to $\mathfrak{E}_n$. Existence of such nonlocal transformation is in the core of our approach to classifying nonlocal symmetries of systems of evolution equations.

It is not but natural to ask whether there are other types of nonlocal transformations of the class $\mathfrak{E}_n$ that can be utilized to generate nonlocal symmetries. Remarkably, such nonlocal transformations do exist. Sokolov [26] put forward the idea of group approach to generating such transformations for a single evolution equation. It is straightforward to modify his approach to handle systems of evolution equations, as well. As an illustration, we consider system (4.1). It is invariant under the m-dimensional Lie algebra $\mathcal{L}_m = \langle \partial_{u^1}, \ldots, \partial_{u^m} \rangle$. The simplest set of $(m+2)$ functionally-independent invariants of the algebra $\mathcal{L}_m$ can be chosen as follows: t, x, u_x^1, …, u_x^m. Now, we define the transformation

$$\bar{t} = T(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots), \qquad \bar{x} = X(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots), \qquad \bar{\mathbf{u}} = \mathbf{U}(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots), \tag{4.2}$$

so that T, X, $\mathbf{U}$ are invariants of the symmetry group of the system under study. In the case under consideration, we have $T = t$, $X = x$, and $\mathbf{U} = \mathbf{u}_x$. As we established in Section 2, applying this transformation to any equation of the form (4.1) yields system of evolution equations that belongs to $\mathfrak{E}_n$. What is more, is that Lie symmetry group of (4.1) is mapped into symmetry group of the transformed system and some of the basis operators of the latter become nonlocal ones.

Consider as the next example system of evolution equations

$$\mathbf{u}_t = f(t, x, \mathbf{u}_2, \ldots, \mathbf{u}_n), \quad n \geq 3. \tag{4.3}$$

This system is invariant under the $2m$-dimensional Lie algebra $\mathcal{L}_{2m} = \langle \partial_{u^1}, \ldots, \partial_{u^m}, x\partial_{u^1}, \ldots, x\partial_{u^m} \rangle$. The simplest set of $m+2$ functionally independent first integrals reads as $t, x, u^1_{xx}, \ldots, u^m_{xx}$. Consequently, change of variables (4.2) takes the form

$$t = t, \qquad x = x, \qquad \mathbf{v} = \mathbf{u}_{xx}. \tag{4.4}$$

Note that we dropped the bars and replaced $\overline{\mathbf{u}}$ with $\mathbf{v}$.

Transforming (4.3) according to (4.4) we get

$$\left(\left(\partial_x^{-1} \right)^2 \mathbf{v} \right)_t = f(t, x, \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-2}) \tag{4.5}$$

or, equivalently,

$$\left(\partial_x^{-1} \right)^2 \left(\mathbf{v}_t - \partial_x^2 f(t, x, \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-2}) \right) = 0. \tag{4.6}$$

Integrating twice yields

$$\mathbf{v}_t = \partial_x^2 f(t, x, \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-2}). \tag{4.7}$$

Note that integration constants $\mathbf{w}^1(t)x + \mathbf{w}^2(t)$ are absorbed by the function $\mathbf{v}$.

So that nonlocal transformation (4.4) maps a subset of equations from $\mathfrak{E}_n$ into $\mathfrak{E}_n$. Consequently, it can be used to generate nonlocal symmetries of the initial system (4.3).

Let system (4.3) be invariant under the Lie transformation group

$$t' = T(t, x, \mathbf{u}, \theta), \qquad x = X(t, x, \mathbf{u}, \theta), \qquad \mathbf{u} = \mathbf{U}(t, x, \mathbf{u}, \theta). \tag{4.8}$$

Computing the second prolongation of the above formulas, we get the transformation law for the functions $\mathbf{v} = \mathbf{u}_{xx}$,

$$\mathbf{v}' = \mathbf{V}(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{v}, \theta). \tag{4.9}$$

