Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2012, Article ID 692030, 41 pages
doi:10.1155/2012/692030

Research Article

Spacetime Deformation-Induced Inertia Effects

Gagik Ter-Kazarian

Division of Theoretical Astrophysics, Byurakan Astrophysical Observatory, 378433 Byurakan, Armenia

Correspondence should be addressed to Gagik Ter-Kazarian, gago_50@yahoo.com

Received 4 March 2012; Revised 18 April 2012; Accepted 22 April 2012

Academic Editor: Giorgio Kaniadakis

We construct a toy model of spacetime deformation-induced inertia effects, in which we prescribe to each and every particle individually a new fundamental constituent of hypothetical 2D, so-called *master space* (MS), subject to certain rules. The MS, embedded in the background 4D-spacetime, is an indispensable companion to the particle of interest, without relation to every other particle. The MS is not measurable directly, but we argue that a deformation (distortion of local internal properties) of MS is the origin of inertia effects that can be observed by us. With this perspective in sight, we construct the alternative *relativistic theory of inertia*. We go beyond the hypothesis of locality with special emphasis on distortion of MS, which allows to improve essentially the standard metric and other relevant geometrical structures referred to a noninertial frame in Minkowski spacetime for an arbitrary velocities and characteristic acceleration lengths. Despite the totally different and independent physical sources of gravitation and inertia, this approach furnishes justification for the introduction of the *weak* principle of equivalence (WPE), that is, the universality of free fall. Consequently, we relate the inertia effects to the more general post-Riemannian geometry.

1. Introduction

Governing the motions of planets, the fundamental phenomena of gravitation and inertia reside at the very beginning of the physics. More than four centuries passed since the famous far-reaching discovery of Galileo (in 1602–1604) that all bodies fall at the same rate [1], which led to an early empirical version of the suggestion that gravitation and inertia may somehow result from a single mechanism. Besides describing these early gravitational experiments, Newton in *Principia Mathematica* [2] has proposed a comprehensive approach to studying the relation between the gravitational and inertial masses of a body. In Newtonian mechanics, masses are simply *placed* in absolute space and time, which remain external to them. That is, the internal state of a Newtonian point particle, characterized by its inertial mass, has no immediate connection with the particles' external state in absolute space and time, characterized by its position and velocity. Ever since, there is an ongoing quest to understand the reason for the universality of the gravitation and inertia, attributing to the WPE, which establishes the independence of free-fall trajectories of the internal composition and structure of bodies. In other words, WPE states that all bodies at the same spacetime point in a given gravitational field will undergo the same acceleration. However, the nature of the relationship of gravity and inertia continues to elude us and, beyond the WPE, there has been little progress in discovering their true relation. Such interesting aspects, which deserve further investigations, unfortunately, have attracted little attention in subsequent developments. Only hypothesis, which in some extent relates inertia and matter, is the Mach principle, see for example, [3–15], but in the same time it is a subject to many uncertainties. The Mach's ideas on inertial induction were proposed as the theoretical mechanism for generating the inertial forces felt during acceleration of a reference frame. The ensuing problem of the physical origin of inertial forces led Mach to hypothesize that inertial forces were to be of gravitational origin, occurring only during acceleration relative to the *fixed stars*. In this model the ratio of inertial to gravitational mass will depend on the average distribution of mass in the universe, in effect making gravitational constant a function of the mass distribution in the universe. The general relativity (GR), which preserves the idea of relativity of all kinds of motion, is built on the so-called *strong* principle (SPE) that the only influence of gravity is through the metric and can thus (apart from tidal effects) be locally, approximately transformed away by going to an appropriately accelerated reference frame. Despite the advocated success of GR, it is now generally acknowledged, however, that what may loosely be termed Mach principle is not properly incorporated into GR. In particular, the origin of inertia remains essentially the same as in Newtonian physics. Brans thorough analysis [4–6] has shown that no extra inertia is induced in a body as a result of the presence of other bodies. Various attempts at the resolution of difficulties that are encountered in linking Machs principle with Einsteins theory of gravitation have led to many interesting investigations. For example, by [14] is shown that the GR can be locally embedded in a Ricci-flat 5D manifold such that every solution of GR in 4D can be locally embedded in a Ricci-flat 5D manifold and that the resulting inertial mass of a test particle varies in space time. Anyhow, the difficulty is brought into sharper focus by considering the laws of inertia, including their quantitative aspects. That is, Mach principle and its modifications do not provide a quantitative means for computing the inertial forces. At present, the variety of consequences of the precision experiments from astrophysical observations makes it possible to probe this fundamental issue more deeply by imposing the constraints of various analyses. Currently, the observations performed in the Earth-Moon-Sun system [16–35], or at galactic and cosmological scales [36–41], probe more deeply both WPE and SPE. The intensive efforts have been made, for example, to clear up whether the rotation state would affect the trajectory of test particle. Shortly after the development of the work by [22], in which is reported that, in weighing gyros, it would be a violation of WPE, the authors of [23–26] performed careful weighing experiments on gyros with improved precision but found only null results which are in disagreement with the report of [22]. The interferometric free-fall experiments by [27, 28] again found null results in disagreement with [22]. For rotating bodies, the ultraprecise Gravity Probe B experiment [29–34], which measured the frame-dragging effect and geodetic precession on four quartz gyros, has the best accuracy. GP-B serves as a starting point for the measurement of the gyrogravitational factor of particles, whereas the gravitomagnetic field, which is locally equivalent to a *Coriolis field* and generated by the absolute rotation of a body, has been measured too. This, with its superb accuracy, verifies WPE for unpolarized bodies to an ultimate precision—a four-order improvement on the noninfluence of rotation on the trajectory, and ultraprecision on the rotational equivalence [35]. Moreover, the theoretical models may indicate cosmic polarization rotations which are being looked for and tested in the CMB experiments [40]. To look into the future, measurement of the gyrogravitational ratio of particle would be a further step, see [41] and references therein, towards probing the microscopic origin of gravity. Also, the inertia effects in fact are of vital interest for the phenomenological aspects of the problem of neutrino oscillations; see, for example, [42–56]. All these have evoked the study of the inertial effects in an accelerated and rotated frame. In doing this, it is a long-established practice in physics to use the hypothesis of locality for extension of the Lorentz invariance to accelerated observers in Minkowski spacetime [57, 58]. This in effect replaces the accelerated observer by a continuous infinity of hypothetical momentarily comoving inertial observers along its wordline. This assumption, as well as its restricted version, so-called clock hypothesis, which is a hypothesis of locality only concerned about the measurement of time, is reasonable only if the curvature of the wordline could be ignored. As long as all relevant length scales in feasible experiments are very small in relation to the huge acceleration lengths of the tiny accelerations we usually experience, the curvature of the wordline could be ignored and that the differences between observations by accelerated and comoving inertial observers will also be very small. In this line, in 1990, Hehl and Ni proposed a framework to study the relativistic inertial effects of a Dirac particle [59], in agreement with [60–62]. Ever since this question has become a major preoccupation of physicists; see, for example, [63–84]. Even this works out, still, it seems quite clear that such an approach is a work in progress, which reminds us of a puzzling underlying reality of inertia and that it will have to be extended to describe physics for arbitrary accelerated observers. Beyond the WPE, there is nothing convincing in the basic postulates of physics for the origin and nature of inertia to decide on the issue. Despite our best efforts, all attempts to obtain a true knowledge of the geometry related to the noninertial reference frames of an arbitrary observer seem doomed, unless we find *a physical principle the inertia might refer to*, and that a working alternative *relativistic theory of inertia* is formulated. Otherwise one wanders in a darkness. The problem of inertia stood open for nearly four centuries, and the physics of inertia is still an unknown exciting problem to be challenged and allows various attempts. In particular, the inertial forces are not of gravitational origin within GR as it was proposed by Einstein in 1918 [85], because there are many controversies to question the validity of such a description [57, 58, 60–91]. The experiments by [87–90], for example, tested the key question of anisotropy of inertia stemming from the idea that the matter in our galaxy is not distributed isotropically with respect to the earth, and hence if the inertia is due to gravitational interactions, then the inertial mass of a body will depend on the direction of its acceleration with respect to the direction towards the center of our galaxy. However, these experiments do not found such anisotropy of mass. The most sensitive test is obtained in [88, 89] from a nuclear magnetic resonance experiment with an Li^7 nucleus of spin $I = 3/2$. The magnetic field was of about 4700 gauss. The south direction in the horizontal plane points within 22 degrees towards the center of our galaxy, and 12 hours later this same direction along the earth's horizontal plane points 104 degrees away from the galactic center. If the nuclear structure of Li^7 is treated as a single $P_{3/2}$ proton in a central nuclear potential, the variation Δm of mass with direction, if it exists, was found to satisfy $\Delta m/m \leq 10^{-20}$. This is by now very strong evidence that there is no anisotropy of mass which is due to the effects of mass in our galaxy. Another experimental test [91] using nuclear-spin-polarized $^9Be^+$ ions also gives null result on spatial anisotropy and thus supporting local Lorentz invariance. This null result represents a decrease in the limits set by [88–90] on a spatial anisotropy by a factor of about 300. Finally, another theoretical objection is that if the curvature of Riemannian space is associated with gravitational interaction, then it would indicate a universal feature equally suitable for action on all the matter fields at once. The source of the curvature as conjectured in GR is the energy-momentum tensor of matter, which is rather applicable for gravitational fields but not for inertia, since the inertia is dependent solely on the state of motion of individual test particle or coordinate frame of interest. In case of accelerated motion, unlike gravitation, the curvature of spacetime might arise entirely due to the inertial properties of the Lorentz-rotated frame of interest, that is, a "fictitious gravitation" which can be globally removed by appropriate coordinate transformations [57]. This refers to the particle of interest itself, without relation to other systems or matter fields.

On the other hand, a general way to deform the spacetime metric with constant curvature has been explicitly posed by [92–94]. The problem was initially solved only for 3D spaces, but consequently it was solved also for spacetimes of any dimension. It was proved that any semi-Riemannian metric can be obtained as a deformation of constant curvature matric, this deformation being parameterized by a 2-form. A novel definition of spacetime metric deformations, parameterized in terms of scalar field matrices, is proposed by [95]. In a recent paper [96], we construct the *two-step spacetime deformation* (TSSD) theory which generalizes and, in particular cases, fully recovers the results of the conventional theory of spacetime deformation [92–95]. All the fundamental gravitational structures in fact—the metric as much as the coframes and connections—acquire the TSSD-induced theoretical interpretation. The TSSD theory manifests its virtue in illustrating how the curvature and torsion, which are properties of a connection of geometry under consideration, come into being. Conceptually and techniquewise this method is versatile and powerful. For example, through a nontrivial choice of explicit form of a *world-deformation tensor*, which we have at our disposal, in general, we have a way to deform that the spacetime displayed different connections, which may reveal different post-Riemannian spacetime structures as a corollary, whereas motivated by physical considerations, we address the essential features of the theory of teleparallel gravity-TSSD-$GR_{\|}$ and construct a consistent TSSD-U_4 Einstein-Cartan (EC) theory, with a dynamical torsion. Moreover, as a preliminary step, in the present paper we show that by imposing different appropriate physical constraints upon the spacetime deformations, in this framework we may reproduce the term in the well-known Lagrangian of pseudoscalar-photon interaction theory, or terms in the Lagrangians of pseudoscalar theories [41, 97–101], or in modification of electrodynamics with an additional external constant vector coupling [102, 103], as well as in case of intergrand for topological invariant [104] or in case of pseudoscalar-gluon coupling occurred in QCD in an effort to solve the strong CP problem [105–107]. Next, our purpose is to carry out some details of this program to probe the origin and nature of the phenomenon of inertia. We ascribe the inertia effects to the geometry itself but as having a nature other than gravitation. In doing this, we note that aforementioned examples pose a problem for us that physical space has intrinsic geometrical and inertial properties beyond 4D spacetime derived from the matter contained therein. Therefore, we should conceive of two different spaces: one would be 4D background spacetime, and another one should be 2D so-called *master space* (MS), which, embedded in the 4D background space, is an indispensable individual companion to the particle, without relation to the other matter. That is, the key to our construction procedure is an assignment in which we prescribe to each and every particle individually a new fundamental constituent of hypothetical MS, subject to certain rules. *In the contrary to Mach principle, the particle has to live with MS companion as an intrinsic property devoid of any external influence*. The geometry of MS is a new physical entity, with degrees of freedom and a dynamics of its own. This together with the idea that the inertia effects arise as a deformation (distortion of local internal properties) of MS is the highlights of the alternative *relativistic theory of inertia* (RTI), whereas we build up the *distortion complex* (DC), yielding a distortion of MS, and show how DC restores the *world-deformation tensor*, which still has to be put in [96] by hand. Within this scheme, the MS was presumably allowed to govern the motion of a particle of interest in the background space. In simple case, for example, of motion of test particle in the free 4D Minkowski space, the suggested heuristic inertia scenario is reduced to the following: unless a particle was acted upon by an unbalanced force, the MS is being flat. This causes a free particle in 4D Minkowski space to tend to stay in motion of uniform speed in a straight line or a particle at rest to tend to stay at rest. As we will see, an alteration of uniform motion of a test particle under the unbalanced net force has as an inevitable consequence of a distortion of MS. This becomes an immediate cause of arising both the universal *absolute* acceleration of test particle and associated *inertial force* in M_4 space. This, we might expect, holds on the basis of an intuition founded on past experience limited to low velocities, and these two features were implicit in the ideas of Galileo and Newton as to the nature of inertia. Thereby the major premise is that the centrifugal endeavor of particles to recede from the axis of rotation is directly proportional to the quantity of the *absolute* circular acceleration, which, for example, is exemplified by the concave water surface in Newton's famous rotating bucket experiments. In other words, it takes force to disturb an inertia state, that is, to make an *absolute* acceleration. In this framework, the *relative* acceleration (in Newton's terminology) (both magnitude and direction), to the contrary, cannot be the cause of a distortion of MS and, thus, it does not produce the inertia effects. The real *inertia* effects, therefore, can be an empirical indicator of *absolute* acceleration. The treatment of deformation/distortion of MS is instructive because it contains the essential quantitative elements for computing the *relativistic inertial force* acting on an arbitrary observer. On the face of it, the hypothesis of locality might be somewhat worrisome, since it presents strict restrictions, replacing the distorted MS by the flat MS. Therefore, it appears natural to go beyond the hypothesis of locality with spacial emphasis on distortion of MS. This, we might expect, will essentially improve the standard metric, and so forth, referred to a noninertial system of an arbitrary observer in Minkowski spacetime. Consequently, we relate the inertia effects to the more general post-Riemannian geometry. The crucial point is to observe that, in spite of totally different and independent physical sources of gravitation and inertia, the RTI furnishes justification for the introduction of the WPE [108, 109]. However, this investigation is incomplete unless it has conceptual problems for further motivation and justification of introducing the fundamental concept of MS. The way we assigned such a property to the MS is completely ad hoc and there are some obscure aspects of this hypothesis. All these details will be further motivated and justified in subsequent paper. The outline of the rest of the present paper is as follows. In Section 2 we briefly revisit the theory of TSSD and show how it can be useful for the theory of electromagnetism and charged particles. In Section 3, we explain our view of what is the MS and lay a foundation of the RLI. A general deformation/distortion of MS is described in Section 4. In Section 5, we construct the RTI in the background 4D Minkowski space. In Section 6, we go beyond the hypothesis of locality, whereas we compute the improved metric and other relevant geometrical structures in noninertial system of arbitrary accelerating and rotating observer in Minkowski spacetime. The case of semi-Riemann background space V_4 is dealt with in Section 7, whereby we give justification for the introduction of the WPE on the theoretical basis. In Section 8, we relate the RTI to more general post-Riemannian geometry. The concluding remarks are presented in Section 9. We will be brief and often ruthlessly suppress the indices without notice. Unless otherwise stated we take natural units, $h = c = 1$.

2. TSSD Revisited: Preliminaries

For the benefit of the reader, this section contains some of the necessary preliminaries on generic of the key ideas behind the TSSD [96], which needs one to know in order to understand the rest of the paper. We adopt then its all ideas and conventions. The interested reader is invited to consult the original paper for further details. It is well known that the notions of space and connections should be separated; see, for example, [110–113]. The curvature and torsion are in fact properties of a connection, and many different connections are allowed to exist in the same spacetime. Therefore, when considering several connections with different curvatures and torsions, one takes spacetime simply as a manifold and connections as additional structures. From this view point in a recent paper [96] we have tackled the problem of spacetime deformation. In order to relate local Lorentz symmetry to curved spacetime, there is, however, a need to introduce the soldering tools, which are the linear frames and forms in tangent fiber-bundles to the external curved space, whose components are so-called tetrad (vierbein) fields. To start with, let us consider the semi-Riemann space, V_4, which has at each point a tangent space, $\breve{T}_{\breve{x}}V_4$, spanned by the anholonomic orthonormal frame field, $\breve{e}$, as a shorthand for the collection of the 4-tuplet $(\breve{e}_0, \dots, \breve{e}_3)$, where $\breve{e}_a = \breve{e}^{\mu}_{a}\breve{\partial}_{\mu}$. All magnitudes related to the space, V_4, will be denoted by an over "$\breve{}$". These then define a dual vector, $\breve{\vartheta}$, of differential forms, $\breve{\vartheta} = \begin{pmatrix} \breve{\vartheta}^0 \\ \vdots \\ \breve{\vartheta}^3 \end{pmatrix}$, as a shorthand for the collection of the $\breve{\vartheta}^b = \breve{e}^b_{\mu} d\breve{x}^{\mu}$, whose values at every point form the dual basis, such that $\breve{e}_a \rfloor \breve{\vartheta}^b = \delta^b_a$, where $\rfloor$ denotes the interior product, namely, this is a C^{∞}-bilinear map $\rfloor : \Omega^1 \to \Omega^0$ where Ω^p denotes the C^{∞}-modulo of differential p-forms on V_4. In components $\breve{e}^{\mu}_a \breve{e}^b_{\mu} = \delta^b_a$. On the manifold, V_4, the tautological tensor field, $i\breve{d}$, of type $(1,1)$ can be defined which assigns to each tangent space the identity linear transformation. Thus for any point $\breve{x} \in V_4$, and any vector $\breve{\xi} \in \breve{T}_{\breve{x}}V_4$, one has $i\breve{d}(\breve{\xi}) = \breve{\xi}$. In terms of the frame field, the $\breve{\vartheta}^a$ give the expression for $i\breve{d}$ as $i\breve{d} = \breve{e}\breve{\vartheta} = \breve{e}_0 \otimes \breve{\vartheta}^0 + \cdots \breve{e}_3 \otimes \breve{\vartheta}^3$, in the sense that both sides yield $\breve{\xi}$ when applied to any tangent vector $\breve{\xi}$ in the domain of definition of the frame field. One can also consider general transformations of the linear group, $GL(4, R)$, taking any base into any other set of four linearly independent fields. The notation $\{\breve{e}_a, \breve{\vartheta}^b\}$ will be used hereinafter for general linear frames. The holonomic metric can be defined in the semi-Riemann space, V_4, as

$$\breve{g} = \breve{g}_{\mu\nu}\breve{\vartheta}^{\mu} \otimes \breve{\vartheta}^{\nu} = \breve{g}(\breve{e}_{\mu}, \breve{e}_{\nu})\breve{\vartheta}^{\mu} \otimes \breve{\vartheta}^{\nu}, \tag{2.1}$$

with components $\breve{g}_{\mu\nu} = \breve{g}(\breve{e}_{\mu}, \breve{e}_{\nu})$ in the dual holonomic base $\{\breve{\vartheta}^{\mu} \equiv d\breve{x}^{\mu}\}$. The anholonomic orthonormal frame field, $\breve{e}$, relates $\breve{g}$ to the tangent space metric, $o_{ab} = \mathrm{diag}(+---)$, by $o_{ab} = \breve{g}(\breve{e}_a, \breve{e}_b) = \breve{g}_{\mu\nu}\breve{e}^{\mu}_a \breve{e}^{\nu}_b$, which has the converse $\breve{g}_{\mu\nu} = o_{ab}\breve{e}^a_{\mu}\breve{e}^b_{\nu}$ because $\breve{e}^{\mu}_a \breve{e}^a_{\nu} = \delta^{\mu}_{\nu}$.