Combining (4.8) and (4.9) yields the symmetry group of system of evolution equations (4.7),

$$t' = T(t, x, \mathbf{u}, \theta), \qquad x = X(t, x, \mathbf{u}, \theta), \qquad \mathbf{v}' = \mathbf{V}(t, x, \mathbf{u}, \mathbf{u}_x, \mathbf{v}, \theta), \tag{4.10}$$

where $\mathbf{u} = (\partial_x^{-1})^2 \mathbf{v}$ are nonlocal variables. Now, if one of the derivatives

$$\frac{\partial T}{\partial u^i}, \quad \frac{\partial X}{\partial u^i}, \quad \frac{\partial \mathbf{V}}{\partial u^i}, \quad \frac{\partial \mathbf{V}}{\partial u^i_x} \tag{4.11}$$

does not vanish identically, then (4.10) is the nonlocal symmetry group of system of evolution equations (4.7).

It is important to emphasize that the symmetry algebra $\mathcal{L}_m$ is not obliged to be commuting. The necessary condition is that the corresponding transformation group has to preserve the temporal variable, t, that is, basis elements of $\mathcal{L}_m$ have to be of the form

$$Q = \xi(t, x, \mathbf{u})\partial_x + \sum_{j=1}^{m} \eta^j(t, x, \mathbf{u})\partial_{u^j}. \tag{4.12}$$

As an illustration, consider the following system of second-order evolution equations:

$$u_t^i = u_x^i f^i\left(t, x, \frac{u_{xx}^1}{u_x^1}, \ldots, \frac{u_{xx}^m}{u_x^m}\right), \quad i = 1, \ldots, m. \tag{4.13}$$

This system is invariant under the $2m$-dimensional Lie algebra $\mathcal{L}_{2m} = \langle \partial_{u^1}, \ldots, \partial_{u^m}, u^1\partial_{u^1}, \ldots, u^m\partial_{u^m}\rangle$. Note that the algebra $\mathcal{L}_{2m}$ is not commutative. The set of $m+2$ invariants of the algebra $\mathcal{L}_{2m}$ can be chosen as follows:

$$t, \quad x, \quad \frac{u_{xx}^1}{u_x^1}, \ldots, \frac{u_{xx}^m}{u_x^m}. \tag{4.14}$$

Making the change of variables

$$t = t, \quad x = x, \quad v^1 = \frac{u_{xx}^1}{u_x^1}, \ldots, v^m = \frac{u_{xx}^m}{u_x^m}, \tag{4.15}$$

we rewrite (4.13) in the form

$$\frac{\partial}{\partial t}\left(\partial_x^{-1} \exp\left(\partial_x^{-1} v^i\right)\right) = \exp\left(\partial_x^{-1} v^i\right) f^i\left(t, x, v^1, \ldots, v^m\right), \quad i = 1, \ldots, m. \tag{4.16}$$

Taking into account that the operators $\partial/\partial t$ and ∂_x^{-1} commute, differentiating (4.16) with respect to x, and replacing $\mathbf{v}$ with $\mathbf{w}_x$, we finally get

$$w_t^i = w_x^i f^i\left(t, x, w_x^1, \ldots, w_x^m\right) + \frac{\partial}{\partial x} f^i\left(t, x, w_x^1, \ldots, w_x^m\right), \quad i = 1, \ldots, m. \tag{4.17}$$

The above system is obtained from the initial one through the change of dependent variables $u^i = \partial_x^{-1} \exp(w^i)$, $i = 1, \ldots, m$. Consequently, if system (4.13) admits symmetry (4.8), then system (4.17) admits the following transformation group:

$$t' = T(t, x, \mathbf{u}, \theta), \qquad x = X(t, x, \mathbf{u}, \theta), \qquad \mathbf{w} = \mathbf{W}(t, x, \mathbf{u}, \mathbf{w}, \theta) \tag{4.18}$$

with $u^i = \partial_x^{-1} \exp(w^i)$, $i = 1, \ldots, m$. Again, if one of the derivatives

$$\frac{\partial T}{\partial u^i}, \quad \frac{\partial X}{\partial u^i}, \quad \frac{\partial \mathbf{V}}{\partial u^i}, \quad \frac{\partial \mathbf{W}}{\partial u_x^i} \tag{4.19}$$

does not vanish identically, then (4.18) is the nonlocal invariance group of system of evolution equations (4.16).