For reasons that will become clear in the sequel, next we write the norm, ds, of the infinitesimal displacement, dx^{μ}, on the general smooth differential 4D-manifold, $\mathcal{M}_4$, in terms of the spacetime structures of V_4, as

$$ds = e\vartheta = \Omega^{\nu}_{\mu}\breve{e}_{\nu} \otimes \breve{\vartheta}^{\mu} = \Omega^{a}_{b}\breve{e}_a \otimes \breve{\vartheta}^b = e_{\rho} \otimes \vartheta^{\rho} = e_a \otimes \vartheta^a \in \mathcal{M}_4, \tag{2.2}$$

where $\Omega^{\nu}_{\mu} = \pi^{\rho}_{\mu}\pi^{\nu}_{\rho}$ is the world *deformation tensor*, $e = \{e_a = e^{\rho}_a e_{\rho}\}$ is the frame field, and $\vartheta = \{\vartheta^a = e^a_{\rho}\vartheta^{\rho}\}$ is the coframe field defined on $\mathcal{M}_4$, such that $e_a \rfloor \vartheta^b = \delta^b_a$, or in components,

$e^{\mu}_{a}e^{b}_{\mu} = \delta^{b}_{a}$; also the procedure can be inverted, $e^{a}_{\rho}e^{\sigma}_{a} = \delta^{\sigma}_{\rho}$. Hence the deformation tensor, $\Omega^{a}_{b} = \pi^{a}_{c}\pi^{c}_{b} = \Omega^{\nu}_{\mu}\check{e}^{a}_{\nu}\check{e}^{\mu}_{b}$, yields local tetrad deformations:

$$e_c = \pi^{a}_{c}\check{e}_a, \qquad \vartheta^c = \pi^{c}_{b}\check{\vartheta}^b, \qquad e\vartheta = e_a \otimes \vartheta^a = \Omega^{a}_{b}\check{e}_a \otimes \check{\vartheta}^b. \tag{2.3}$$

The components of the general spin connection then transform *inhomogeneously* under a local tetrad deformations (2.3):

$$\omega^{a}_{b\mu} = \pi^{a}_{c}\check{\omega}^{c}_{d\mu}\pi^{d}_{b} + \pi^{a}_{c}\partial_{\mu}\pi^{c}_{b}. \tag{2.4}$$

This is still a passive transformation, but with inverted factor ordering. The matrices $\pi(x) := (\pi^{a}_{b})(x)$ are called *first deformation matrices*, and the matrices $\gamma_{cd}(x) = o_{ab}\pi^{a}_{c}(x)\pi^{b}_{d}(x)$, *second deformation matrices*. The matrices $\pi^{a}_{c}(x) \in \mathrm{GL}(4,R)$ for all x, in general, give rise to right cosets of the Lorentz group; that is, they are the elements of the quotient group $\mathrm{GL}(4,R)/\mathrm{SO}(3,1)$, because the Lorentz matrices, Λ^{a}_{c}, leave the Minkowski metric invariant. A right-multiplication of $\pi(x)$ by a Lorentz matrix gives an other deformation matrix. If we deform the tetrad according to (2.3), in general, we have two choices to recast metric as follows: either writing the deformation of the metric in the space of tetrads or deforming the tetrad field:

$$g = o_{ab}\pi^{a}_{c}\pi^{b}_{d}\check{\vartheta}^c \otimes \check{\vartheta}^d = \gamma_{cd}\check{\vartheta}^c \otimes \check{\vartheta}^d = o_{ab}\vartheta^a \otimes \vartheta^b. \tag{2.5}$$

In the first case, the contribution of the Christoffel symbols, constructed by the metric γ_{ab}, reads

$$\begin{aligned}\Gamma^{a}_{bc} &= \frac{1}{2}\left(\check{C}^{a}_{bc} - \gamma^{aa'}\gamma_{bb'}\check{C}^{b'}_{a'c} - \gamma^{aa'}\gamma_{cc'}\check{C}^{c'}_{a'b}\right) \\ &\quad + \frac{1}{2}\gamma^{aa'}\left(\check{e}_c\rfloor d\gamma_{ba'} - \check{e}_b\rfloor d\gamma_{ca'} - \check{e}_{a'}\rfloor d\gamma_{bc}\right),\end{aligned} \tag{2.6}$$

with $\check{C}^{e}_{df}$ representing the curls of the base members in the semi-Riemann space:

$$\check{C}^{e}_{df} = \check{e}_f \rfloor \check{e}_d \rfloor \check{C}^e = -\check{\vartheta}^e\left([\check{e}_d,\check{e}_f]\right) = \check{e}^{\mu}_{d}\check{e}^{\nu}_{f}\left(\check{\partial}_{\mu}\check{e}^{e}_{\nu} - \check{\partial}_{\nu}\check{e}^{e}_{\mu}\right) = -\check{e}^{e}_{\mu}\left[\check{e}_d\left(\check{e}^{\mu}_{f}\right) - \check{e}_f\left(\check{e}^{\mu}_{d}\right)\right], \tag{2.7}$$

where $\check{C}^a := d\check{\vartheta}^a = (1/2)\check{C}^{a}_{bc}\check{\vartheta}^b \wedge \check{\vartheta}^c$ is the anfolonomity 2-form. The deformed metric can be split as follows [96]:

$$g_{\mu\nu}(\pi) = \Upsilon^2(\pi)\breve{g}_{\mu\nu} + \gamma_{\mu\nu}(\pi), \tag{2.8}$$

where $\Upsilon(\pi) = \pi^{a}_{a}$, and $\gamma_{\mu\nu}(\pi) = [\gamma_{ab} - \Upsilon^2(\pi)o_{ab}]\check{e}^{a}_{\mu}\check{e}^{b}_{\nu}$. In the second case, we may write the commutation table for the anholonomic frame, $\{e_a\}$,

$$[e_a, e_b] = -\frac{1}{2}C^{c}_{ab}e_c, \tag{2.9}$$

and define the anholonomy objects:

$$C^{a}_{bc} = \pi^{a}_{e}\pi^{-1}_{b}{}^{d}\pi^{-1}_{c}{}^{f}\breve{C}^{e}_{df} + 2\pi^{a}_{f}\breve{e}^{\mu}_{g}\left(\pi^{-1}_{[b}{}^{g}\partial_{\mu}\pi^{-1}_{[c}{}^{f}\right). \tag{2.10}$$

The usual Levi-Civita connection corresponding to the metric (2.8) is related to the original connection by the following relation:

$$\Gamma^{\mu}_{\rho\sigma} = \breve{\Gamma}^{\mu}_{\rho\sigma} + \Pi^{\mu}_{\rho\sigma}, \tag{2.11}$$

provided that

$$\Pi^{\mu}_{\rho\sigma} = 2g^{\mu\nu}\breve{g}_{\nu(\rho}\nabla_{\sigma)}\Upsilon - \breve{g}_{\rho\sigma}g^{\mu\nu}\nabla_{\nu}\Upsilon + \frac{1}{2}g^{\mu\nu}\left(\nabla_{\rho}\gamma_{\nu\sigma} + \nabla_{\sigma}\gamma_{\rho\nu} - \nabla_{\nu}\gamma_{\rho\sigma}\right), \tag{2.12}$$

where the controvariant deformed metric, $g^{\nu\rho}$, is defined as the inverse of $g_{\mu\nu}$, such that $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}$. Hence, the connection deformation $\Pi^{\mu}_{\rho\sigma}$ acts like a force that deviates the test particles from the geodesic motion in the space V_4 (for more details see [96]). Next, we deal with the spacetime deformation $\pi(x)$, to be consisted of two ingredient deformations ($\overset{\bullet}{\pi}(\overset{\bullet}{x})$, $\sigma(x)$). Provided, we require that the first deformation matrix, $\overset{\bullet}{\pi}(\overset{\bullet}{x}) := (\overset{\bullet}{\pi}{}^{a}_{b})(\overset{\bullet}{x})$, satisfies the following peculiar condition:

$$\overset{\bullet}{\pi}{}^{a}_{c}\left(\overset{\bullet}{x}\right)\overset{\bullet}{\partial}_{\mu}\overset{\bullet}{\pi}{}^{-1}_{b}{}^{c}\left(\overset{\bullet}{x}\right) = \breve{\omega}^{a}_{b\mu}(\breve{x}), \tag{2.13}$$

where $\breve{\omega}^{a}_{b\mu}(\breve{x})$ is the spin connection defined in the semi-Riemann space. By virtue of (2.13), the general deformed spin connection vanishes, and a general linear connection, $\overset{\bullet}{\Gamma}{}^{\mu}_{\rho\sigma}$, is related to the corresponding spin connection $\overset{\bullet}{\omega}{}^{a}_{b\mu}$, through the inverse

$$\overset{\bullet}{\Gamma}{}^{\mu}_{\rho\sigma} = \overset{\bullet}{e}{}^{\mu}_{a}\overset{\bullet}{\partial}_{\sigma}\overset{\bullet}{e}{}^{a}_{\rho} + \overset{\bullet}{e}{}^{\mu}_{a}\overset{\bullet}{\omega}{}^{a}_{b\sigma}\overset{\bullet}{e}{}^{b}_{\rho} = \overset{\bullet}{e}{}^{\mu}_{a}\overset{\bullet}{\partial}_{\sigma}\overset{\bullet}{e}{}^{a}_{\rho}, \tag{2.14}$$

which is the Weitzenböck connection revealing the Weitzenböck spacetime W_4 of the teleparallel gravity. Thus, $\overset{\bullet}{\pi}(\overset{\bullet}{x})$ can be referred to as the Weitzenböck deformation matrix. All magnitudes related to the teleparallel gravity will be denoted by an over "•". The components of the general spin connection then transform *inhomogeneously* under a local tetrad deformations:

$$\omega'^{a}_{b\mu} = \sigma^{a}_{c}\overset{\bullet}{\omega}{}^{c}_{d\mu}\sigma^{d}_{b} + \sigma^{a}_{c}\partial_{\mu}\sigma^{c}_{b}, \tag{2.15}$$

such that

$$\overset{(\sigma)}{\omega}{}^{a}_{b\mu} := \omega'^{a}_{b\mu} = \sigma^{a}_{c}\partial_{\mu}\sigma^{c}_{b} \tag{2.16}$$

is referred to as the *deformation-related frame connection*, which represents the *deformed properties of the frame* only. Then, it follows that the affine connection Γ transforms *inhomogeneously* through

$$\Gamma^{\mu}_{\rho\sigma} = e^{\mu}_{a}\partial_{\sigma}e^{a}_{\rho} + e^{\mu}_{a}\overset{(\pi)}{\omega}{}^{a}_{b\sigma}\, e^{b}_{\rho} = \sigma^{\mu}_{a}\partial_{\sigma}\sigma^{a}_{\rho} + \sigma^{\mu}_{a}\overset{(\sigma)}{\omega}{}^{a}_{b\sigma}\, \sigma^{b}_{\rho}, \tag{2.17}$$

where we have $\sigma^{\mu}_{a}\sigma^{b}_{\mu} = \delta^{b}_{a}$, also the procedure can be inverted, $\sigma^{\mu}_{a}\sigma^{a}_{\nu} = \delta^{\mu}_{\nu}$, and

$$\overset{(\mu)}{\omega}{}^{a}_{b\mu} := \omega^{a}_{b\mu} = \pi^{a}_{c}\breve{\omega}^{c}_{d\mu}\pi^{db} + \pi^{a}_{c}\partial_{\mu}\pi^{cb} \tag{2.18}$$

is the spin connection. For our convenience, hereinafter the notation $\{\overset{(A)}{e}_{a}, \overset{(A)}{\vartheta}{}^{b}\}(A = \pi, \sigma)$ will be used for general linear frames:

$$\left\{\overset{(A)}{e}_{a}, \overset{(A)}{\vartheta}{}^{b}\right\} = \left\{\left(\overset{(\pi)}{e}_{a}, \overset{(\sigma)}{e}_{a}\right), \left(\overset{(\pi)}{\vartheta}{}^{b}, \overset{(\sigma)}{\vartheta}{}^{b}\right)\right\} \equiv \left\{\left(e_{a}, \overset{\bullet}{e}_{a}\right), \left(\vartheta^{b}, \overset{\bullet}{\vartheta}{}^{b}\right)\right\}, \tag{2.19}$$

where $\overset{(A)}{e}_{a}\rfloor\overset{(A)}{\vartheta}{}^{b} = \delta^{b}_{a}$, or in components, $\overset{(A)}{e}{}^{\mu}_{a}\overset{(A)}{e}{}^{b}_{\mu} = \delta^{b}_{a}$; also the procedure can be inverted, $\overset{(A)}{e}{}^{a}_{\rho}\overset{(A)}{e}{}^{\sigma}_{a} = \delta^{\sigma}_{\rho}$, provided that

$$\overset{(A)}{e}{}^{\mu}_{a} = \left(\overset{(A)}{e}{}^{\mu}_{a}, \overset{(\sigma)}{e}{}^{\mu}_{a}\right) \equiv \left(e^{\mu}_{a}, \sigma^{\mu}_{a}\right). \tag{2.20}$$

Hence, the affine connection (2.17) can be rewritten in the abbreviated form:

$$\overset{(A)}{\Gamma}{}^{\mu}_{\rho\sigma} = \overset{(A)}{e}{}^{\mu}_{a}\,\partial_{\sigma}\overset{(A)}{e}{}^{a}_{\rho} + \overset{(A)}{e}{}^{\mu}_{a}\,\overset{(A)}{\omega}{}^{a}_{b\sigma}\,\overset{(A)}{e}{}^{b}_{\rho}. \tag{2.21}$$

Since the first deformation matrices $\pi(x)$ and $\sigma(x)$ are arbitrary functions, the *inhomogeneously* transformed general spin connections $\overset{(\pi)}{\omega}(x)$ and $\overset{(\sigma)}{\omega}(x)$, as well as the affine connection (2.21), are independent of tetrad fields and their derivatives. In what follows, therefore, we will separate the notions of space and connections—the metric-affine formulation of gravity. A metric-affine space (M_4, g, Γ) is defined to have a metric and a linear connection that need not be dependent on each other. The lifting of the constraints of metric-compatibility and symmetry yields the new geometrical property of the spacetime, which are the *nonmetricity* 1-form $\overset{(A)}{N}_{ab}$ and the affine *torsion* 2-form $\overset{(A)}{T}{}^{a}$ representing a translational misfit (for a comprehensive discussion see [114–118]. These, together with the *curvature* 2-form $\overset{(A)}{S}_{a}{}^{b}$, symbolically can be presented as [119, 120]

$$\left(\overset{(A)}{N}_{ab}, \overset{(A)}{T}{}^{a}, \overset{(A)}{S}_{a}{}^{b}\right) \sim \overset{(A)}{\mathfrak{D}}\left(\overset{(A)}{g}_{ab}, \overset{(A)}{\vartheta}{}^{a}, \Gamma_{\text{a}}{}^{b}\right), \tag{2.22}$$

where for a tensor-valued p-form density of representation type $\rho(L^b_a)$, the $GL(4,R)$-covariant exterior derivative reads $\overset{(A)}{\mathfrak{D}}:=d+\overset{(A)}{\Gamma}{}_a{}^b\,\rho(L^b_a)\wedge$ and $\overset{(A)}{\Gamma}{}_a{}^b=\overset{(A)}{\Gamma}{}_{\mu a}{}^b\,dx^\mu$ is the general nonmetricity connection. This notation will be used instead of $\overset{(A)}{\omega}{}_a{}^b=\overset{(A)}{\omega}{}_{\mu a}{}^b\,dx^\mu$, such that $\overset{(A)}{\Gamma}{}_{\mu a}{}^b=\overset{(A)}{e}{}_a{}^\nu\overset{(A)}{e}{}_\sigma{}^b\,\Gamma^\sigma_{\mu\nu}+\overset{(A)}{e}{}_a{}^\nu\,\partial_\mu\overset{(A)}{e}{}_\nu{}^b$. In what follows, however, we may still maintain the former notation $\overset{(A)}{\omega}{}_a{}^b$ to be referred to as corresponding connection constrained by the metricity condition. We may introduce the affine *contortion* 1-form $\overset{(A)}{K}_{ab}=-\overset{(A)}{K}_{ab}$ given in terms of the torsion 2-form $\overset{(A)}{T}{}^a=\overset{(A)}{\vartheta}{}^b\wedge\overset{(A)}{K}{}^a_b$. In tensor components we have $\overset{(A)}{K}_{\mu\nu}{}^\rho=2\overset{(A)}{Q}{}_{(\mu\nu)}{}^\rho+\overset{(A)}{Q}{}_{\mu\nu}{}^\rho$, where the torsion tensor $\overset{(A)}{Q}{}_{\mu\nu}{}^\rho=(1/2),\ \overset{(A)}{T}{}_{\mu\nu}{}^\rho=\overset{(A)}{\Gamma}{}_{[\mu\nu]}{}^\rho$ given with respect to a holonomic frame, $d\overset{(A)}{\vartheta}{}^\rho=0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components. The TSSD-U_4 theory (see [96]) considers curvature and torsion as representing independent degrees of freedom. The RC manifold, U_4, is a particular case of general metric-affine manifold M_4, restricted by the metricity condition, when a nonsymmetric linear connection is said to be metric compatible. Taking the antisymmetrized derivative of the metric condition gives an identity between the curvature of the spin-connection and the curvature of the Christoffel connection:

$$\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right)\overset{(A)}{e}{}_{\rho b}-R^\sigma_{\rho\mu\nu}(\Gamma)\overset{(A)}{e}{}_\sigma{}^a=0, \tag{2.23}$$

where

$$\begin{gathered}\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right)=\partial_\mu\overset{(A)}{\omega}{}_\nu{}^{ab}-\partial_\nu\overset{(A)}{\omega}{}_\mu{}^{ab}+\overset{(A)}{\omega}{}_\mu{}^{ac}\,\overset{(A)}{\omega}{}_{\nu c}{}^b-\overset{(A)}{\omega}{}_\nu{}^{ac}\,\overset{(A)}{\omega}{}_{\mu c}{}^b,\\ R^\sigma_{\rho\mu\nu}(\Gamma)=\partial_\mu\Gamma^\sigma_{\nu\rho}-\partial_\nu\Gamma^\sigma_{\mu\rho}-\Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}+\Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}.\end{gathered} \tag{2.24}$$

Hence, the relations between the scalar curvatures for an U_4 manifold read

$$\overset{(A)}{R}\left(\overset{(A)}{\omega}\right)\equiv\overset{(A)}{e}{}_a{}^\mu\overset{(A)}{e}{}_b{}^\nu\,\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right)=R(g,\Gamma)\equiv g^{\rho\nu}R^\mu_{\rho\mu\nu}(\Gamma). \tag{2.25}$$

This means that the Lorentz and diffeomorphism invariant scalar curvature, R, becomes either a function of $\overset{(A)}{e}{}_a{}^\mu$ only or a function of $g_{\mu\nu}$ only. Certainly, it can be seen by noting that the Lorentz gauge transformations can be used to fix the six antisymmetric components of $\overset{(A)}{e}{}_a{}^\mu$ to vanish. Then in both cases diffeomorphism invariance fixes four more components out of the six $g_{\mu\nu}$, with the four components $g_{0\mu}$ being non dynamical, obviously, leaving only two dynamical degrees of freedom. This shows that the equivalence of the vierbein and metric formulations holds. According to (2.25), the relations between the Ricci scalars read

$$\overset{\circ}{R}\equiv\overset{\circ\,(\sigma)}{R}_{cd}\wedge\overset{\bullet}{\vartheta}{}^c\wedge\overset{\bullet}{\vartheta}{}^d=\overset{\circ\,(\pi)}{R}_{cd}\wedge\vartheta^c\wedge\vartheta^d. \tag{2.26}$$

To recover the TSSD-U_4 theory, one can choose the EC Lagrangian, L_{EC}, as

$$L_{\text{EC}} = -\frac{1}{2\ell^2} \overset{(A)}{R}{}^{ab} \wedge \overset{(A)}{\eta}_{ab} + \frac{1}{2}\Lambda \overset{(A)}{\eta} + \frac{1}{2} \overset{(A)}{N}_{ab} \wedge \lambda^{ab}, \tag{2.27}$$

where Λ is the cosmological constant, $\overset{(A)}{R}{}_a{}^b$ is the curvature tensor, λ^{ab} is the *Lagrange multiplier*, and $(1/2) \overset{(A)}{R}{}^{ab} \wedge \overset{(A)}{\eta}_{ab} = \overset{(A)}{R} \overset{(A)}{\eta}$. The η-basis is consisting in the Hodge dual of exterior products of tetrads by means of the Levi-Civita object: $\eta^{abcd} := {}^{\star}\vartheta^{abcd}$, which yields $\eta^{ab} := {}^{\star}\vartheta^{ab} = (1/2!)\ \eta^{ab}_{cd} \wedge \vartheta^{cd}$ and $\eta := {}^{\star}1 = (1/4!)\ \eta_{abcd}\ \vartheta^{abcd}$, where we used the abbreviated notations for the wedge product monomials, $\overset{(A)}{\vartheta}{}^{\mu\nu\alpha\cdots} = \overset{(A)}{\vartheta}{}^{\mu} \wedge \overset{(A)}{\vartheta}{}^{\nu} \wedge \overset{(A)}{\vartheta}{}^{\alpha} \wedge \cdots$, and $\star$ denotes the Hodge dual. The variation of the total action

$$S = S_{\text{EC}} + S_m^{(\pi)}, \tag{2.28}$$

given by the sum of the gravitational field action, $S_{\text{EC}} = \int L_{\text{EC}}$, with the Lagrangian (2.27) and the macroscopic matter sources, $S_m^{(\pi)}$, with respect to the $\overset{(A)}{\vartheta}{}^{a}$, 1-form $\overset{(A)}{\omega}{}^{ab}$ and Ψ, which is a p-form representing a matter field (fundamentally a representation of the SL$(4, R)$ or of some of its subgroups), gives

$$\begin{gathered}
(1)\ \frac{1}{2} \overset{\circ\,(A)}{R}_{ca} \wedge \overset{(A)}{\vartheta}{}^{c} + \Lambda \overset{(A)}{\eta}_{a} = \ell^2 \overset{(A)}{\theta}_{a}, \qquad (2)\ \overset{(A)}{\Theta}_{ab}{}^{a'b'} \wedge \star \overset{(A)}{\mathcal{T}}_{a'b'} = \ell^2 \star \overset{(\pi)}{\Sigma}_{ab}, \\
(3)\ \frac{\delta L_m^{(\pi)}}{\delta \Psi} = 0, \qquad \frac{\delta L_m^{(\pi)}}{\delta \overline{\Psi}} = 0,
\end{gathered} \tag{2.29}$$

where ℓ is the Planck length, $\overset{(A)}{\Theta}_{a'b'}{}^{ab}\ (\pi(x),\ \sigma(x)) = ((\partial \overset{(A)}{\omega}{}^{a'b'})/\partial \overset{(\pi)}{\omega}{}^{ab})$, and $\star \overset{(\pi)}{\Sigma}_{ab} = -\star \overset{(\pi)}{\Sigma}_{ba}$ is the dual 3-form corresponding to the canonical spin tensor, which is identical with the dynamical spin tensor $\overset{(\pi)}{S}_{abc}$, namely,

$$\star \overset{(\pi)}{\Sigma}_{ab} = \overset{(\pi)}{S}_{ab}{}^{\mu}\ \varepsilon_{\mu\nu\alpha\beta} \overset{(\pi)}{\vartheta}{}^{\nu\alpha\beta}, \tag{2.30}$$

provided that,

$$\star \overset{(A)}{T}_{ab} := \frac{1}{2} \star \left(\overset{(A)}{Q}_{a} \wedge \overset{(A)}{e}_{b} \right) = \overset{(A)}{Q}{}^{c} \wedge \overset{(A)}{\vartheta}{}^{d}\ \varepsilon_{cdab} = \frac{1}{2} \overset{(A)}{Q}_{\mu\nu}{}^{c} \wedge \overset{(A)}{e}{}^{d}_{\alpha}\ \varepsilon_{abcd} \overset{(A)}{\vartheta}{}^{\mu\nu\alpha}, \tag{2.31}$$

and that

$$\overset{(A)}{Q}{}^{a} = \overset{(A)}{D} \overset{(A)}{\vartheta}{}^{a} = d \overset{(A)}{\vartheta}{}^{a} + \overset{(A)}{\omega}{}^{a}_{b} \wedge \overset{(A)}{\vartheta}{}^{b}. \tag{2.32}$$

To obtain some feeling for the tensor language in a holonomic frame then we may recast the first two field equations in (2.29) in the tensorial form:

$$(1)\ \overset{\circ}{G}_{\mu\nu} + \Lambda\ g_{\mu\nu} = \ell^2\ \overset{(A)}{\theta}_{\mu\nu}, \qquad (2)\ \frac{\partial \overset{(A)}{\omega}{}_{\nu'}^{\mu'\rho'}}{\partial \overset{(\pi)}{\omega}{}_{\nu}^{\mu\rho}} \left(\overset{(A)}{T}\right) \overset{(A)}{T}{}_{\mu'\rho'}^{\ \ \nu'} = \ell^2 \overset{(\pi)}{S}{}_{\mu\rho}^{\ \ \nu}, \tag{2.33}$$

where $\overset{\circ}{G}_{\mu\nu} \equiv \overset{\circ}{R}_{\mu\nu} - (1/2) g_{\mu\nu}\overset{\circ}{R}$ is Einstein's tensor, and the *modified torsion* reads

$$\overset{(A)}{T}{}_{\mu\rho}^{\ \ \nu} := \overset{(A)}{Q}{}_{\mu\rho}^{\ \ \nu} + \delta_\mu^\nu\ \overset{(A)}{Q}_\rho - \delta_\rho^\nu\ \overset{(A)}{Q}_\mu\ . \tag{2.34}$$

Thus, the equations of the standard EC theory can be recovered for $A = \pi$. However, these equations can be equivalently replaced by the set of *modified EC equations* for $A = \sigma$:

$$(1)\ \overset{\circ}{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \ell^2\ \overset{(\sigma)}{\theta}_{\mu\nu}, \qquad (2)\ \overset{(\sigma)}{\Theta}{}_{\nu'\mu\rho}^{\mu'\rho'\nu} \left(\overset{(\sigma)}{T}\right) \overset{(\sigma)}{T}{}_{\mu'\rho'}^{\ \ \nu'} = \ell^2 \overset{(\pi)}{S}{}_{\mu\rho}^{\ \ \nu}\ . \tag{2.35}$$

We may impose different physical constraints upon the spacetime deformation $\sigma(x)$, which will be useful for the theory of electromagnetism and charged particles:

$$\Theta_{\nu'\mu\rho}^{\mu'\rho'\nu}(\pi(x),\ \sigma(x)) \equiv \Theta_{\nu'\mu\rho}^{\mu'\rho'\nu}\left(\overset{(\sigma)}{T}\right) = 2\varphi_{,\,\sigma}\ \varepsilon^{\sigma\nu}_{\ \ \mu\rho}\left(\overset{(\sigma)}{T}{}^{-1}\right)_{\nu'}^{\mu'\rho'}, \tag{2.36}$$

with φ as a scalar or pseudoscalar function of relevant variables. Here $\Theta_{\nu'\mu\rho}^{\mu'\rho'\nu}(\overset{(A)}{T}) = ((\partial \overset{(A)}{\omega}{}_{\nu'}^{\mu'\rho'})/\partial \overset{(\pi)}{\omega}{}_{\nu}^{\mu\rho}(\overset{(A)}{T}))$. Then we obtain

$$\overset{(\pi)}{T}{}_{\mu\rho}^{\ \ \nu} = \Theta_{\nu'\mu\rho}^{\mu'\rho'\nu}(\pi(x),\ \sigma(x)) \overset{(\sigma)}{T}{}_{\mu'\rho'}^{\ \ \nu'} = 2\varphi_{,\,\sigma}\varepsilon^{\sigma\nu}_{\ \ \mu\rho}, \tag{2.37}$$

which recovers the term in the Lagrangian of pseudoscalar-photon interaction theory [41, 97–101], such that the nonmetric part of the Lagrangian can be put in the well-known form of the $\chi - g$ framework:

$$L_I^{(\pi)NM} = 2(-g)^{1/2} A_\nu A_{\mu,\rho} \overset{(\pi)}{\mathcal{T}}{}^{\nu\mu\rho} = 4(-g)^{1/2}\varphi_{,\,\sigma}\varepsilon^{\sigma\nu\mu\rho} A_\nu A_{\mu,\rho},\ (\mathrm{mod\ \ div}), \tag{2.38}$$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ has the usual meaning for electromagnetism. This is equivalent, up to integration by parts in the action integral (modulo a divergence), to the Lagrangian

$$L_I^{(\pi)NM} = (-g)^{1/2}\varphi\varepsilon^{\sigma\nu\mu\rho} F_{\sigma\nu} F_{\mu\rho}. \tag{2.39}$$

According to (2.39), the gravitational constitutive tensor $\chi^{\sigma\nu\mu\rho} = \chi^{\mu\rho\sigma\nu} = -\chi^{\mu\rho\nu\sigma}$ [40] of the gravitational fields (e.g., metric $g_{\mu\nu}$, (pseudo)scalar field $\varphi_{,}$ etc.) reads

$$\chi^{\sigma\nu\mu\rho} = (-g)^{1/2}\left[\frac{1}{2}g^{\sigma\mu}g^{\nu\rho} - \frac{1}{2}g^{\sigma\rho}g^{\mu\nu} + \varphi\varepsilon^{\sigma\nu\mu\rho}\right]. \tag{2.40}$$

The special case $\varphi_{,\,\sigma} = \text{constant} = V_\sigma$ is considered by [102, 103], for modification of electrodynamics with an additional external constant vector coupling. Imposing other appropriate constraints upon the spacetime deformation $\sigma(x)$, in the framework of TSSD-U_4 theory we may reproduce the various terms in the Lagrangians of pseudoscalar theories, for example, as intergrand for topological invariant [104], or pseudoscalar-gluon coupling occurred in QCD in an effort to solve the strong CP problem [105–107].

3. The Hypothetical Flat MS companion: A Toy Model

As a preliminary step we now conceive two different spaces: one would be 4D background Minkowski space, M_4, and another one should be MS embedded in the M_4, which is an indispensable individual companion to the particle, without relation to the other matter. This theory is mathematically somewhat similar to the more recent membrane theory. The flat MS in suggested model is assumed to be 2D Minkowski space, M_2:

$$M_2 = R^1_{(+)} \oplus R^1_{(-)}. \tag{3.1}$$

The ingredient 1D-space R^1_A is spanned by the coordinates η^A, where we use the *naked* capital Latin letters $A, B, \ldots = (\pm)$ to denote the world indices related to M_2. The metric in M_2 is

$$\overline{g} = \overline{g}(\overline{e}_A, \overline{e}_B)\overline{\vartheta}^A \otimes \overline{\vartheta}^B, \tag{3.2}$$

where $\overline{\vartheta}^A = d\eta^A$ is the infinitesimal displacement. The basis $\overline{e}_A$ at the point of interest in M_2 consists of two real *null vectors*:

$$\overline{g}(\overline{e}_A, \overline{e}_B) \equiv \langle \overline{e}_A, \overline{e}_B \rangle = {}^*o_{AB}, \quad ({}^*o_{AB}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

The norm, $i\overline{d} \equiv d\hat{\eta}$, given in this basis reads $i\overline{d} = \overline{e}\overline{\vartheta} = \overline{e}_A \otimes \overline{\vartheta}^A$, where $i\overline{d}$ is the tautological tensor field of type (1,1), $\overline{e}$ is a shorthand for the collection of the 2-tuplet $(\overline{e}_{(+)}, \ \overline{e}_{(-)})$, and $\overline{\vartheta} = \begin{pmatrix} \overline{\vartheta}^{(+)} \\ \overline{\vartheta}^{(-)} \end{pmatrix}$. We may equivalently use a temporal $q^0 \in T^1$ and a spatial $q^1 \in R^1$ variables $q^r(q^0, q^1)$ $(r = 0, 1)$, such that

$$M_2 = R^1 \oplus T^1. \tag{3.4}$$

The norm, $i\overline{d}$, now can be rewritten in terms of displacement, dq^r, as

$$i\overline{d} = d\widehat{q} = e_0 \otimes dq^0 + e_1 \otimes dq^1, \tag{3.5}$$

where e_0 and e_1 are, respectively, the temporal and spatial basis vectors:

$$\begin{gathered} e_0 = \frac{1}{\sqrt{2}}\left(\overline{e}_{(+)} + \overline{e}_{(-)}\right), \qquad e_1 = \frac{1}{\sqrt{2}}\left(\overline{e}_{(+)} - \overline{e}_{(-)}\right), \\ \overline{g}(e_r, e_s) \equiv \langle e_r, e_s \rangle = o_{rs}, \qquad (o_{rs}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{3.6}$$

The MS companion (M_2) of this particle is assumed to be smoothly (injective and continuous) embedded in the M_4. Suppose that the position of the particle in the background M_4 space is specified by the coordinates $x^l(s)$ $(l = 0, 1, 2, 3)(x^0 = t)$ with respect to the axes of the inertial system $S_{(4)}$. Then, a smooth map $f : M_2 \to M_4$ is defined to be an immersion—an embedding is a function that is a homeomorphism onto its image:

$$q^0 = \frac{1}{\sqrt{2}}\left(\eta^{(+)} + \eta^{(-)}\right) = t, \qquad q^1 = \frac{1}{\sqrt{2}}\left(\eta^{(+)} - \eta^{(-)}\right) = |\vec{x}|. \tag{3.7}$$

In fact, we assume that the particle has to be moving simultaneously in the parallel *individual* M_2 space and the ordinary 4D background space (either Minkowskian or Riemannian). Let the nonaccelerated observer uses the inertial coordinate frame $S_{(2)}$ for the position q^r of a free test particle in the flat M_2. We may choose the system $S_{(2)}$ in such a way as the time axis e_0 lies along the time axis of a comoving inertial frame S_4, such that the time coordinates in the two systems are taken the same, $q^0 = t$. For the case at hand,

$$v^{(\pm)} = \frac{d\eta^{(\pm)}}{dq^0} = \frac{1}{\sqrt{2}}\left(1 \pm v_q\right), \qquad v_q = \frac{dq^1}{dq^0} = |\vec{v}| = \left|\frac{d\vec{x}}{dt}\right| = \text{const} \geq 0. \tag{3.8}$$

Hence, given the inertial frames $S_{(4)}, S'_{(4)}, S''_{(4)}, \ldots$ in the M_4, in this manner we may define the corresponding inertial frames $S_{(2)}, S'_{(2)}, S''_{(2)}, \ldots$ in the M_2.

Continuing on our quest, we next define the concepts of *absolute* and *relative* states of the ingredient spaces R^1_A. The measure for these states is the very magnitude of the velocity components v^A of the particle.

Definition 3.1. The ingredient space R^1_A of the individual MS companion of the particle is said to be in

$$\text{absolute (abs) state if } v^A = 0, \qquad \text{relative (rel) state if } v^A \neq 0. \tag{3.9}$$

Therefore, the MS can be realized either in the *semiabsolute* state (rel, abs), or (abs, rel), or in the *total relative* state (rel, rel). It is remarkable that the *total-absolute* state, (abs, abs), which is equivalent to the unobservable Newtonian *absolute* two-dimensional spacetime, cannot be realized because of the relation $v^{(+)} + v^{(-)} = \sqrt{2}$. An existence of the *absolute* state of the R^1_A is an immediate cause of the light traveling in empty space R^1 along the q-axis with a maximal velocity $v_q = c$ (we reinstate the factor (c)) in the $(+)$-direction corresponding to the state $(v^{(+)}, 0) \Leftrightarrow$ (rel, abs), and in the $(-)$-direction corresponding to the state $(0, v^{(-)}) \Leftrightarrow$ (abs, rel). The *absolute* state of R^1_A manifests its *absolute* character in the important for SR fact that the resulting velocity of light in the empty space R^1 is the same in all inertial frames $S_{(2)}$, $S'_{(2)}$, $S''_{(2)}, \ldots$; that is, in empty space light propagates independently of the state of motion of the source—if $v^A = 0$, then $v^{A'} = v^{A''} = \cdots = 0$. Since the v^A is the very key measure of a deviation from the *absolute* state, we might expect that this has a substantial effect in an alteration of the particle motion under the unbalanced force. This observation allows us to lay forth the foundation of the fundamental RLI as follows.

Conjecture 1 (RIL conjecture). *The nonzero local rate $\varrho(\eta, m, f)$ of instantaneously change of a constant velocity v^A (both magnitude and direction) of a massive (m) test particle under the unbalanced net force $(\vec{f})$ is the immediate cause of a deformation (distortion of the local internal properties) of MS: $M_2 \rightarrow \widetilde{\mathcal{M}}_2$.*

We can conclude therefrom that, unless MS is flat, a free particle in 4D background space in motion of uniform speed in a straight line tends to stay in this motion and a particle at rest tends to stay at rest. In this way, the MS companion, therefore, abundantly serves to account for the state of motion of the particle in the 4D background space. The MS companion is not measurable directly, but in going into practical details, in Section 4 we will determine the function $\varrho(\eta, m, \vec{f})$ and show that a deformation (distortion of local internal properties) of MS is the origin of inertia effects that can be observed by us. Before tempting to build realistic model of accelerated motion and inertial effects, for the benefit of the reader, we briefly turn back to physical discussion of why the MS is two dimensional and not higher. We have first to recall the salient features of MS which admittedly possesses some rather unusual properties; namely, the basis at the point of interest in MS, embedded in the 4D spacetime, would be consisted of the real *null vectors*, which just allows only two-dimensional constructions (3.3). Next, note that the immediate cause of inertia effects is the *nonlinear* process of deformation (distortion of local internal properties) of MS, which yields the resulting *linear* relation $\vec{f}_{\text{in}} = -\vec{f}$ (see (2.19)–(5.35)) with respect to the components of inertial force $\vec{f}_{\text{in}}$ in terms of the relativistic force $\vec{f}$ acting on a purely classical particle in M_4. This ultimately requires that MS should only be two dimensional, because to resolve the afore-mentioned relationship of *nonlinear* and *linear* processes we may choose the system $S_{(2)}$ in only allowed way as the time axis e_0 lies along the time axis of a comoving inertial frame S_4, in order that the time coordinates in the two systems are taken the same, $q^0 = t$ and that another axis $\vec{e}_q$ lies along the net 3-acceleration $(\vec{e}_q \| \vec{e}_a)$, $(\vec{e}_a = \vec{a}_{\text{net}}/|\vec{a}_{\text{net}}|)$ (5.26).

4. The General Spacetime Deformation/Distortion Complex

For the self-contained arguments, we now extend just necessary geometrical ideas of the spacetime deformation framework described in Section 2, without going into the subtleties, as applied to the 2D deformation $M_2 \rightarrow \widetilde{\mathcal{M}}_2$. To start with, let V_2 be 2D semi-Riemann space, which has at each point a tangent space, $\check{T}_{\check{\eta}} V_2$, spanned by the anholonomic orthonormal frame field, $\check{e}$, as a shorthand for the collection of the 2-tuplet $(\check{e}_{(+)}, \check{e}_{(-)})$, where $\check{e}_a = \check{e}^{\tilde{A}}_a \check{e}_{\tilde{A}}$, where the holonomic frame is given as $\check{e}_{\tilde{A}} = \check{\partial}_{\tilde{A}}$. Here, we use the first half of Latin alphabet

$a, b, c, \ldots = (\pm)$ to denote the anholonomic indices to related the tangent space, and the capital Latin letters with an over "$\,\tilde{}\,$"—$\tilde{A}, \tilde{B}, \ldots = (\pm)$, to denote the holonomic world indices related to either the space V_2 or $\widetilde{\mathcal{M}}_2$. All magnitudes referred to the space, V_2, will be denoted by an over "$\,\breve{}\,$". These then define a dual vector, $\breve{\vartheta}$, of differential forms, $\breve{\vartheta} = \begin{pmatrix} \breve{\vartheta}^{(+)} \\ \breve{\vartheta}^{(-)} \end{pmatrix}$, as a shorthand for the collection of the $\breve{\vartheta}^b = \breve{e}^b_{\tilde{A}} \breve{\vartheta}^{\tilde{A}}$, whose values at every point form the dual basis, such that $\breve{e}_a \rfloor \breve{\vartheta}^b = \delta^b_a$. In components $\breve{e}^{\tilde{A}}_a \breve{e}^b_{\tilde{A}} = \delta^b_a$. On the manifold, V_2, the tautological tensor field, $\breve{id}$, of type (1,1) can be defined which assigns to each tangent space the identity linear transformation. Thus for any point $\breve{\eta} \in V_2$, and any vector $\breve{\xi} \in \breve{T}_{\breve{\eta}} V_2$, one has $\breve{id}(\breve{\xi}) = \breve{\xi}$. In terms of the frame field, the $\breve{\vartheta}^a$ give the expression for $\breve{id}$ as $\breve{id} = \breve{e}\breve{\vartheta} = \breve{e}_{(+)} \otimes \breve{\vartheta}^{(+)} + \breve{e}_{(-)} \otimes \breve{\vartheta}^{(-)}$, in the sense that both sides yield $\breve{\xi}$ when applied to any tangent vector $\breve{\xi}$ in the domain of definition of the frame field. We may consider general transformations of the linear group, $\mathrm{GL}(2, R)$, taking any base into any other set of four linearly independent fields. The notation $\{\breve{e}_a, \breve{\vartheta}^b\}$ will be used hereinafter for general linear frames. The holonomic metric can be defined in the semi-Riemann space, V_2, as

$$\breve{g} = \breve{g}_{\tilde{A}\tilde{B}} \breve{\vartheta}^{\tilde{A}} \otimes \breve{\vartheta}^{\tilde{B}} = \breve{g}\left(\breve{e}_{\tilde{A}}, \breve{e}_{\tilde{B}}\right) \breve{\vartheta}^{\tilde{A}} \otimes \breve{\vartheta}^{\tilde{B}}, \tag{4.1}$$

with components $\breve{g}_{\tilde{A}\tilde{B}} = \breve{g}(\breve{e}_{\tilde{A}}, \breve{e}_{\tilde{B}})$ in the dual holonomic base $\{\breve{\vartheta}^{\tilde{A}}\}$. The anholonomic orthonormal frame field, $\breve{e}$, relates $\breve{g}$ to the tangent space metric, ${}^*o_{ab}$, by ${}^*o_{ab} = \breve{g}(\breve{e}_a, \breve{e}_b) = \breve{g}_{\tilde{A}\tilde{B}} \breve{e}^{\tilde{A}}_a \breve{e}^{\tilde{B}}_b$, which has the converse $\breve{g}_{\tilde{A}\tilde{B}} = {}^*o_{ab} \breve{e}^a_{\tilde{A}} \breve{e}^b_{\tilde{B}}$ because of the relation $\breve{e}^{\tilde{A}}_a \breve{e}^a_{\tilde{B}} = \delta^{\tilde{A}}_{\tilde{B}}$. With this provision, we build up a general *distortion-complex*, yielding a distortion of the flat space M_2, and show how it recovers the *world-deformation tensor* $\tilde{\Omega}$, which still has to be put in [96] by hand. The DC members are the invertible distortion matrix D, the tensor Y, and the *flat-deformation tensor* Ω. Symbolically,

$$\mathrm{DC} \sim \left(\breve{D}, \breve{Y}, \Omega\right) \longrightarrow \tilde{\Omega}. \tag{4.2}$$

The principle foundation of a *distortion of local internal properties of MS* comprises then two steps.