The procedure for calculation of nonlocal symmetries of system (1.1) suggested in the previous section yields those nonlocal symmetries which are potential, since the nonlocal transformation was chosen *a priori*. Allowing for a nonlocal transformation to be determined by symmetry group of the system under study yields a more general algorithm for constructing nonlocal symmetries.

Let system of evolution equations (1.1) be invariant under N-dimensional Lie symmetry algebra $\mathcal{L}_N$. Then the following procedure can be used to construct nonlocal symmetries of (1.1).

Procedure 2. Classification of Nonlocal Symmetries of (1.1)

(1) Calculate inequivalent subalgebras $\mathcal{M}$ of the algebra $\mathcal{L}_N$.

(2) Select those subalgebras $\widetilde{\mathcal{M}}$, which contain basis elements $e_1, \ldots, e_r$ of the form (4.12).

(3) For each $\widetilde{\mathcal{M}}$ construct $r+2$ functionally independent invariants $\omega^t(t, x, \mathbf{u}, \mathbf{u}_x, \ldots)$, $\omega^x(t, x, \mathbf{u}, \mathbf{u}_x, \ldots), \omega^1(t, x, \mathbf{u}, \mathbf{u}_x, \ldots), \ldots, \omega^r(t, x, \mathbf{u}, \mathbf{u}_x, \ldots)$ and make change of variables

$$\bar{t} = \omega^t, \qquad \bar{x} = \omega^x, \qquad \bar{u}^i = \omega^i, \quad i = 1, \ldots, r. \tag{4.20}$$

(4) Eliminate "old" dependent variables $\mathbf{u}$ from (4.20) in order to derive symmetry group $\mathcal{G}$ of the transformed system of evolution equations.

(5) Verify that there is, at least, one function from the list $\{\omega^t, \omega^x, \omega^1, \ldots, \omega^r\}$ that depends on u^i for some $1 \le i \le r$. If this is the case, then $\mathcal{G}$ is the nonlocal symmetry of (3.11).

5. Conclusion

One of the principal results of the paper is Theorem 3.1 stating that any potential symmetry of system of evolution equations (1.1) reduces to a Lie symmetry by an appropriate nonlocal transformation of dependent and independent variables. The nonlocal transformation in question is a superposition of the local change of variables

$$\bar{t} = t, \qquad \bar{x} = X(t, x, \mathbf{u}), \qquad \bar{\mathbf{u}} = \mathbf{U}(t, x, \mathbf{u}) \tag{5.1}$$

and of the nonlocal change of dependent variables

$$\mathbf{v} = \bar{\mathbf{u}}_x. \tag{5.2}$$

The explicit form of transformations (5.1) is defined by the Lie symmetry admitted by the corresponding system (1.1).

We obtain as a by-product exhaustive characterization of systems (1.1) that can be represented in the form of conservation law(s), in terms of the Lie symmetries preserving the temporal variable, t,

$$t' = t, \qquad x' = X(t, x, \mathbf{u}, \theta), \qquad \mathbf{u}' = \mathbf{U}(t, x, \mathbf{u}, \theta) \tag{5.3}$$

(see Theorems 2.3 and 2.4).

In Section 4, we generalize the above reasoning in order to obtain nonlocal symmetries which are not potential. The basic idea is replacing (5.2) with a more general nonlocal transformation. This transformation is determined by invariants of the Lie symmetry algebra of the system under study.

We intend to devote one of our future publications to systematic study of nonlocal symmetries of systems of nonlinear evolution equations (1.1) within the framework of the approach developed in Section 4.

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