(1) The first is to assume that the linear frame $(\overline{e}_A; \overline{\vartheta}^A)$, at given point $(p \in M_2)$, undergoes the *distortion* transformations, conducted by $(\breve{D}, \breve{Y})$ and (D, Y), respectively, relating to V_2 and $\widetilde{\mathcal{M}}_2$, recast in the form

$$\breve{e}_{\tilde{A}} = \breve{D}^B_{\tilde{A}} \overline{e}_B, \qquad \breve{\vartheta}^{\tilde{A}} = \breve{Y}^{\tilde{A}}_B \overline{\vartheta}^B, \qquad e_{\tilde{A}} = D^B_{\tilde{A}} \overline{e}_B, \qquad \vartheta^{\tilde{A}} = Y^{\tilde{A}}_B \overline{\vartheta}^B. \tag{4.3}$$

(2) Then, the norm $d\tilde{\eta} \equiv id$ of the infinitesimal displacement $d\tilde{\eta}^{\tilde{A}}$ on the general smooth differential 2D-manifold $\widetilde{\mathcal{M}}_2$ can be written in terms of the spacetime structures of V_2 and M_2:

$$id = e\vartheta = \tilde{\Omega}^{\tilde{B}}_{\tilde{A}} \breve{e}_{\tilde{B}} \otimes \breve{\vartheta}^{\tilde{A}} = \Omega^a_b \breve{e}_a \otimes \breve{\vartheta}^b = e_{\tilde{C}} \otimes \vartheta^{\tilde{C}} = e_a \otimes \vartheta^a = \Omega^B_A \overline{e}_B \otimes \overline{\vartheta}^A \in \widetilde{\mathcal{M}}_2, \tag{4.4}$$

where $e = \{e_a = e^{\tilde{C}}_a e_{\tilde{C}}\}$ is the frame field and $\vartheta = \{\vartheta^a = e^a_{\tilde{C}}\vartheta^{\tilde{C}}\}$ is the coframe field defined on $\widetilde{\mathcal{M}}_2$, such that $e_a \rfloor \vartheta^b = \delta^b_a$. The deformation tensors $\tilde{\Omega}^{\tilde{B}}_{\tilde{A}} = \pi^{\tilde{C}}_{\tilde{A}}\pi^{\tilde{B}}_{\tilde{C}}$ and Ω^B_A imply

$$\tilde{\Omega}^{\tilde{B}}_{\tilde{A}} = \check{D}^C_{\tilde{A}}\Omega^D_C\check{Y}^{\tilde{B}}_D, \qquad \Omega^B_A = Y^{\tilde{C}}_A D^B_{\tilde{C}}, \tag{4.5}$$

provided that

$$D^A_{\tilde{C}} = \pi^{\tilde{B}}_{\tilde{C}}\check{D}^A_{\tilde{B}}, \qquad Y^{\tilde{C}}_B = \pi^{\tilde{C}}_{\tilde{A}}\check{Y}^{\tilde{A}}_B, \tag{4.6}$$

such that

$$e_{\tilde{C}} = \pi^{\tilde{B}}_{\tilde{C}}\check{e}_{\tilde{B}} \equiv \tilde{\partial}_{\tilde{C}}, \qquad \vartheta^{\tilde{C}} = \pi^{\tilde{C}}_{\tilde{A}}\check{\vartheta}^{\tilde{V}} \equiv d\tilde{\eta}^{\tilde{C}}, \qquad \tilde{\eta}^{\tilde{C}} \in \mathcal{U} \in \widetilde{\mathcal{M}}_2. \tag{4.7}$$

Hence the anholonomic deformation tensor, $\Omega^a_b = \pi^a_c\pi^c_b = \tilde{\Omega}^{\tilde{B}}_{\tilde{A}}\check{e}^a_{\tilde{B}}\check{e}^{\tilde{A}}_b$, yields local tetrad deformations:

$$e_c = \pi^a_c\check{e}_a, \qquad \vartheta^c = \pi^c_b\check{\vartheta}^b, \qquad e\vartheta = e_a \otimes \vartheta^a = \Omega^a_b\check{e}_a \otimes \check{\vartheta}^b. \tag{4.8}$$

The matrices $\pi(\tilde{\eta}) := (\pi^a_b)(\tilde{\eta})$ are referred to as the *first deformation matrices*, and the matrices $\gamma_{cd}(\tilde{\eta}) = {}^*o_{ab}\pi^a_c(\tilde{\eta})\pi^b_d(\tilde{\eta})$, *second deformation matrices*. The matrices $\pi^a_c(\tilde{\eta}) \in \mathrm{GL}(2, R)$ for all $\tilde{\eta}$, in general, give rise to right cosets of the Lorentz group; that is, they are the elements of the quotient group $\mathrm{GL}(2, R)/\mathrm{SO}(1,1)$, because the Lorentz matrices, Λ^r_s, $(r, s = 1, 0)$ leave the Minkowski metric invariant. A right multiplication of $\pi(\tilde{\eta})$ by a Lorentz matrix gives another deformation matrix. So, all the fundamental geometrical structures on deformed/distorted MS in fact—the metric as much as the coframes and connections—acquire a *deformation/distortion*-induced theoretical interpretation. If we deform the tetrad according to (4.8), in general, we have two choices to recast metric as follows: either writing the deformation of the metric in the space of tetrads or deforming the tetrad field:

$$g = {}^*o_{ab}\pi^a_c\pi^b_d\check{\vartheta}^c \otimes \check{\vartheta}^d = \gamma_{cd}\check{\vartheta}^c \otimes \check{\vartheta}^d = {}^*o_{ab}\vartheta^a \otimes \vartheta^b. \tag{4.9}$$

In the first case, the contribution of the Christoffel symbols, constructed by the metric γ_{ab}, reads

$$\begin{aligned}\Gamma^a_{bc} = &\frac{1}{2}\left(\check{C}^a_{bc} - \gamma^{aa'}\gamma_{bb'}\check{C}^{b'}_{a'c} - \gamma^{aa'}\gamma_{cc'}\check{C}^{c'}_{a'b}\right) \\ &+ \frac{1}{2}\gamma^{aa'}\left(\check{e}_c \rfloor d\gamma_{ba'} - \check{e}_b \rfloor d\gamma_{ca'} - \check{e}_{a'} \rfloor d\gamma_{bc}\right).\end{aligned} \tag{4.10}$$

The deformed metric can be split as follows [96]:

$$g_{\tilde{A}\tilde{B}}(\pi) = \Upsilon^2(\pi)\check{g}_{\tilde{A}\tilde{B}} + \gamma_{\tilde{A}\tilde{B}}(\pi), \tag{4.11}$$

where $\Upsilon(\pi) = \pi^a_a$, and

$$\gamma_{\tilde{A}\tilde{B}}(\pi) = \left[\gamma_{ab} - \Upsilon^2(\pi)\,{}^*o_{ab}\right]\check{e}^a_{\tilde{A}}\check{e}^b_{\tilde{B}}. \tag{4.12}$$

In the second case, we may write the commutation table for the anholonomic frame, $\{e_a\}$,

$$[e_a, e_b] = -\frac{1}{2}\,C^c_{ab}e_c, \tag{4.13}$$

and define the anholonomy objects:

$$C^a_{bc} = \pi^a_e \pi^{-1}{}_b{}^d \pi^{-1}{}_c{}^f \check{C}^e_{df} + 2\pi^a_f \check{e}^{\tilde{A}}_g\left(\pi^{-1}{}_{[b}{}^g \partial_{\tilde{A}} \pi^{-1}{}_{c]}{}^f\right). \tag{4.14}$$

The usual Levi-Civita connection corresponding to the metric (4.11) is related to the original connection by the following relation:

$$\Gamma^{\tilde{A}}_{\tilde{C}\tilde{D}} = \check{\Gamma}^{\tilde{A}}_{\tilde{C}\tilde{D}} + \Pi^{\tilde{A}}_{\tilde{C}\tilde{D}}, \tag{4.15}$$

provided that

$$\begin{aligned}\Pi^{\tilde{A}}_{\tilde{C}\tilde{D}} &= 2g^{\tilde{A}\tilde{B}}\check{g}_{\tilde{B}(\tilde{C}}\,\nabla_{\tilde{D})}\Upsilon - \check{g}_{\tilde{C}\tilde{D}}g^{\tilde{A}\tilde{B}}\nabla_{\tilde{B}}\Upsilon \\ &\quad + \frac{1}{2}g^{\tilde{A}\tilde{B}}\left(\nabla_{\tilde{C}}\gamma_{\tilde{B}\tilde{D}} + \nabla_{\tilde{D}}\gamma_{\tilde{C}\tilde{B}} - \nabla_{\tilde{B}}\gamma_{\tilde{C}\tilde{D}}\right),\end{aligned} \tag{4.16}$$

where the controvariant deformed metric, $g^{\tilde{B}\tilde{C}}$, is defined as the inverse of $g_{\tilde{A}\tilde{B}}$, such that $g_{\tilde{A}\tilde{B}}g^{\tilde{B}\tilde{C}} = \delta^{\tilde{C}}_{\tilde{A}}$. That is, the connection deformation $\Pi^{\tilde{A}}_{\tilde{C}\tilde{D}}$ acts like a force that deviates the test particles from the geodesic motion in the space, V_2. Taking into account (4.4), the metric (4.9) can be alternatively written in a general form of the spacetime or frame objects:

$$\begin{aligned} g &= g_{\tilde{A}\tilde{B}}\vartheta^{\tilde{A}} \otimes \vartheta^{\tilde{B}} = \left(\tilde{\Omega}^{\tilde{B}}_{\tilde{A}}\,\tilde{\Omega}^{\tilde{D}}_{\tilde{C}}\right)\check{g}_{\tilde{B}\tilde{D}}\check{\vartheta}^{\tilde{A}} \otimes \check{\vartheta}^{\tilde{C}} \\ &= {}^*o_{ab}\vartheta^a \otimes \vartheta^b = \left(\Omega^c_a\,\Omega^d_b\right){}^*o_{cd}\check{\vartheta}^a \otimes \check{\vartheta}^b \\ &= \gamma_{cd}\check{\vartheta}^c \otimes \check{\vartheta}^d = \left(\Omega^C_A\,\Omega^D_B\right){}^*o_{CD}\overline{\vartheta}^A \otimes \overline{\vartheta}^B. \end{aligned} \tag{4.17}$$

A significantly more rigorous formulation of the spacetime deformation technique with different applications as we have presented it may be found in [96].

5. Model Building in the 4D Background Minkowski Spacetime

In this section we construct the RTI in particular case when the relativistic test particle accelerated in the Minkowski 4D background flat space, M_4, under an unbalanced net force other than gravitational. Here and henceforth we simplify DC for our use by imposing the constraints

$$D^{A}_{\widetilde{C}} = \check{D}^{A}_{\widetilde{B}}, \qquad \check{Y}^{\widetilde{A}}_{B} = \check{D}^{\widetilde{A}}_{B}, \tag{5.1}$$

and, therefore,

$$\mathrm{DC} \sim (D, \Omega) \longrightarrow \widetilde{\Omega}. \tag{5.2}$$

The (4.5), by virtue of (4.4) and (5.1), gives

$$\widetilde{\Omega}^{\widetilde{B}}_{\widetilde{A}} = \check{D}^{C}_{\widetilde{A}} \Omega^{D}_{C} \check{D}^{\widetilde{B}}_{D} = \pi^{\widetilde{B}}_{\widetilde{A}}, \qquad Y^{\widetilde{C}}_{B} = \widetilde{\Omega}^{\widetilde{C}}_{\widetilde{A}} \check{D}^{\widetilde{A}}_{B}, \tag{5.3}$$

where the deformation tensor, $\widetilde{\Omega}^{\widetilde{B}}_{\widetilde{A}}$, yields the partial holonomic frame transformations:

$$e_{\widetilde{C}} = \check{e}_{\widetilde{C}}, \qquad \vartheta^{\widetilde{C}} = \widetilde{\Omega}^{\widetilde{C}}_{\widetilde{A}} \check{\vartheta}^{\widetilde{V}}, \tag{5.4}$$

or, respectively, the Ω^{a}_{b} yields the partial local tetrad deformations:

$$e_{c} = \check{e}_{c}, \qquad \vartheta^{c} = \Omega^{c}_{b} \check{\vartheta}^{b}, \qquad e\vartheta = e_{a} \otimes \vartheta^{a} = \Omega^{a}_{b} \check{e}_{a} \otimes \check{\vartheta}^{b}. \tag{5.5}$$

Hence, (4.4) defines a diffeomorphism $\widetilde{\eta}^{\widetilde{A}}(\eta) : M_2 \longrightarrow \widetilde{\mathcal{M}}_2$:

$$e_{\widetilde{A}} Y^{\widetilde{A}}_{B} = \Omega^{A}_{B} \overline{e}_{A}, \tag{5.6}$$

where $Y^{\widetilde{A}}_{B} = \partial \widetilde{\eta}^{\widetilde{A}} / \partial \eta^{B}$. The conditions of integrability, $\partial_{A} Y^{\widetilde{C}}_{B} = \partial_{B} Y^{\widetilde{C}}_{A}$, and nondegeneracy, $\det|Y^{\widetilde{A}}_{B}| \neq 0$, immediately define a general form of the *flat-deformation tensor* $\Omega^{A}_{B} := D^{A}_{\widetilde{C}} \partial_{B} \Theta^{\widetilde{C}}$, where $\Theta^{\widetilde{C}}$ is an arbitrary holonomic function. To make the remainder of our discussion a bit more concrete, it proves necessary to provide, further, a constitutive ansatz of simple, yet tentative, linear *distortion transformations*, which, according to RLI conjecture, can be written in terms of *local rate* $\varrho(\eta, m, f)$ of instantaneously change of the measure v^{A} of massive (m) test particle under the unbalanced net force (f):

$$\begin{aligned} e_{(\widetilde{+})}(\varrho) &= D^{\ B}_{(\widetilde{+})}(\varrho) \overline{e}_{B} = \overline{e}_{(+)} - \varrho(\eta, m, f) v^{(-)} \overline{e}_{(-)}, \\ e_{(\widetilde{-})}(\varrho) &= D^{\ B}_{(\widetilde{-})}(\varrho) \overline{e}_{B} = \overline{e}_{(-)} + \varrho(\eta, m, f) v^{(+)} \overline{e}_{(+)}. \end{aligned} \tag{5.7}$$

Clearly, these transformations imply a violation of the relation (3.3) ($e^2_{\widetilde{A}}(\varrho) \neq 0$) for the null vectors $\overline{e}_A$. Now we can use (4.4) to observe that for dual vectors of differential forms $\vartheta = \begin{pmatrix} \vartheta^{(\widetilde{+})} \\ \vartheta^{(\widetilde{-})} \end{pmatrix}$ and $\overline{\vartheta} = \begin{pmatrix} \overline{\vartheta}^{(+)} \\ \overline{\vartheta}^{(-)} \end{pmatrix}$ we may obtain

$$\vartheta = \begin{pmatrix} \Omega^{\,C}_{(+)}\left\langle e^{(\widetilde{+})}, \overline{e}_C \right\rangle & \Omega^{\,C}_{(-)}\left\langle e^{(\widetilde{+})}, \overline{e}_C \right\rangle \\ \Omega^{\,C}_{(+)}\left\langle e^{(\widetilde{-})}, \overline{e}_C \right\rangle & \Omega^{\,C}_{(-)}\left\langle e^{(\widetilde{-})}, \overline{e}_C \right\rangle \end{pmatrix} \overline{\vartheta}. \tag{5.8}$$

We parameterize the tensor Ω^A_B in terms of the parameters τ_1 and τ_2 as

$$\begin{gathered} \Omega^{(+)}_{(+)} = \Omega^{(-)}_{(-)} = \tau_1\left(1 + \tau_2 \overline{\varrho}^2\right), \qquad \Omega^{(-)}_{(+)} = -\tau_1(1-\tau_2)\varrho v^{(-)}, \\ \Omega^{(+)}_{(-)} = \tau_1(1-\tau_2)\varrho v^{(+)}, \end{gathered} \tag{5.9}$$

where $\overline{\varrho}^2 = v^2\varrho^2$, $v^2 = v^{(+)}v^{(-)} = 1/2\gamma_q^2$ and $\gamma_q = (1 - v_q^2)^{-1/2}$. Then, the relation (5.8) can be recast in an alternative form:

$$\vartheta = \tau_1 \begin{pmatrix} 1 & -\tau_2\varrho v^{(+)} \\ \tau_2 \varrho v^{(-)} & 1 \end{pmatrix} \overline{\vartheta}. \tag{5.10}$$

Suppose that a second observer, who makes measurements using a frame of reference $\widetilde{S}_{(2)}$ which is held stationary in deformed/distorted space $\widetilde{\mathcal{M}}_2$, uses for the test particle the corresponding spacetime coordinates $\widetilde{q}^{\,\widetilde{r}}((\widetilde{q}^{\widetilde{0}}, \widetilde{q}^{\widetilde{1}}) \equiv (\widetilde{t}, \widetilde{q}))$. The (4.4) can be rewritten in terms of spacetime variables as

$$id = e\vartheta \equiv d\widetilde{\widetilde{q}} = \widetilde{e}_0 \otimes d\widetilde{t} + \widetilde{e}_q \otimes d\widetilde{q}, \tag{5.11}$$

where $\widetilde{e}_0$ and $\widetilde{e}_q$ are, respectively, the temporal and spatial basis vectors:

$$\widetilde{e}_0(\varrho) = \frac{1}{\sqrt{2}}\left[e_{(\widetilde{+})}(\varrho) + e_{(\widetilde{-})}(\varrho)\right], \qquad \widetilde{e}_q(\varrho) = \frac{1}{\sqrt{2}}\left[e_{(\widetilde{+})}(\varrho) - e_{(\widetilde{-})}(\varrho)\right]. \tag{5.12}$$

The transformation equation for the coordinates, according to (5.10), becomes

$$\vartheta^{(\widetilde{\pm})} = \tau_1\left(\overline{\vartheta}^{(\pm)} \mp \tau_2\ \varrho v^{(\pm)} \overline{\vartheta}^{(\mp)}\right) = \tau_1\left(v^{(\pm)} \mp \tau_2\ \varrho v^2\right) dt, \tag{5.13}$$

which gives the general transformation equations for spatial and temporal coordinates as follows ($\vec{e}_q \equiv e_1, q \equiv q^1$):

$$d\widetilde{t} = \tau_1 dt, \qquad d\widetilde{q} = \tau_1\left[dq\left(1 + \frac{\tau_2 \varrho v_q}{\sqrt{2}}\right) - \frac{\tau_2 \varrho}{\sqrt{2}} dt\right] = \tau_1\left(dq - \frac{\tau_2 \varrho}{\sqrt{2}\gamma_q^2} dt\right). \tag{5.14}$$

Hence, the general metric (4.17) in $\widetilde{\mathcal{M}}_2$ reads

$$g \equiv d\widetilde{s}_q^2 = g_{\widetilde{r}\widetilde{s}} d\widetilde{q}^{\widetilde{r}} \otimes d\widetilde{q}^{\widetilde{s}} = \left[\left(\Omega_{(+)}^{(+)}\right)^2 + \Omega_{(-)}^{(+)}\Omega_{(+)}^{(-)}\right] ds_q^2 + \Omega_{(+)}^{(+)}\left(\Omega_{(-)}^{(+)} + \Omega_{(+)}^{(-)}\right)(dt \otimes dt + dq \otimes dq) - 2\Omega_{(+)}^{(+)}\left(\Omega_{(-)}^{(+)} - \Omega_{(+)}^{(-)}\right) dt \otimes dq, \tag{5.15}$$

provided that

$$g_{\widetilde{0}\widetilde{0}} = \left(1 + \frac{\varrho v_q}{\sqrt{2}}\right)^2 - \frac{\varrho^2}{2}, \qquad g_{\widetilde{1}\widetilde{1}} = -\left(1 - \frac{\varrho v_q}{\sqrt{2}}\right)^2 + \frac{\varrho^2}{2}, \qquad g_{\widetilde{1}\widetilde{0}} = g_{\widetilde{0}\widetilde{1}} = -\sqrt{2}\varrho. \tag{5.16}$$

The difference of the vector, $d\widehat{q} \in M_2$ (3.5), and the vector, $d\widetilde{q} \in \widetilde{\mathcal{M}}_2$ (5.11), can be interpreted by the second observer as being due to the deformation/distortion of flat space M_2. However, this difference with equal justice can be interpreted by him as a definite criterion for the *absolute* character of his own state of acceleration in M_2, rather than to any absolute quality of a deformation/distortion of M_2. To prove this assertion, note that the transformation equations (5.14) give a reasonable change at low velocities $v_q \simeq 0$, as

$$d\widetilde{t} = \tau_1 dt, \qquad d\widetilde{q} \simeq \tau_1\left(dq - \frac{\tau_2\varrho}{\sqrt{2}} dt\right), \tag{5.17}$$

thereby

$$\Omega_{(+)}^{(+)} = \Omega_{(-)}^{(-)} = \tau_1\left(1 + \tau_2\overline{\varrho}^2\right), \qquad \Omega_{(-)}^{(+)} = -\Omega_{(+)}^{(-)} = \tau_1(1 - \tau_2)\overline{\varrho}. \tag{5.18}$$

Then (5.17) becomes conventional transformation equations to accelerated ($a_{\rm net} \neq 0$) axes if we assume that $d(\tau_2\varrho)/\sqrt{2}dt = a_{\rm net}$ and $\tau_1(v_q \simeq 0) = 1$, where $a_{\rm net}$ is a magnitude of proper net acceleration. In high-velocity limit $v_q \simeq 1$, $\overline{\varrho} \simeq 0$ ($d\eta^{(-)} = v^{(-)}dt \simeq 0$, $v^{(+)} \simeq v \simeq \sqrt{2}$), we have

$$\Omega_{(+)}^{(+)} = \Omega_{(-)}^{(-)} = \tau_1, \qquad \Omega_{(+)}^{(-)} = 0, \qquad \Omega_{(-)}^{(+)} = \tau_1(1 - \tau_2)\sqrt{2}\varrho, \tag{5.19}$$

and so (5.14) and (5.15), respectively, give

$$d\widetilde{t} = \tau_1 dt \simeq \tau_1 dq \simeq d\widetilde{q}, \tag{5.20}$$

$$d\widetilde{s}_q^2 \simeq \left[\left(1 + \frac{\varrho}{\sqrt{2}}\right)^2 - \frac{\varrho^2}{2}\right] d\widetilde{t} \otimes dt + \left[-\left(1 - \frac{\varrho}{\sqrt{2}}\right)^2 + \frac{\varrho^2}{2}\right] d\widetilde{q} \otimes d\widetilde{q} - 2\sqrt{2}\varrho\, d\widetilde{t} \otimes d\widetilde{q} \simeq \tau_1^2 ds_q^2 = 0. \tag{5.21}$$

To this end, *the inertial effects become zero*. Let $\vec{a}_{\text{net}}$ be a local net 3-acceleration of an arbitrary observer with proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ measured in the rest frame:

$$\vec{a}_{\text{net}} = \frac{d\vec{u}}{ds} = \vec{a} \wedge \vec{u} + \vec{\omega} \times \vec{u}, \tag{5.22}$$

where $\mathbf{u}$ is the 4-velocity. A magnitude of $\vec{a}_{\text{net}}$ can be computed as the simple invariant of the absolute value $|d\mathbf{u}/ds|$ as measured in rest frame:

$$|\mathbf{a}| = \left|\frac{d\mathbf{u}}{ds}\right| = \left(\frac{du^l}{ds}, \frac{du_l}{ds}\right)^{1/2}. \tag{5.23}$$

Following [57, 58], let us define an orthonormal frame $e_{\hat{a}}$, carried by an accelerated observer, who moves with proper linear 3-acceleration and $\vec{a}(s)$ and proper 3-rotation $\vec{\omega}(s)$. Particular frame components are denoted by hats, $\hat{0}, \hat{1},$ and so forth. Let the zeroth leg of the frame $e_{\hat{0}}$ be 4-velocity $\mathbf{u}$ of the observer that is tangent to the worldline at a given event $x^l(s)$ and we parameterize the remaining spatial triad frame vectors $e_{\hat{i}}$, orthogonal to $e_{\hat{0}}$, also by (s). The spatial triad $e_{\hat{i}}$ rotates with proper 3-rotation $\vec{\omega}(s)$. The 4-velocity vector naturally undergoes Fermi-Walker transport along the curve C, which guarantees that $e_{\hat{0}}(s)$ will always be tangent to C determined by $x^l = x^l(s)$:

$$\frac{de_{\hat{a}}}{ds} = -\Omega e_{\hat{a}}, \tag{5.24}$$

where the antisymmetric rotation tensor Ω splits into a Fermi-Walker transport part Ω_{FW} and a spatial rotation part Ω_{SR}:

$$\Omega_{\text{FW}}^{lk} = a^l u^k - a^k u^l, \qquad \Omega_{\text{SR}}^{lk} = u_m \omega_n \varepsilon^{mnlk}. \tag{5.25}$$

The 4-vector of rotation ω^l is orthogonal to 4-velocity u^l, therefore, in the rest frame it becomes $\omega^l(0, \vec{\omega})$, and ε^{mnlk} is the Levi-Civita tensor with $\varepsilon^{0123} = -1$. Then (5.17) immediately indicates that we may introduce the very concept of the local *absolute acceleration* (in Newton's terminology) brought about via the Fermi-Walker transported frames as

$$\vec{a}_{\text{abs}} \equiv \vec{e}_q \frac{d(\tau_2 \varrho)}{\sqrt{2}ds_q} = \vec{e}_q \left|\frac{de_{\hat{0}}}{ds}\right| = \vec{e}_q |\mathbf{a}|, \tag{5.26}$$

where we choose the system $S_{(2)}$ in such a way as the axis $\vec{e}_q$ lies along the net 3-acceleration $(\vec{e}_q \parallel \vec{e}_a)$, $(\vec{e}_a = \vec{a}_{\text{net}}/|\vec{a}_{\text{net}}|)$. Hereinafter, we may simplify the flat-deformation tensor Ω_A^B by setting $\tau_2 = 1$, such that (5.9) becomes

$$\Omega_{(+)}^{(+)} = \Omega_{(-)}^{(-)} \equiv \Omega(\overline{\varrho}) = 1 + \overline{\varrho}^2, \qquad \Omega_{(+)}^{(-)} = \Omega_{(-)}^{(+)} = 0, \tag{5.27}$$

and the general metric (4.17) in $\widetilde{\mathcal{M}}_2$ reads $d\tilde{s}_q^2 = \Omega^2(\overline{\varrho})ds_q^2$. Hence (5.26) gives

$$\varrho = \sqrt{2}\int_0^{s_q} |\mathbf{a}| ds'_q. \tag{5.28}$$

Combining (5.14) and (5.26), we obtain the key relation between a so-called *inertial acceleration*, arisen due to the curvature of MS,

$$\vec{a}_{\text{in}} = \vec{e}_a a_{\text{in}}, \quad a_{\text{in}} = \frac{d^2\tilde{q}}{d\tilde{s}_q^2} = -\Gamma^1_{\tilde{r}\tilde{s}}(\varrho)\frac{d\tilde{q}^{\tilde{r}}}{d\tilde{s}_q}\frac{d\tilde{q}^{\tilde{s}}}{d\tilde{s}_q} = \frac{1}{\sqrt{2}}\left(\frac{d^2\tilde{\eta}^{(+)}}{d\tilde{s}_q^2} - \frac{d^2\tilde{\eta}^{(-)}}{d\tilde{s}_q^2}\right), \tag{5.29}$$

and a local *absolute acceleration* as follows:

$$\Omega^2(\overline{\varrho})\gamma_q \vec{a}_{\text{in}} = -\vec{a}_{\text{abs}}, \tag{5.30}$$

where $\Gamma^1_{\tilde{r}\tilde{s}}(\varrho)$ are the Christoffel symbols constructed by the metric (5.16). Then (5.30) provides a quantitative means for the *inertial force* $\vec{f}_{(\text{in})}$:

$$\vec{f}_{(\text{in})} = m\vec{a}_{\text{in}} = -m\Gamma^1_{\tilde{r}\tilde{s}}(\varrho)\frac{d\tilde{q}^{\tilde{r}}}{d\tilde{s}_q}\frac{d\tilde{q}^{\tilde{s}}}{d\tilde{s}_q} = -\frac{m\vec{a}_{\text{abs}}}{\Omega^2(\overline{\varrho})\gamma_q}. \tag{5.31}$$

In case of absence of rotation, we may write the local *absolute acceleration* (5.26) in terms of the relativistic force f^l acting on a particle with coordinates $x^l(s)$:

$$f^l\left(f^0, \vec{f}\right) = m\frac{d^2x^l}{ds^2} = \Lambda^l_k(\vec{v})F^k. \tag{5.32}$$

Here $F^k(0, \vec{F})$ is the force defined in the rest frame of the test particle, and $\Lambda^l_k(\vec{v})$ is the Lorentz transformation matrix $(i, j = 1, 2, 3)$:

$$\Lambda^i_j = \delta_{ij} - (\gamma - 1)\frac{v_i v_j}{|\vec{v}|^2}, \qquad \Lambda^0_i = \gamma v_i, \tag{5.33}$$

where $\gamma = (1 - \vec{v}^2)^{-1/2}$. So

$$|\mathbf{a}| = \frac{1}{m}\left|f^l\right| = \frac{1}{m}\left(f^l f_l\right)^{1/2} = \frac{1}{m\gamma}\left|\vec{f}\right|, \tag{5.34}$$

and hence (5.31), (5.26), and (5.34) give

$$\vec{f}_{(\text{in})} = -\frac{1}{\Omega^2(\overline{\varrho})\gamma_q\gamma}\left[\vec{F} + (\gamma - 1)\frac{\vec{v}\left(\vec{v}\cdot\vec{F}\right)}{|\vec{v}|^2}\right]. \tag{5.35}$$

At low velocities $v_q \simeq |\vec{v}| \simeq 0$ and tiny accelerations we usually experience, one has $\Omega(\overline{\varrho}) \simeq 1$; therefore (5.35) reduces to the conventional nonrelativistic law of inertia:

$$\vec{f}_{(\mathrm{in})} = -m\vec{a}_{\mathrm{abs}} = -\vec{F}. \tag{5.36}$$

At high velocities $v_q \simeq |\vec{v}| \simeq 1$ $(\Omega(\overline{\varrho}) \simeq 1)$, if $(\vec{v} \cdot \vec{F}) \neq 0$, the inertial force (5.35) becomes

$$\vec{f}_{(\mathrm{in})} \simeq -\frac{1}{\gamma}\vec{e}_v\left(\vec{e}_v \cdot \vec{F}\right), \tag{5.37}$$

and, in agreement with (5.21), it vanishes in the limit of the photon ($|\vec{v}| = 1,\ m = 0$). Thus, it takes force to disturb an inertia state, that is, to make the *absolute acceleration* ($\vec{a}_{\mathrm{abs}} \neq 0$). The *absolute acceleration* is due to the real deformation/distortion of the space M_2. The *relative* $(d(\tau_2\varrho)/ds_q = 0)$ acceleration (in Newton's terminology) (both magnitude and direction), to the contrary, has nothing to do with the deformation/distortion of the space M_2 and, thus, it cannot produce inertia effects.

6. Beyond the Hypothesis of Locality

The standard geometrical structures, referred to a noninertial coordinate frame of accelerating and rotating observer in Minkowski spacetime, were computed on the base of the hypothesis of locality [59–66], which in effect replaces an accelerated observer at each instant with a momentarily comoving inertial observer along its wordline. This assumption represents strict restrictions, because, in other words, it approximately replaces a noninertial frame of reference $\widetilde{S}_{(2)}$, which is held stationary in the deformed/distorted space $\widetilde{\mathcal{M}}_2 \equiv V_2^{(\varrho)}$ $(\varrho \neq 0)$, with a continuous infinity set of the inertial frames $\{S_{(2)},\ S'_{(2)},\ S''_{(2)},\ldots\}$ given in the flat $M_2(\varrho = 0)$. In this situation the use of the hypothesis of locality is physically unjustifiable. Therefore, it is worthwhile to go beyond the hypothesis of locality with special emphasis on distortion of MS, which, we might expect, will essentially improve the standard results. The notation will be slightly different from the previous section. We denote the orthonormal frame $e_{\hat{a}}$ (5.24), carried by an accelerated observer, with the over "*breve*" such that

$$\breve{e}_{\hat{a}} = \overline{e}^{\mu}_{\hat{a}}\overline{e}_{\mu} = \breve{e}^{\mu}_{\hat{a}}\breve{e}_{\mu}, \qquad \breve{\vartheta}^{\hat{b}} = \overline{e}^{\hat{b}}_{\mu}\overline{\vartheta}^{\mu} = \breve{e}^{\hat{b}}_{\mu}\breve{\vartheta}^{\mu}, \tag{6.1}$$

with $\overline{e}_{\mu} = \partial_{\mu} = \partial/(\partial x^{\mu})$, $\breve{e}_{\mu} = \breve{\partial}_{\mu} = \partial/\partial\breve{x}^{\mu}$, and $\overline{\vartheta}^{\mu} = dx^{\mu}$, $\breve{\vartheta}^{\mu} = d\breve{x}$. Here, following [58, 64], we introduced a geodesic coordinate system $\breve{x}^{\mu}$—*coordinates relative to the accelerated observer* (laboratory coordinates)—in the neighborhood of the accelerated path. The coframe members $\{\breve{\vartheta}^{\hat{b}}\}$ are the objects of dual counterpart: $\breve{e}_{\hat{a}}\rfloor\breve{\vartheta}^{\hat{b}} = \delta^{b}_{a}$. We choose the zeroth leg of the frame, $\breve{e}_{\hat{0}}$, as before, to be the unit vector $\mathbf{u}$ that is tangent to the worldline at a given event $x^{\mu}(s)$, where (s) is a proper time measured along the accelerated path by the standard (static inertial) observers in the underlying global inertial frame. The condition of orthonormality for the frame field $\bar{e}^{\mu}_{\hat{a}}$ reads $\eta_{\mu\nu}\bar{e}^{\ \mu}_{\hat{a}}\bar{e}^{\ \nu}_{\hat{b}} = o_{\hat{a}\hat{b}} = \mathrm{diag}(+ - - -)$. The antisymmetric acceleration tensor Φ_{ab} [64–68, 121–125] is given by

$$\Phi^{b}_{a} := \bar{e}^{\hat{b}}_{\mu}\frac{d\bar{e}^{\mu}_{\hat{a}}}{ds} = \bar{e}^{\hat{b}}_{\mu}u^{\lambda}\breve{\nabla}_{\lambda}\bar{e}^{\mu}_{\hat{a}} = u\rfloor\breve{\Gamma}^{b}_{a}, \tag{6.2}$$

provided that $\breve{\Gamma}^{b}_{a} = \breve{\Gamma}^{b}_{a\mu}d\breve{x}^{\mu}$, where $\breve{\Gamma}^{b}_{a\mu}$ is the metric compatible, torsion-free Levi-Civita connection. According to (5.24) and (5.25), and in analogy with the Faraday tensor, one can identify $\Phi_{ab} \rightarrow (-\mathbf{a}, \omega)$, with $\mathbf{a}(s)$ as the translational acceleration $\Phi_{0i} = -a_i$ and $\omega(s)$ as the frequency of rotation of the local spatial frame with respect to a nonrotating (Fermi-Walker transported) frame $\Phi_{ij} = -\varepsilon_{ijk}\omega^{k}$. The invariants constructed out of Φ_{ab} establish the acceleration scales and lengths. The hypothesis of locality holds for huge proper acceleration lengths $|I|^{-1/2} \gg 1$ and $|I^{*}|^{-1/2} \gg 1$, where the scalar invariants are given by $I = (1/2)\,\Phi_{ab}\Phi^{ab} = -\vec{a}^{2} + \vec{\omega}^{2}$ and $I^{*} = (1/4)\,\Phi^{*}_{ab}\Phi^{ab} = -\vec{a}\cdot\vec{\omega}$ ($\Phi^{*}_{ab} = \varepsilon_{abcd}\Phi^{cd}$) [64–66, 121–125]. Suppose that the displacement vector $z^{\mu}(s)$ represents the position of the accelerated observer. According to the hypothesis of locality, at any time (s) along the accelerated worldline the hypersurface orthogonal to the worldline is Euclidean space and we usually describe some event on this hypersurface (*local coordinate system*) at x^{μ} to be at $\breve{x}^{\mu}$, where x^{μ} and $\breve{x}^{\mu}$ are connected via $\breve{x}^{0} = s$ and

$$x^{\mu} = z^{\mu}(s) + \breve{x}^{i}\bar{e}^{\mu}_{\hat{i}}(s). \tag{6.3}$$

Let $\breve{q}^{r}(\breve{q}^{0}, \breve{q}^{1})$ be *coordinates relative to the accelerated observer* in the neighborhood of the accelerated path in MS, with spacetime components implying

$$d\breve{q}^{0} = d\breve{x}^{0}, \qquad d\breve{q}^{1} = \left|d\vec{\breve{x}}\right|, \qquad \vec{\breve{e}} = \frac{d\vec{\breve{x}}}{d\breve{q}^{1}} = \frac{d\vec{\breve{x}}}{\left|d\vec{\breve{x}}\right|}, \qquad \vec{\breve{e}}\cdot\vec{\breve{e}} = 1. \tag{6.4}$$

As long as a locality assumption holds, we may describe, with equal justice, the event at x^{μ} (6.3) to be at point $\breve{q}^{r}$, such that x^{μ} and $\breve{q}^{r}$, in full generality, are connected via $\breve{q}^{0} = s$ and

$$x^{\mu} = z^{\mu}_{q}(s) + \breve{q}^{1}\bar{\beta}^{\mu}_{\hat{1}}(s), \tag{6.5}$$

where the displacement vector from the origin reads $dz^{\mu}_{q}(s) = \bar{\beta}^{\mu}_{\hat{0}}d\breve{q}^{0}$, and the components $\bar{\beta}^{\mu}_{\hat{r}}$ can be written in terms of $\bar{e}^{\ \mu}_{\hat{a}}$. Actually, from (6.3) and (6.5) we may obtain

$$\begin{aligned} dx^{\mu} &= dz^{\mu}_{q}(s) + d\breve{q}^{1}\bar{\beta}^{\mu}_{\hat{1}}(s) + \breve{q}^{1}d\bar{\beta}^{\mu}_{\hat{1}}(s) \\ &= \left[\bar{\beta}^{\mu}_{\hat{0}}\left(1 + \breve{q}^{1}\breve{\varphi}_{0}\right) + \bar{\beta}^{\mu}_{\hat{1}}\breve{q}^{1}\breve{\varphi}_{1}\right]d\breve{q}^{0} \\ &\quad + \bar{\beta}^{\mu}_{\hat{1}}d\breve{q}^{1} \equiv dz^{\mu}(s) + d\breve{x}^{i}\bar{e}^{\mu}_{\hat{i}}(s) + \breve{x}^{i}d\bar{e}^{\mu}_{\hat{i}}(s) \\ &= \left[\bar{e}^{\mu}_{\hat{0}}\left(1 + \breve{x}^{i}\Phi^{0}_{i}\right) + \bar{e}^{\mu}_{\hat{j}}\breve{x}^{i}\Phi^{j}_{i}\right)\right]d\breve{x}^{0} + \bar{e}^{\mu}_{\hat{i}}d\breve{x}^{i}, \end{aligned} \tag{6.6}$$

where $d\overline{\beta}^{\mu}_{\hat{1}}(s)$ is written in the basis $\overline{\beta}^{\mu}_{\hat{a}}$ as $d\overline{\beta}^{\mu}_{\hat{1}} = (\breve{\varphi}_0\overline{\beta}^{\mu}_{\hat{0}} + \breve{\varphi}_1\overline{\beta}^{\mu}_{\hat{1}})d\breve{q}^0$. Equation (6.6) holds by identifying

$$\overline{\beta}^{\mu}_{\hat{0}}(1+\breve{q}^1\breve{\varphi}_0) \equiv \overline{e}^{\mu}_{\hat{0}}(1+\breve{x}^i\Phi^0_i), \qquad \overline{\beta}^{\mu}_{\hat{1}}\breve{q}^1\breve{\varphi}_1 \equiv \overline{e}^{\mu}_{\hat{j}}\breve{x}^i\Phi^j_i, \qquad \overline{\beta}^{\mu}_{\hat{1}}d\breve{q}^1 \equiv \overline{e}^{\mu}_{\hat{i}}d\breve{x}^i. \tag{6.7}$$

Choosing $\overline{\beta}_{\hat{0}}^{\ \mu} \equiv \overline{e}_{\hat{0}}^{\ \mu}$, we have then

$$\breve{q}^1\breve{\varphi}_0 = \breve{x}^i\Phi^0_i, \qquad \overline{\beta}^{\mu}_{\hat{1}} = \overline{e}^{\mu}_{\hat{i}}\breve{e}^i, \qquad \breve{q}^1\breve{\varphi}_1 = \breve{x}^i\Phi^j_i\breve{e}_j^{-1}, \tag{6.8}$$

with $\breve{e}^j\breve{e}_i^{-1} = \delta^j_i$. Consequently, (6.6) yields the standard metric of semi-Riemannian 4D background space $V_4^{(0)}$, in noninertial system of the accelerating and rotating observer, computed on the base of hypothesis of locality:

$$\begin{aligned}\breve{g} &= \eta_{\mu\nu}dx^\mu \otimes dx^\nu \\ &= \left[\left(1+\vec{a}\cdot\vec{\breve{x}}\right)^2 + \left(\vec{\omega}\cdot\vec{\breve{x}}\right)^2 - (\vec{\omega}\cdot\vec{\omega})\left(\vec{\breve{x}}\cdot\vec{\breve{x}}\right)\right]d\breve{x}^0\otimes d\breve{x}^0 \\ &\quad - 2\left(\vec{\omega}\wedge\vec{\breve{x}}\right)\cdot d\vec{\breve{x}}\otimes d\breve{x}^0 - d\vec{\breve{x}}\otimes d\vec{\breve{x}}.\end{aligned} \tag{6.9}$$

This metric was derived by [59] and [63], in agreement with [99] and [62] (see also [64–66]). We see that the hypothesis of locality leads to the 2D semi-Riemannian MS space: $V_2^{(0)}$ with the incomplete metric $\breve{g}$ ($\varrho = 0$):

$$\begin{aligned}\breve{g} \;&= \left[(1+\breve{q}^1\breve{\varphi}_0)^2 - (\breve{q}^1\breve{\varphi}_1)^2\right]d\breve{q}^0\otimes d\breve{q}^0 \\ &\quad -2\ (\breve{q}^1\breve{\varphi}_1)d\breve{q}^1\otimes d\breve{q}^0 - d\breve{q}^1\otimes d\breve{q}^1.\end{aligned} \tag{6.10}$$

Therefore, our strategy now is to deform the metric (6.10) by carrying out an additional deformation of semi-Riemannian 4D background space $V_4^{(0)} \rightarrow \widetilde{\mathcal{M}}_4 \equiv V_4^{(\varrho)}$, which, as a corollary, will recover the complete metric $g(\varrho \neq 0)$ (5.15) of the distorted MS-$V_2^{(\varrho)}$. According to (2.3), this means that we should find the first deformation matrices, $\pi(\varrho) := (\pi^{\hat{b}}_{\hat{a}})(\varrho)$, which yield the local tetrad deformations:

$$e_{\hat{c}} = \pi^{\hat{a}}_{\hat{c}}\breve{e}_{\hat{a}}, \qquad \vartheta^{\hat{c}} = \pi^{\hat{c}}_{\hat{b}}\breve{\vartheta}^{\hat{b}}, \qquad e\vartheta = e_{\hat{a}}\otimes\vartheta^{\hat{a}} = \Omega^a_{\hat{b}}\breve{e}_{\hat{a}}\otimes\breve{\vartheta}^{\hat{b}}, \tag{6.11}$$

where $\Omega^{\hat{a}}_{\hat{b}}(\varrho) = \pi^{\hat{a}}_{\hat{c}}(\varrho)\pi^{\hat{c}}_{\hat{b}}(\varrho)$ is referred to as the anholonomic *deformation tensor* and that the resulting deformed metric of the space $V_4^{(\varrho)}$ can be split as

$$g_{\mu\nu}(\varrho) = \Upsilon^2(\varrho)\breve{g}_{\mu\nu} + \gamma_{\mu\nu}(\varrho), \tag{6.12}$$

provided that

$$\gamma_{\mu\nu}(\varrho) = \left[\gamma_{\hat{a}\hat{b}} - \Upsilon^2(\varrho)o_{\hat{a}\hat{b}}\right]\breve{e}^{\hat{a}}_{\mu}\breve{e}^{\hat{b}}_{\nu}, \qquad \gamma_{\hat{c}\hat{d}} = o_{\hat{a}\hat{b}}\pi^{\hat{a}}_{\hat{c}}\pi^{\hat{b}}_{\hat{d}}, \tag{6.13}$$

where $\Upsilon(\varrho) = \pi_{\hat{a}}^{\hat{a}}(\varrho)$ and $\gamma_{\hat{a}\hat{b}}(\breve{x})$ are the second deformation matrices. Let the Latin letters $\hat{r}, \hat{s}, \ldots = 0, 1$ be the anholonomic indices referred to the anholonomic frame $e_{\hat{r}} = e_{\hat{r}}^{s} \partial_{\tilde{s}}$, defined on the $V_2^{(\varrho)}$, with $\partial_{\tilde{s}} = \partial / \partial \tilde{q}^{\tilde{s}}$ as the vectors tangent to the coordinate lines. So, a smooth differential 2D-manifold $V_2^{(\varrho)}$ has at each point $\tilde{q}^s$ a tangent space $\widetilde{T}_{\tilde{q}} V_2^{(\varrho)}$, spanned by the frame, $\{e_{\hat{r}}\}$, and the coframe members $\vartheta^{\hat{r}} = e_{\tilde{s}}^{\hat{r}} d\tilde{q}^{\tilde{s}}$, which constitute a basis of the covector space $\widetilde{T}_{\tilde{q}}^{\star} V_2^{(\varrho)}$. All this nomenclature can be given for $V_2^{(0)}$ too. Then, we may calculate corresponding vierbein fields $\breve{e}_r^{\hat{s}}$ and $e_r^{\hat{s}}$ from

$$\breve{g}_{rs} = \breve{e}_r^{\hat{r}'} \breve{e}_s^{\hat{s}'} o_{\hat{r}'\hat{s}'}, \qquad g_{\tilde{r}\tilde{s}} = e_r^{\hat{r}'} e_s^{\hat{s}'} o_{\hat{r}'\hat{s}'}, \tag{6.14}$$

with $\breve{g}_{rs}$ and $g_{\tilde{r}\tilde{s}}$ given by (6.10) and (5.16), respectively. Hence

$$\begin{gathered}
\breve{e}_0^{\hat{0}} = 1 + \vec{a} \cdot \vec{\breve{x}}, \qquad \breve{e}_0^{\hat{1}} = \vec{\omega} \wedge \vec{\breve{x}}, \qquad \breve{e}_1^{\hat{0}} = 0, \qquad \breve{e}_1^{\hat{1}} = 1, \\
e_0^{\hat{0}} = 1 + \frac{\varrho v_q}{\sqrt{2}}, \qquad e_0^{\hat{1}} = \frac{\varrho}{\sqrt{2}}, \qquad e_1^{\hat{0}} = -\frac{\varrho}{\sqrt{2}}, \qquad e_1^{\hat{1}} = 1 - \frac{\varrho v_q}{\sqrt{2}}.
\end{gathered} \tag{6.15}$$

Since a distortion of MS may affect only the MS part of the components $\overline{\beta}_{\hat{r}}^{\mu}$, without relation to the background spacetime part, therefore, a deformation $V_4^{(0)} \rightarrow V_4^{(\varrho)}$ is equivalent to a straightforward generalization $\overline{\beta}_{\hat{r}}^{\mu} \rightarrow \beta_{\hat{r}}^{\mu}$, where

$$\beta_{\hat{r}}^{\mu} = E_{\hat{r}}^{\hat{s}} \overline{\beta}_{\hat{s}}^{\mu}, \qquad E_{\hat{r}}^{\hat{s}} := e_{\hat{r}}^{r'} \breve{e}_{r'}^{\hat{s}}. \tag{6.16}$$

Consequently, (6.16) gives a generalization of (6.3) as

$$x^{\mu} \longrightarrow x_{(\varrho)}^{\mu} = z_{(\varrho)}^{\mu}(s) + \breve{x}^{i} e_{\hat{i}}^{\mu}(s), \tag{6.17}$$

provided that, as before, $\breve{x}^{\mu}$ denotes the coordinates relative to the accelerated observer in 4D background space $V_4^{(\varrho)}$, and according to (6.7), we have

$$e_{\hat{0}}^{\mu} = \beta_{\hat{0}}^{\mu}, \qquad e_{\hat{i}}^{\mu} = \beta_{\hat{1}}^{\mu} \breve{e}_{i}^{-1}. \tag{6.18}$$

A displacement vector from the origin is then $dz^\mu_\varrho(s) = e^\mu_{\hat{0}} d\check{x}^0$. Combining (6.16) and (6.18), and inverting $e^{\hat{s}}_r$ (6.15), we obtain $e^\mu_{\hat{a}} = \pi^{\hat{b}}_{\hat{a}}(\varrho)\bar{e}^\mu_{\hat{b}}$, where

$$\begin{gathered}
\pi^{\hat{0}}_{\hat{0}}(\varrho) \equiv \left(1+\frac{\varrho^2}{2\gamma_q^2}\right)^{-1}\left(1-\frac{\varrho v_q}{\sqrt{2}}\right)\left(1+\vec{a}\cdot\vec{\check{x}}\right),\\
\pi^{\hat{i}}_{\hat{0}}(\varrho) \equiv -\left(1+\frac{\varrho^2}{2\gamma_q^2}\right)^{-1}\frac{\varrho}{\sqrt{2}}\check{e}^i\left(1+\vec{a}\cdot\vec{\check{x}}\right),\\
\pi^{\hat{0}}_{\hat{i}}(\varrho) \equiv \left(1+\frac{\varrho^2}{2\gamma_q^2}\right)^{-1}\left[\left(\vec{\omega}\wedge\vec{\check{x}}\right)\left(1-\frac{\varrho v_q}{\sqrt{2}}\right)-\frac{\varrho}{\sqrt{2}}\right]\check{e}^{-1}_i,\\
\pi^{\hat{j}}_{\hat{i}}(\varrho) = \delta^j_i\pi(\varrho),\\
\pi(\varrho) \equiv \left(1+\frac{\varrho^2}{2\gamma_q^2}\right)^{-1}\left[\left(\vec{\omega}\wedge\vec{\check{x}}\right)\frac{\varrho}{\sqrt{2}}+1+\frac{\varrho v_q}{\sqrt{2}}\right].
\end{gathered}\tag{6.19}$$

Thus,

$$dx^\mu_\varrho = dz^\mu_\varrho(s) + d\check{x}^i e^\mu_{\hat{i}} + \check{x}^i de^\mu_{\hat{i}}(s) = \left(\tau^{\hat{b}} d\check{x}^0 + \pi^{\hat{b}}_{\hat{i}} d\check{x}^i\right)\bar{e}^\mu_{\hat{b}},\tag{6.20}$$

where

$$\tau^{\hat{b}} \equiv \pi^{\hat{b}}_{\hat{0}} + \check{x}^i\left(\pi^{\hat{a}}_{\hat{i}}\Phi^b_a + \frac{d\pi^{\hat{b}}_{\hat{i}}}{ds}\right).\tag{6.21}$$

Hence, in general, the metric in noninertial frame of arbitrary accelerating and rotating observer in Minkowski spacetime is

$$g(\varrho) = \eta_{\mu\nu}dx^\mu_\varrho \otimes dx^\nu_\varrho = W_{\mu\nu}(\varrho)d\check{x}^\mu \otimes d\check{x}^\nu,\tag{6.22}$$

which can be conveniently decomposed according to

$$\begin{gathered}
W_{00}(\varrho) = \pi^2\left[\left(1+\vec{a}\cdot\vec{\check{x}}\right)^2+\left(\vec{\omega}\cdot\vec{\check{x}}\right)^2-(\vec{\omega}\cdot\vec{\omega})\left(\vec{\check{x}}\cdot\vec{\check{x}}\right)\right]+\gamma_{00}(\varrho),\\
W_{0i}(\varrho) = -\pi^2\left(\vec{\omega}\wedge\vec{\check{x}}\right)^i+\gamma_{0i}(\varrho), \qquad W_{ij}(\varrho) = -\pi^2\delta_{ij}+\gamma_{ij}(\varrho),
\end{gathered}\tag{6.23}$$

and also

$$\begin{gathered}
\gamma_{00}(\varrho) = \pi\left[\left(1+\vec{a}\cdot\vec{\check{x}}\right)\zeta^0-\left(\vec{\omega}\wedge\vec{\check{x}}\right)\cdot\vec{\zeta}\right]+\left(\zeta^0\right)^2-\left(\vec{\zeta}\right)^2,\\
\gamma_{0i}(\varrho) = -\pi\zeta^i+\tau^{\hat{0}}\pi^{\hat{0}}_{\hat{i}},\\
\gamma_{ij}(\varrho) = \pi^{\hat{0}}_{\hat{i}}\pi^{\hat{0}}_{\hat{j}}, \qquad \zeta^0 = \pi\left(\tau^{\hat{0}}-1-\vec{a}\cdot\vec{\check{x}}\right), \qquad \vec{\zeta} = \pi\left(\vec{\tau}-\vec{\omega}\wedge\vec{\check{x}}\right).
\end{gathered}\tag{6.24}$$

As we expected, according to (6.22)–(6.24), the matric $g(\varrho)$ is decomposed in the form of (4.11):

$$g(\varrho) = \pi^2(\varrho)\breve{g} + \gamma(\varrho), \tag{6.25}$$

where $\gamma(\varrho) = \gamma_{\mu\nu}(\varrho)d\breve{x}^\mu \otimes d\breve{x}^\nu$ and $\Upsilon(\varrho) = \pi^{\hat{a}}_{\hat{a}}(\varrho) = \pi(\varrho)$. In general, the geodesic coordinates are admissible as long as

$$\left(1 + \vec{a}\cdot\vec{\breve{x}} + \frac{\zeta^0}{\pi}\right)^2 > \left(\vec{\omega}\wedge\vec{\breve{x}} + \frac{\vec{\zeta}}{\pi}\right)^2. \tag{6.26}$$

Equations (6.9) and (6.22) say that the vierbein fields with entries $\eta_{\mu\nu}\overline{e}^{\,\mu}_{\hat{a}}\overline{e}^{\,\nu}_{\hat{b}} = o_{\hat{a}\hat{b}}$ and $\eta_{\mu\nu}e^\mu_{\hat{a}}e^\nu_{\hat{b}} = \gamma_{\hat{a}\hat{b}}$ lead to the relations

$$\begin{gathered}\breve{g} = o_{\hat{a}\hat{b}}\breve{\vartheta}^{\hat{a}}\otimes\breve{\vartheta}^{\hat{b}},\\ g = o_{\hat{a}\hat{b}}\vartheta^{\hat{a}}\otimes\vartheta^{\hat{b}} = \gamma_{\hat{a}\hat{b}}\breve{\vartheta}^{\hat{a}}\otimes\breve{\vartheta}^{\hat{b}} = \left(\Omega^{\hat{c}}_{\hat{a}}\Omega^{\hat{d}}_{\hat{b}}o_{\hat{c}\hat{d}}\right)\overline{\vartheta}^{\hat{a}}\otimes\overline{\vartheta}^{\hat{b}},\end{gathered} \tag{6.27}$$

and that (6.6) and (6.20) readily give the coframe fields:

$$\begin{aligned}\breve{\vartheta}^{\hat{b}} &= \overline{e}^{\,\hat{b}}_{\mu}dx^\mu = \breve{e}^{\hat{b}}_{\mu}d\breve{x}^\mu, \qquad \breve{e}^{\hat{b}}_0 = N^b_0, \qquad \breve{e}^{\hat{b}}_i = N^b_i,\\ \vartheta^{\hat{b}} &= \overline{e}^{\,\hat{b}}_{\mu}dx^\mu_\varrho = e^{\hat{b}}_{\mu}d\breve{x}^\mu = \pi^{\hat{b}}_{\hat{a}}\breve{\vartheta}^{\hat{a}}, \qquad e^{\hat{b}}_0 = \tau^{\hat{b}}, \qquad e^{\hat{b}}_i = \pi^{\hat{b}}_{\hat{i}},\end{aligned} \tag{6.28}$$

where $N^0_0 = N \equiv (1 + \vec{a}\cdot\vec{\breve{x}})$, $N^0_i = 0$, $N^i_0 = N^i \equiv (\vec{\omega}\cdot\vec{\breve{x}})^i$, and $N^j_i = \delta^j_i$. In the standard (3 + 1)-decomposition of spacetime, N and N^i are known as *lapse function* and *shift vector*, respectively [126, 127]. Hence, we may easily recover the frame field $e_{\hat{a}} = e^\mu_{\hat{a}}\breve{e}_\mu = \pi^{\hat{b}}_{\hat{a}}\breve{e}_{\hat{b}}$ by inverting (6.28):

$$\begin{aligned}e_{\hat{0}} &= \frac{\pi}{\pi\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\tau^{\hat{k}}}\breve{e}_0 - \frac{\tau^{\hat{i}}}{\pi\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\tau^{\hat{k}}}\breve{e}_i,\\ e_{\hat{i}} &= -\frac{\pi^{\hat{0}}_{\hat{i}}}{\pi\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\tau^{\hat{k}}}\breve{e}_0 + \pi^{-1}\left[\delta^j_i + \frac{\tau^j\pi^{\hat{0}}_{\hat{i}}}{\pi\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\tau^{\hat{k}}}\right]\breve{e}_j.\end{aligned} \tag{6.29}$$

A *generalized transport* for deformed frame $e_{\hat{a}}$, which includes both the Fermi-Walker transport and distortion of MS, can be written in the following form:

$$\frac{de^\mu_{\hat{a}}}{ds} = \widetilde{\Phi}^b_{\ a}e^\mu_{\hat{b}}, \tag{6.30}$$

where a *deformed acceleration tensor* $\widetilde{\Phi}_a^b$ concisely is given by

$$\widetilde{\Phi} = \left(\frac{d\ln\pi}{ds}\right) + \pi\Phi\pi^{-1}. \tag{6.31}$$

Although the results (6.29)–(6.31) are obtained in the framework of purely classical physics, nevertheless on this base we may straightforwardly put the special-relativistic Dirac equation into a noninertial reference frame by standard method similar to [59]. But we will forbear to write it out here as it is somewhat lengthy and evidently irrelevant to the problem in quest in this paper. It will be interesting topic for another publication.

7. Involving the Background Semi-Riemann Space V_4: Justification for the Introduction of the WPE

We can always choose *natural coordinates* $X^\alpha(T, X, Y, Z) = (T, \vec{X})$ with respect to the axes of the local free-fall coordinate frame $S_4^{(l)}$ in an immediate neighbourhood of any spacetime point $(\breve{x}_p) \in V_4$ in question of the background semi-Riemann space, V_4, over a differential region taken small enough so that we can neglect the spatial and temporal variations of gravity for the range involved. The values of the metric tensor $\breve{g}_{\mu\nu}$ and the affine connection $\breve{\Gamma}^\lambda_{\mu\nu}$ at the point $(\breve{x}_p)$ are necessarily sufficient information for determination of the natural coordinates $X^\alpha(\breve{x}^\mu)$ in the small region of the neighbourhood of the selected point [128]. Then the whole scheme outlined in Section 4 will be held in the frame $S_4^{(l)}$. The relativistic gravitational force $\breve{f}_g^\mu(\breve{x})$ exerted on the test particle of the mass (m) is given by

$$\breve{f}_g^\mu(\breve{x}) = m\frac{d^2\breve{x}^\mu}{d\breve{s}^2} = -m\breve{\Gamma}^\mu_{\nu\lambda}(a)\frac{d\breve{x}^\nu}{d\breve{s}}\frac{d\breve{x}^\lambda}{d\breve{s}}. \tag{7.1}$$

The frame $S_4^{(l)}$ will be valid if only the gravitational force given in this coordinate frame

$$f^\alpha_{g(l)} = \frac{\partial X^\alpha}{\partial\breve{x}^\mu}f^\mu_g \tag{7.2}$$

could be removed by the inertial force, whereas, as before, the two systems S_2 and $S_4^{(l)}$ can be chosen in such a way as the axis $\vec{e}_q$ of $S_{(2)}$ lies $(\vec{e}_q = \vec{e}_f)$ along the acting net force $\vec{f} = \vec{f}_{(l)} + \vec{f}_{g(l)}$, where $\vec{f}_{(l)}$ is the SR value of the unbalanced relativistic force other than gravitational in the frame $S_4^{(l)}$, while the time coordinates in the two systems are taken the same, $q^0 = t = X^0 = T$. Then (5.34) now can be replaced by

$$\frac{1}{\sqrt{2}}\frac{d(\tau_2\varrho)}{ds_q} = \frac{1}{m}\left|f^\alpha_{(l)} + f^\alpha_{g(l)}\right|, \tag{7.3}$$

and according to (5.31), the general *inertial force* reads

$$\breve{\vec{f}}_{(\text{in})} = m\vec{a}_{\text{in}} = -\frac{m\vec{a}_{\text{abs}}}{\Omega^2(\overline{\varrho})\gamma_q} = -\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}\left|f^\alpha_{(l)} - m\frac{\partial X^\alpha}{\partial\breve{x}^\sigma}\breve{\Gamma}^\sigma_{\mu\nu}\frac{d\breve{x}^\mu}{dS}\frac{d\breve{x}^\nu}{dS}\right|. \tag{7.4}$$

Despite totally different and independent sources of gravitation and inertia, at $f^{\alpha}_{(l)} = 0$, (7.4) establishes the independence of free-fall ($v_q = 0$) trajectories of the mass, internal composition, and structure of bodies. This furnishes a justification for the introduction of the WPE. A remarkable feature is that although the inertial force has a nature different than the gravitational force, nevertheless both are due to a distortion of the local inertial properties of, respectively, 2D MS and 4D-background space. The nonvanishing inertial force acting on the photon of energy $h\nu$ and that of effective mass $(h\nu/c^2)$, after inserting units (h, c) which so far was suppressed, can be obtained from (7.4) ($f^{\alpha}_{(l)} = 0$) as

$$\begin{aligned}\breve{\vec{f}}_{(\text{in})} &= -\left(\frac{h\nu}{c^2\Omega^2(\overline{\varrho})}\right)\vec{e}_f\left|\frac{\partial X^{\alpha}}{\partial\breve{x}^{\sigma}}\Gamma^{\sigma}_{\mu\nu}\frac{d\breve{x}^{\mu}}{dT}\frac{d\breve{x}^{\nu}}{dT}\right| \\ &= -\left(\frac{h\nu}{c^2\Omega^2(\overline{\varrho})}\right)\vec{e}_f\left|\left(\frac{d^2\tilde{t}}{dT^2}\right)\frac{dX^{\alpha}}{d\tilde{t}} + \left(\frac{d\tilde{t}}{dT}\right)^2\frac{\partial X^{\alpha}}{\partial\breve{x}^{i}}\frac{du_i}{d\tilde{t}}\right|,\end{aligned} \tag{7.5}$$

provided that $\vec{e}_f = (\vec{X}/|\vec{X}|)$, $v_q = (\vec{e}_f\cdot\breve{\vec{u}}) = |\breve{\vec{u}}|$, $(\gamma_q = \gamma)$ where $\breve{\vec{u}}$ is the velocity of a photon, $(d\breve{\vec{u}}/d\tilde{t})$ is the acceleration, and, $\breve{g}_{\mu\nu}(d\breve{x}^{\mu}/dT)\otimes(d\breve{x}^{\nu}/dT) = 0$. Note that Nordtvedt, Will, and others [129–132] were led to provide rigorous underpinnings to the operational significance of various theories, especially in solar system context, developing the parameterized post-Newtonian (PPN) formalism as a theoretical standard for expressing the predictions of relativistic gravitational theories in terms which could be directly related to experimental observations. To obtain some feeling for this, in the PPN approximation we may calculate the inertial force exerted on the photon in a gravitating system of particles that are bound together by their mutual gravitational attraction to order $\overline{v}^2 \sim G_N\overline{M}/\overline{r}$ of a small parameter, where $\overline{v}$, $\overline{M}$, and $\overline{r}$ are typically the average values of their velocities, masses, and separations, respectively. To this aim, we may expand the metric tensor to the following order: $\breve{g}_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \cdots$, $\breve{g}_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \cdots$, $\breve{g}_{i0} = \overset{3}{g}_{i0} + \overset{5}{g}_{i0} + \cdots$, where $\overset{N}{g}_{\mu\nu}$ denotes the term of order $\overline{v}^N$. Taking into account the standard expansions of the affine connection [128]: $\breve{\Gamma}^{\sigma}_{\mu\nu} = \overset{2}{\Gamma}{}^{\sigma}_{\mu\nu} + \overset{4}{\Gamma}{}^{\sigma}_{\mu\nu} + \cdots$ for the components $\breve{\Gamma}^{i}_{00}$, $\breve{\Gamma}^{i}_{jk}$, and $\breve{\Gamma}^{0}_{0i}$, and that $\breve{\Gamma}^{\sigma}_{\mu\nu} = \overset{3}{\Gamma}{}^{\sigma}_{\mu\nu} + \overset{5}{\Gamma}{}^{\sigma}_{\mu\nu} + \cdots$ for the components $\breve{\Gamma}^{i}_{0j}$, $\breve{\Gamma}^{0}_{00}$, and $\breve{\Gamma}^{0}_{ij}$, where $\overset{2}{\Gamma}{}^{i}_{00} = \overset{2}{\Gamma}{}^{0}_{0i} = -(1/2)(\partial\overset{2}{g}_{00}/\partial\breve{x}^{\,i})$, and so forth; hence to the required accuracy we obtain

$$\begin{aligned}\breve{\vec{f}}^{(2)}_{(\text{in})} &= -\left(\frac{h\nu}{c^2}\right)\vec{e}_f\left|\left(\frac{\partial\overset{1}{X}{}^{\alpha}}{\partial\breve{x}^{\sigma}}\right)\left(\frac{d^2\overset{2}{\breve{x}}{}^{\sigma}}{dT^2}\right)\right| = -\left(\frac{h\nu}{c^2}\right)\left(\frac{d\overset{2}{\breve{\vec{u}}}}{d\tilde{t}}\right) \\ &= -\left(\frac{h\nu}{\gamma c^2}\right)\left[-2\vec{\nabla}\phi + 4\breve{\vec{u}}\left(\breve{\vec{u}}\cdot\vec{\nabla}\phi\right) + O\left(\overline{v}^3\right)\right],\end{aligned} \tag{7.6}$$

where ϕ is the Newton potential, such that $\overset{2}{g}_{00} = 2\phi$, $\overset{2}{g}_{ij} = 2\delta_{ij}\phi$, and $|\breve{\vec{u}}| = 1 + 2\phi + O(\overline{v}^3)$.

8. RTI in the Background Post-Riemannian Geometry

According to (2.21) and (2.22), if the nonmetricity tensor $N_{\lambda\mu\nu} = -\mathfrak{D}_\lambda g_{\mu\nu} \equiv -g_{\mu\nu;\lambda}$ does not vanish, the general formula for the affine connection written in the spacetime components is (also see [118])

$$\Gamma^{\rho}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu} + K^{\rho}_{\mu\nu} - N^{\rho}_{\mu\nu} + \frac{1}{2}N^{\rho}_{(\mu\nu)}, \tag{8.1}$$

where the metric alone determines the torsion-free Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}$, and $K^{\rho}_{\mu\nu} := 2Q_{(\mu\nu)}^{\ \ \ \ \rho} + Q^{\rho}_{\mu\nu}$ is the non-Riemann part—the affine *contortion tensor*. The torsion, $Q^{\rho}_{\mu\nu} = (1/2)T^{\rho}_{\mu\nu} = \Gamma^{\rho}_{[\mu\nu]}$ given with respect to a holonomic frame, $d\vartheta^{\rho} = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components.

8.1. The Principle of Equivalence in the RC Space

The RC manifold, U_4, is a particular case of general metric-affine manifold $\widetilde{M}_4$, restricted by the metricity condition $N_{ab} = 0$, when a nonsymmetric linear connection, Γ, is said to be metric compatible. To avoid any possibility of confusion, here and throughout we again use the first half of Latin alphabet $(a, b, c, \ldots = 0, 1, 2, 3$ rather than $(\pm))$ now to denote the anholonomic indices referred to the tangent space, which is endowed with the Lorentzian metric $o_{ab} := \mathrm{diag}(+---)$. The space, U_4, also locally has the structure of M_4, as has been first pointed out by [133] and developed by [134–137]. In the case of the RC space there also exist orthonormal reference frames which realize an "anholonomic" free-fall elevator. In Hartley's formulation [137], this reads as follows. *For any single point $P \in U_4$, there exist coordinates $\{x^{\mu}\}$ and an orthonormal frame $\{e_a\}$ in a neighborhood of P such that*

$$\begin{matrix} e_a = \delta^{\mu}_a \partial_{x^{\mu}} \\ \Gamma_a^{\ b} = 0 \end{matrix} \quad \text{at } P, \tag{8.2}$$

where Γ^b_a are the connection 1-forms referred to the frame $\{e_a\}$. Therefore the existence of torsion does not violate the PE. Note that since $\nabla\mathbf{g} = 0$ holds in U_4, the arguments showing that $\mathbf{g}$ can be transformed to o at any point P in U_4 are the same as in the case of V_4, while the treatment of the connection must be different: the antisymmetric part of ω can be eliminated only by a suitable choice for the relative orientation of neighbouring tetrads. Actually, let us choose new local coordinates at P, $dx^{\mu} \to dx^a = e^a_{\mu} dx^{\mu}$, related to an inertial frame. Then,

$$g'_{ab} = e^{\mu}_a e^{\nu}_b g_{\mu\nu} = o_{ab}, \qquad \Gamma'^{b}_{ac} = e^b_{\mu} e^{\nu}_a e^{\lambda}_c \left(\Delta^{\mu}_{\nu\lambda} + K^{\mu}_{\nu\lambda}\right) \equiv e^{\lambda}_c \omega^{b}_{a\lambda}. \tag{8.3}$$

As it is argued in [138], the metricity condition ensures that this can be done consistently at every point in spacetime. Suppose that we have a tetrad $\{e_a(x)\}$ at the point P and a tetrad $\{e_a(x + dx)\}$ at another point in a neighbourhood of P; then, we can apply a suitable Lorentz rotation to $e_a(x + dx)$, so that it becomes parallel to $e_a(x)$. Given a vector v at P, it follows that the components $v_c = v \cdot e_c$ do not change under parallel transport from x to $x + dx$, provided that the metricity condition holds. Hence, the connection coefficients $\omega^{ab}_{\mu}(x)$ at P, defined with respect to this particular tetrad field, vanish: $\omega_{\mu}^{ab}(P) = 0$. This property is compatible with $g'_{ab} = o_{ab}$, since Lorentz rotation does not influence the value of the metric at a given point. In more general geometries, where the symmetry of the tangent space is higher than the Poincare group, the usual form of the PE is violated and local physics differs from SR.

8.2. The Generalized Inertial Force Exerted on the Extended Spinning Body in the U_4

We now compute the relativistic inertial force for the motion of the matter, which is distributed over a small region in the U_4 space and consists of points with the coordinates x^{μ}, forming an extended body whose motion in the space, U_4, is represented by a world tube in spacetime. Suppose that the motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $\widetilde{X}^{\mu}(\tau)$, where τ is the proper time on γ. Define

$$\delta x^{\mu} = x^{\mu} - \widetilde{X}^{\mu}, \qquad \delta x^{0} = 0, \qquad u^{\mu} = \frac{d\widetilde{X}^{\mu}}{ds}. \tag{8.4}$$

The *Papapetrou equation of motion for the modified momentum* (see [118, 139–143]) is

$$\frac{\overset{\circ}{\mathfrak{D}}\Theta^{\nu}}{\mathfrak{D}s} = -\frac{1}{2}\overset{\circ}{R}{}^{\nu}_{\mu\sigma\rho}u^{\mu}J^{\sigma\rho} - \frac{1}{2}N_{\mu\rho\lambda}K^{\mu\rho\lambda:\nu}, \tag{8.5}$$

where $K^{\mu}_{\nu\lambda}$ is the contortion tensor,

$$\Theta^{\nu} = P^{\nu} + \frac{1}{u^{0}}\overset{\circ}{\Gamma}{}^{\nu}_{\mu\rho}\left(u^{\mu}J^{\rho 0} + N^{0\mu\rho}\right) - \frac{1}{2u^{0}}K_{\mu\rho}^{\ \ \nu}N^{\mu\rho 0} \tag{8.6}$$

is referred to as the *modified 4-momentum*, $P^{\lambda} = \int \tau^{\lambda 0}d\Omega$ is the ordinary 4-momentum, $d\Omega := dx^{4}$, and the following integrals are defined:

$$\begin{gathered} M^{\mu\rho} = u^{0}\int \tau^{\mu\rho}d\Omega, \qquad M^{\mu\nu\rho} = -u^{0}\int \delta x^{\mu}\tau^{\nu\rho}d\Omega, \qquad N^{\mu\nu\rho} = u^{0}\int s^{\mu\nu\rho}d\Omega, \\ J^{\mu\rho} = \int\left(\delta x^{\mu}\tau^{\rho 0} - \delta x^{\rho}\tau^{\mu 0} + s^{\mu\rho 0}\right)d\Omega = \frac{1}{u^{0}}\left(-M^{\mu\rho 0} + M^{\rho\mu 0} + N^{\mu\rho 0}\right), \end{gathered} \tag{8.7}$$

where $\tau^{\mu\rho}$ is the energy-momentum tensor for particles, and $s^{\mu\nu\rho}$ is the spin density. The quantity $J^{\mu\rho}$ is equal to $\int(\delta x^{\mu}\tau^{kl} - \delta x^{\rho}\tau^{\mu\lambda} + s^{\mu\rho\lambda})\, dS_{\lambda}$ taken for the volume hypersurface, so it is a tensor, which is called the *total spin tensor*. The quantity $N^{\mu\nu\rho}$ is also a tensor. The relation $\delta x^{0} = 0$ gives $M^{0\nu\rho} = 0$. It was assumed that the dimensions of the body are small, so integrals with two or more factors δx^{μ} multiplying $\tau^{\nu\rho}$ and integrals with one or more factors δx^μ multiplying $s^{\nu\rho\lambda}$ can be neglected. The *Papapetrou equations of motion for the spin* (see [118, 139–143]) are

$$\frac{\overset{\circ}{\mathfrak{D}}}{\mathfrak{D}s}J^{\lambda\nu} = u^\nu\Theta^\lambda - u^\lambda\Theta^\nu + K^{\lambda}_{\mu\rho}N^{\nu\mu\rho} + \frac{1}{2}K_{\mu\rho}^{\;\;\lambda}N^{\mu\nu\rho} - K^{\nu}_{\mu\rho}N^{\lambda\mu\rho} - \frac{1}{2}K_{\mu\rho}^{\;\;\nu}N^{\mu\rho\lambda}. \tag{8.8}$$

Calculating from (8.5) the particle 4-acceleration is

$$\frac{1}{m}f^\mu_g(x) = \frac{d^2x^\mu}{d\widetilde{s}^2} = -\Gamma^\mu_{\nu\lambda}\left[u^\nu u^\lambda + \frac{1}{u^0}\overset{\circ}{\Gamma}{}^{\mu}_{\nu\rho}\left(u^\nu J^{\rho 0} + N^{0\nu\rho}\right)\right] + \frac{1}{2u^0}K^\mu_{\nu\rho}N^{\nu\rho 0} - \frac{1}{2}\overset{\circ}{R}{}^{\mu}_{\nu\sigma\rho}u^\nu J^{\sigma\rho} - \frac{1}{2}N_{\nu\rho\lambda}K^{\nu\rho\lambda:\mu}. \tag{8.9}$$

Thus, the relativistic inertial force, exerted on the extended spinning body moving in the RC space U_4, can be found to be

$$\begin{aligned}\vec{f}_{(\mathrm{in})}(x) = m\vec{a}_{\mathrm{in}}(x) &= -\frac{m\vec{a}_{\mathrm{abs}}(x)}{\Omega^2(\overline{\varrho})\gamma_q}\\ &= -m\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}\Bigg|\frac{1}{m}f^\alpha_{(l)} - \frac{\partial X^\alpha}{\partial x^\mu}\\ &\quad\times\left[\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}u^\nu u^\lambda + \frac{1}{u^0}\overset{\circ}{\Gamma}{}^{\mu}_{\nu\rho}\left(u^\nu J^{\rho 0} + N^{0\nu\rho}\right)\right.\\ &\quad\left.-\frac{1}{2u^0}K_{\nu\rho}^{\;\;\mu}N^{\nu\rho 0} + \frac{1}{2}\overset{\circ}{R}{}^{\mu}_{\nu\sigma\rho}u^\nu J^{\sigma\rho} + \frac{1}{2}N_{\nu\rho\lambda}K^{\nu\rho\lambda:\mu}\right]\Bigg|.\end{aligned} \tag{8.10}$$

In particular, if the spin density vanishes, $s^{\mu\nu\rho} = 0$, from the conservation law we get then $\tau^{\mu\rho} = \tau^{\rho\mu}$, $M^{\mu\rho} = M^{\rho\mu}$, $M^{\mu\nu\rho} = M^{\mu\rho\rho}$, $N^{\mu\nu\rho} = 0$, and

$$J^{\mu\rho} = L^{\mu\rho} = \int\left(\delta x^\mu\tau^{\rho 0} - \delta x^\rho\tau^{\mu 0}\right)d\Omega = \frac{1}{u^0}\left(-M^{\mu\rho 0} + M^{\rho\mu 0}\right), \tag{8.11}$$

where $L^{\mu\rho}$ is the angular momentum tensor. The modified 4-momentum (8.6) reduces to

$$\Theta^\nu = P^\nu + \frac{\overset{\circ}{\mathfrak{D}}}{\mathfrak{D}s}L^{\nu\lambda}u_\lambda. \tag{8.12}$$

Equation (8.8) can be recast in the following form:

$$\frac{\overset{\circ}{\mathfrak{D}}}{\mathfrak{D}s}L^{\lambda\nu} = u^\nu\Theta^\lambda - u^\lambda\Theta^\nu, \tag{8.13}$$

while (8.5) becomes

$$\frac{\overset{\circ}{\mathfrak{D}}\Theta^{\nu}}{\mathfrak{D}s}=-\frac{1}{2}\overset{\circ}{R}{}^{\mu}_{\nu\sigma\rho}u^{\mu}L^{\sigma\rho}, \tag{8.14}$$

which give the relativistic inertial force exerted on the spinless extended body moving in the RC space U_4 as follows:

$$\vec{f}_{(\text{in})}(x)=-m\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}\left|\frac{1}{m}f^{\alpha}_{(l)}-\frac{\partial X^{\alpha}}{\partial x^{\mu}}\left[\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}u^{\nu}u^{\lambda}+\frac{1}{u^0}\overset{\circ}{\Gamma}{}^{\mu}_{\nu\rho}u^{\nu}L^{\rho 0}+\frac{1}{2}\overset{\circ}{R}{}^{\mu}_{\nu\sigma\rho}u^{\nu}L^{\sigma\rho}\right]\right|. \tag{8.15}$$

If the body is not spatially extended, then it is referred to as a *particle*. The corresponding condition $\delta x^{\alpha}=0$ gives $M^{\mu\nu\rho}=0$, and $L^{\mu\rho}=0$. Therefore $(u^{\lambda}/u^{0})N^{\mu\nu 0}-N^{\mu\nu\lambda}=0$, which gives $N^{\mu\nu\rho}=u^{\mu}J^{\nu\rho}$, so $J^{\mu\nu}=S^{\mu\nu}=N^{\mu\nu\rho}u_{\rho}$, where $S^{\mu\nu}$ is the *intrinsic spin tensor*. If the body is spatially extended, then the difference $R^{\mu\rho}=J^{\mu\rho}-S^{\mu\rho}$ is the *rotational spin tensor*. The relativistic inertial force is then

$$\begin{aligned}\vec{f}_{(\text{in})}(x)&\\ =-m\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}&\left|\frac{1}{m}f^{\alpha}_{(l)}-\frac{\partial X^{\alpha}}{\partial x^{\mu}}\left[\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}u^{\nu}u^{\lambda}+\frac{1}{u^0}\overset{\circ}{\Gamma}{}^{\mu}_{\nu\rho}\left(u^{\nu}S^{\rho 0}+\overline{u^{0}}S^{\nu\rho}\right)-\frac{1}{2u^0}K^{\mu}_{\nu\rho}u^{\nu}S^{\rho 0}\right.\right.\\ &\left.\left.+\frac{1}{2}\overset{\circ}{R}{}^{\mu}_{\nu\sigma\rho}u^{\nu}S^{\sigma\rho}+\frac{1}{2}u_{\nu}S_{\rho\lambda}K^{\nu\rho\lambda;\mu}\right]\right|.\end{aligned} \tag{8.16}$$

In case of the Riemann space, V_4 ($\breve{Q}=0$), the relativistic inertial force (7.5) exerted on the extended spinning body can be written in terms of the Ricci coefficient of rotation only:

$$\breve{\vec{f}}_{(\text{in})}(\breve{x})=-m\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}\left|\frac{1}{m}f^{\alpha}_{(l)}-\frac{\partial X^{\alpha}}{\partial \breve{x}^{\mu}}\left[\breve{\Gamma}{}^{\mu}_{\nu\lambda}\breve{u}^{\nu}\breve{u}^{\lambda}+\frac{1}{\breve{u}^0}\breve{\Gamma}{}^{\mu}_{\nu\rho}\left(\breve{u}^{\nu}\breve{J}^{\rho 0}+\breve{N}^{0\nu\rho}\right)+\frac{1}{2}\breve{R}{}^{\mu}_{\nu\sigma\rho}\breve{u}^{\nu}\breve{J}^{\sigma\rho}\right]\right|. \tag{8.17}$$

In case of the Weitzenböck space, W_4 ($\overset{\bullet}{R}=0$), (7.5) reduces to its teleparallel equivalent:

$$\begin{aligned}\overset{\bullet}{\vec{f}}_{(\text{in})}\left(\overset{\bullet}{x}\right)=-m\,\frac{\vec{e}_f}{\Omega^2(\overline{\varrho})\gamma_q}&\left|\frac{1}{m}f^{\alpha}_{(l)}-\frac{\partial X^{\alpha}}{\partial \overset{\bullet}{x}{}^{\mu}}\left[\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}\overset{\bullet}{u}{}^{\nu}\overset{\bullet}{u}{}^{\lambda}+\frac{1}{\overset{\bullet}{u}{}^{0}}\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}\left(\overset{\bullet}{u}{}^{\nu}\overset{\bullet}{J}{}^{\rho 0}+\overset{\bullet}{N}{}^{0\nu\rho}\right)\right.\right.\\ &\left.\left.-\frac{1}{2\overset{\bullet}{u}{}^{0}}\overset{\bullet}{K}{}^{\mu}_{\nu\rho}\overset{\bullet}{N}{}^{\nu\rho 0}+\frac{1}{2}\overset{\bullet}{N}_{\nu\rho\lambda}\overset{\bullet}{K}{}^{\nu\rho\lambda;\mu}\right]\right|.\end{aligned} \tag{8.18}$$

All magnitudes related to the teleparallel gravity are denoted by an over "•". Finally, the nonvanishing inertial force, $\mathbf{f}_{(\mathrm{in})}^{(\mathrm{phot})}(x)$, acting on the photon of energy $h\nu$ in the U_4, can be obtained from (8.16), at $\vec{f}_{(l)} = 0$, as

$$\begin{aligned}\vec{f}_{(\mathrm{in})}^{(\mathrm{phot})}(x) = -\left(\frac{h\nu}{c^2\Omega^2(\overline{\varrho})}\right)\vec{e}_f \Bigg| \frac{\partial X^\alpha}{\partial x^\mu}\Bigg[\overset{\circ}{\Gamma}{}^\mu_{\nu\lambda}\frac{dx^\nu}{dT}\frac{dx^\lambda}{dT} \\ + \frac{dT}{d\bar{t}}\overset{\circ}{\Gamma}{}^\mu_{\nu\rho}\left(\frac{dx^\nu}{dT}S^{\rho 0} + \frac{d\bar{t}}{dT}S^{\nu\rho}\right) - \frac{dT}{2d\bar{t}}K^\mu_{\nu\rho}\frac{dx^\nu}{dT}S^{\rho 0} \\ + \frac{1}{2}\overset{\circ}{R}{}^\mu_{\nu\sigma\rho}\frac{dx^\nu}{dT}S^{\sigma\rho} + \frac{1}{2}\frac{dx_\nu}{dT}\; S_{\rho\lambda}K^{\nu\rho\lambda:\mu}\Bigg]\Bigg|,\end{aligned} \tag{8.19}$$

where $\vec{e}_f = (\vec{X}/|\vec{X}|)$, $v_q = (\vec{e}_f \cdot \overline{\vec{u}}) = |\overline{\vec{u}}|$, $\gamma_q = \gamma$, $\overline{\vec{u}}$ is the velocity of the photon in U_4, $(d\overline{\vec{u}}/d\bar{t})$ is the acceleration, and $g_{\mu\nu}(dx^\mu/dT) \otimes (dx^\nu/dT) = 0$.

9. Concluding Remarks

In the framework of TSSD theory, as a preliminary step, we show that by imposing different appropriate physical constraints upon the spacetime deformations, we may recover the term in the Lagrangian of pseudoscalar-photon interaction theory, or we may reproduce the various terms in the Lagrangians of pseudoscalar theories, for example, as intergrand for topological invariant, or pseudoscalar-gluon coupling occurred in QCD in an effort to solve the strong CP problem. We carry out some details of this program to probe the origin and nature of the phenomenon of inertia. We construct the RTI, which treats the inertia as a distortion of local internal properties of hypothetical 2D, so-called *master space* (MS). The MS is an indispensable companion of individual particle, without relation to the other matter, embedded in the background 4D-spacetime. The RTI allows to compute the *inertial force*, acting on an arbitrary point-like observer or particle due to its *absolute acceleration*. In this framework we essentially improve standard metric and other relevant geometrical structures referred to a noninertial frame for an arbitrary velocities and characteristic acceleration lengths. Despite the totally different and independent physical sources of gravitation and inertia, this approach furnishes justification for the introduction of the WPE. We relate the inertia effects to the more general post-Riemannian geometry. We derive a general expression of the relativistic inertial force exerted on the extended spinning body moving in the Rieman-Cartan space.

Acknowledgments

The author would like to thank W.-T. Ni for drawing his attention to the pseudoscalar-photon interaction theory. The helpful and knowledgable comments from the anonymous referees are much appreciated. Also, this research work was partially supported by the State Committee of Science of Armenia.

References

[1] S. Drake, *Galileo at work*, University of Chicago Press, Chicago, Ill, USA, 1978, http://en.wikipedia.org/wiki/Galileo.

[2] I. Newton, "Philosophiae Naturalis Principia Mathematica," 1687, http://www.tcshelp.com/.

[3] C. Brans and R. H. Dicke, "Mach's principle and a relativistic theory of gravitation," *Physical Review*, vol. 124, pp. 925–935, 1961.

[4] C. H. Brans, "Mach's principle and the locally measured gravitational constant in general relativity," *Physical Review*, vol. 125, pp. 388–396, 1962.

[5] C. H. Brans, "Mach's principle and a relativistic theory of gravitation. II," *Physical Review*, vol. 125, pp. 2194–2201, 1962.

[6] C. H. Brans, "Absence of inertial induction in general relativity," *Physical Review Letters*, vol. 39, pp. 856–857, 1977.

[7] H. Goenner, in *Ernst Mach: Physicist and Philosopher, Boston Studies in the Philosophy of Science*, R. S. Cohen and R. J. Seeger, Eds., vol. 6, Reidel, Dordrecht, The Netherlands, 1970.

[8] M. Reinhardt, *Zeitschrift für Naturforschung*, vol. 28, p. 529, 1973.

[9] D. J. Raine, "Mach's principle and space-time structure," *Reports on Progress in Physics*, vol. 44, no. 11, pp. 1151–1195, 1981.

[10] J. Barbour and H. Pfister, Eds., *Mach's Principle*, vol. 6 of *From Newton's Bucket to Quantum Gravity, Einstein Studies*, Birkhäuser, Boston, Mass, USA, 1995.

[11] C. H. Brans, "Absolute spacetime: the twentieth century ether," *General Relativity and Gravitation*, vol. 31, no. 5, pp. 597–607, 1999.

[12] E. W. Mielke, "Weak equivalence principle from a spontaneously broken gauge theory of gravity," *Physics Letters B*, vol. 702, no. 4, pp. 187–190, 2011.

[13] H.-J. Blome, C. Chicone, F. W. Hehl, and B. Mashhoon, "Nonlocal modification of Newtonian gravity," *Physical Review D*, vol. 81, no. 6, Article ID 065020, 2010.

[14] B. Mashhoon and P. S. Wesson, "Mach, the universe, and foundations of mechanics," *Annalen der Physik*, vol. 524, no. 2, p. A44, 2012.

[15] S. Vladimirov, *Geometrophysics*, BINOM, Moscow, Russia, 2005.

[16] G. T. Gillies, "The Newtonian gravitational constant: recent measurements and related studies," *Reports on Progress in Physics*, vol. 60, p. 151, 1997.

[17] C. M. Will, "The confrontation between general relativity and experiment," *Living Reviews in Relativity*, vol. 9, article 3, 2006.

[18] C. M. Will, "The confrontation between general relativity and experiment: an update," Lecture notes from the 1998 SLAC Summer Institute on Particle Physics, 1998.

[19] S. G. Turyshev, "Experimental tests of general relativity: recent progress and future directions," *Physics-Uspekhi*, vol. 52, no. 1, pp. 1–27, 2009.

[20] M. D. Gabriel and M. P. Haugan, "Testing the einstein equivalence principle: atomic clocks and local lorentz invariance," *Physical Review D*, vol. 41, no. 10, pp. 2943–2955, 1990.

[21] M. P. Haugan and T. F. Kauffmann, "New test of the Einstein equivalence principle and the isotropy of space," *Physical Review D*, vol. 52, no. 6, pp. 3168–3175, 1995.

[22] H. Hayasaka and S. Takeuchi, "Anomalous weight reduction on a gyroscope's right rotation around the vertical axis of the Earth," *Physical Review Letters*, vol. 63, pp. 2701–2704, 1989.

[23] J. E. Faller, W. J. Hollander, P. G. Nelson, and M. P. McHugh, "Gyroscope-weighing experiment with a null result," *Physical Review Letters*, vol. 64, pp. 825–826, 1990.

[24] T. J. Quinn and A. Picard, "The mass of spinning rotors: no dependence on speed or sense of rotation," *Nature*, vol. 343, no. 6260, pp. 732–735, 1990.

[25] J. M. Nitschke and P. A. Wilmarth, "Null result for the weight change of a spinning gyroscope," *Physical Review Letters*, vol. 64, no. 18, pp. 2115–2116, 1990.

[26] A. Imanishi, K. Maruyama, S. Midorikawa, and T. Morimoto, "Observation against the weight reduction of spinning gyroscopes," *Journal of the Physical Society of Japan*, vol. 60, no. 4, pp. 1150–1152, 1991.

[27] J. Luo, Y. X. Nie, Y. Z. Zhang, and Z. B. Zhou, "Null result for violation of the equivalence principle with free-fall rotating gyroscopes," *Physical Review D*, vol. 65, no. 4, Article ID 042005, 2002.

[28] Z. B. Zhou, J. Luo, Q. Yan, Z. G. Wu, Y. Z. Zhang, and Y. X. Nie, "New upper limit from terrestrial equivalence principle test for extended rotating bodies," *Physical Review D*, vol. 66, no. 2, Article ID 022002, 2002.

[29] C. W. F. Everitt, D. B. Debra, B. W. Parkinson et al., "Gravity probe B: final results of a space experiment to test general relativity," *Physical Review Letters*, vol. 106, no. 22, Article ID 221101, 2011.
[30] C. W. F. Everitt, M. Adams, W. Bencze et al., "Gravity probe B data analysis: status and potential for improved accuracy of scientific results," *Space Science Reviews*, vol. 148, no. 1–4, pp. 53–69, 2009.
[31] M. Heifetz, W. Bencze, T. Holmes, A. Silbergleit, and V. Solomonik, "The gravity probe B data analysis filtering approach," *Space Science Reviews*, vol. 148, no. 1–4, pp. 411–427, 2009.
[32] G. M. Keiser, J. Kolodziejczak, and A. S. Silbergleit, "Misalignment and resonance torques and their treatment in the GP-B data analysis," *Space Science Reviews*, vol. 148, no. 1–4, pp. 383–395, 2009.
[33] B. Muhlfelder, M. Adams, B. Clarke et al., "GP-B systematic error determination," *Space Science Reviews*, vol. 148, no. 1–4, pp. 429–439, 2009.
[34] A. Silbergleit, J. Conklin, D. Debra et al., "Polhode motion, trapped flux, and the GP-B science data analysis," *Space Science Reviews*, vol. 148, no. 1–4, pp. 397–409, 2009.
[35] W.-T. Ni, "Rotation, the equivalence principle, and the gravity probe B experiment," *Physical Review Letters*, vol. 107, no. 5, Article ID 051103, 2011.
[36] M. P. Haugan and C. Lämmerzahl, in *Gyrod, Clocks, Interferometers: Testing Relativistic Gravity in Space*, C. Lämmerzahl, C. W. F. Everitt, and F. W. Hehl, Eds., p. 195, Springer, 2001.
[37] C. Lämmerzahl and C. J. Bordé, in *Gyrod, Clocks, Interferometers: Testing Relativistic Gravity in Space*, C. Lämmerzahl, C. W. F. Everitt, and F. W. Hehl, Eds., p. 466, Springer, 2001.
[38] W.-T. Ni, "Probing the microscopic origin of gravity via precision polarization and spin experiments," *Chinese Physics Letters*, vol. 22, no. 1, pp. 33–35, 2005.
[39] W.-T. Ni, "Empirical foundations of the relativistic gravity," *International Journal of Modern Physics D*, vol. 14, no. 6, pp. 901–921, 2005.
[40] W.-T. Ni, "From equivalence principles to cosmology: cosmic polarization rotation, CMB observation, neutrino number asymmetry, Lorentz invariance and CPT," *Progress of Theoretical Physics Supplement*, no. 172, pp. 49–60, 2008.
[41] W.-T. Ni, "Searches for the role of spin and polarization in gravity," *Reports on Progress in Physics*, vol. 73, no. 5, Article ID 056901, 2010.
[42] V. De Sabbata and M. Gasperini, "Nyeicyrillic, shorttrinnyye ostsillyatsii v prisutstvii zakruchivaniya," *Il Nuovo Cimento A*, vol. 65, no. 4, pp. 479–500, 1981.
[43] M. Gasperini, "Testing the principle of equivalence with neutrino oscillations," *Physical Review D*, vol. 38, pp. 2635–2637, 1988.
[44] M. Gasperini, "Experimental constraints on a minimal and nonminimal violation of the equivalence principle in the oscillations of massive neutrinos," *Physical Review D*, vol. 39, no. 12, pp. 3606–3611, 1989.
[45] A. Halprin and C. N. Leung, "Can the Sun shed light on neutrino gravitational interactions?" *Physical Review Letters*, vol. 67, pp. 1833–1835, 1991.
[46] A. Halprin and C. N. Leung, "Solar neutrino oscillations and neutrino-gravity couplings," *Nuclear Physics B*, vol. 28, no. 1, pp. 139–141, 1992.
[47] J. Pantaleone, A. Halprin, and C. N. Leung, "Neutrino mixing due to a violation of the equivalence principle," *Physical Review D*, vol. 47, no. 10, pp. R4199–R4202, 1993.
[48] K. Iida, H. Minakata, and O. Yasuda, *Modern Physics Letters A*, vol. 8, p. 1037, 1993.
[49] D. Piniz, M. Roy, and J. Wudka, *Physical Review D*, vol. 54, p. 1587, 1996.
[50] M. Roy and J. Wudka, "Majorana neutrinos and gravitational oscillation," *Physical Review D*, vol. 56, no. 4, pp. 2403–2407, 1997.
[51] U. Bonse and T. Wroblewski, "Measurement of neutron quantum interference in noninertial frames," *Physical Review Letters*, vol. 51, no. 16, pp. 1401–1404, 1983.
[52] R. Colella, A. W. Overhauser, and S. A. Werner, "Observation of gravitationally induced quantum interference," *Physical Review Letters*, vol. 34, no. 23, pp. 1472–1474, 1975.
[53] D. K. Atwood, M. A. Horne, C. G. Shull, and J. Arthur, "Neutron phase shift in a rotating two-crystal interferometer," *Physical Review Letters*, vol. 52, no. 19, pp. 1673–1676, 1984.
[54] Y. Q. Cai, D. G. Lloyd, and G. Papini, "Helicity precession of accelerated fermions from spin-rotation coupling," *Physics Letters A*, vol. 178, no. 3-4, pp. 225–230, 1993.
[55] S. Capozziello and G. Lambiase, "Inertial effects on neutrino oscillations," *European Physical Journal C*, vol. 12, no. 2, pp. 343–347, 2000.
[56] S. Capozziello and G. Lambiase, "Inertial effects on Berry's phase of neutrino oscillations," *European Physical Journal C*, vol. 16, no. 1, pp. 155–159, 2000.
[57] J. L. Synge, *Relativity: The General Theory*, Series in Physics, North-Holland, Amsterdam, The Netherlands, 1960.

[58] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, Calif, USA, 1973.
[59] F. W. Hehl and W.-T. Ni, "Inertial effects of a Dirac particle," *Physical Review D*, vol. 42, no. 6, pp. 2045–2048, 1990.
[60] W.-T. Ni, *Chinese Journal of Physics*, vol. 15, p. 51, 1977.
[61] W.-Q. Li and W.-T. Ni, *Chinese Journal of Physics*, vol. 16, p. 214, 1978.
[62] W.-T. Ni and M. Zimmermann, "Inertial and gravitational effects in the proper reference frame of an accelerated, rotating observer," *Physical Review D*, vol. 17, no. 5, pp. 1473–1476, 1978.
[63] F. W. Hehl, J. Lemke, and E. W. Mielke, "Two lectures on fermions and gravity," in *Geometry and Theoretical Physics*, J. Debrus and A. C. Hirshfeld, Eds., pp. 56–140, Springer, Berlin, Germany, 1991.
[64] B. Mashhoon, "Length measurement in accelerated systems," *Annalen der Physik*, vol. 11, no. 7, pp. 532–547, 2002.
[65] B. Mashhoon, "Necessity of acceleration-induced nonlocality," *Annalen der Physik*, vol. 523, no. 3, pp. 226–234, 2011.
[66] B. Mashhoon, "Quantum theory in accelerated frames of reference," *Lecture Notes in Physics*, vol. 702, pp. 112–132, 2006.
[67] J. W. Maluf, F. F. Faria, and S. C. Ulhoa, "On reference frames in spacetime and gravitational energy in freely falling frames," *Classical and Quantum Gravity*, vol. 24, no. 10, pp. 2743–2753, 2007.
[68] J. W. Maluf and F. F. Faria, "On the construction of Fermi-Walker transported frames," *Annalen der Physik*, vol. 17, no. 5, pp. 326–335, 2008.
[69] K.-P. Marzlin, "What is the reference frame of an accelerated observer?" *Physics Letters A*, vol. 215, pp. 1–6, 1996.
[70] J. Audretsch, F. W. Hehl, and C. Lammerzahl, "Relativistic gravity research with emphasis on experiments and observations," in *Proceedings of the Bad Honnef School on Gravitation*, J. Ehlers and G. Schafer, Eds., vol. 410 of *Lecture Notes in Physics*, p. 368, Springer, Bad Honnef, Germany, 1992.
[71] Y. Y. Pan and J. Ren, "Spin-rotation coupling in the teleparallelism description in high speed rotation system," *International Journal of Theoretical Physics*, vol. 50, pp. 724–728, 2011.
[72] K. Bakke and C. Furtado, "Bound states for neutral particles in a rotating frame in the cosmic string spacetime," *Physical Review D*, vol. 82, no. 8, Article ID 084025, 2010.
[73] K. Bakke and C. Furtado, "Anandan quantum phase for a neutral particle with Fermi-Walker reference frame in the cosmic string background," *European Physical Journal C*, vol. 69, no. 3, pp. 531–539, 2010.
[74] K. Bakke, "Analog Landau-He-McKellar-Wilkens quantization due to noninertial effects of the Fermi-Walker reference frame," *Physical Review A*, vol. 81, no. 5, Article ID 052117, 2010.
[75] M. Korunur, M. Salti, and I. Acikgoz, "Finding Dirac spin effect in NUT spacetime," *Communications in Theoretical Physics*, vol. 53, no. 5, pp. 864–868, 2010.
[76] L. Ryder, "Spin-rotation coupling and Fermi-Walker transport," *General Relativity and Gravitation*, vol. 40, no. 6, pp. 1111–1115, 2008.
[77] L. Ryder, "Relativistic treatment of inertial spin effects," *Journal of Physics A*, vol. 31, no. 10, pp. 2465–2469, 1998.
[78] M. Arminjon and F. Reifler, "Dirac equation: representation independence and tensor transformation," *Brazilian Journal of Physics*, vol. 38, no. 2, pp. 248–258, 2008.
[79] A. J. Silenko and O. V. Teryaev, "Equivalence principle and experimental tests of gravitational spin effects," *Physical Review D*, vol. 76, no. 6, Article ID 061101, 2007.
[80] B. Mashhoon and H. Kaiser, "Inertia of intrinsic spin," *Physica B*, vol. 385-386, pp. 1381–1383, 2006.
[81] D. Bini, C. Cherubini, and B. Mashhoon, "Spin, acceleration and gravity," *Classical and Quantum Gravity*, vol. 21, no. 16, pp. 3893–3908, 2004.
[82] D. N. Vulcanov and I. I. Cotăescu, "Dirac field, gravity, inertial effects, and computer algebra," *International Journal of Modern Physics C*, vol. 8, no. 2, pp. 345–359, 1997.
[83] I. Vetharaniam and G. E. Stedman, "Accelerated observers: synchronization and tests of local Lorentz invariance," *Classical and Quantum Gravity*, vol. 11, no. 4, pp. 1069–1082, 1994.
[84] J. C. Huang, "Dirac particle, gravity, and inertial effects," *Annalen der Physik*, vol. 3, no. 1, pp. 53–67, 1994.
[85] J. Norton, "What was Einstein's principle of equivalence?" *Studies in History and Philosophy of Science*, vol. 16, no. 3, pp. 203–246, 1985.
[86] T. Damour, "Missions spatiales en physique fondamentale (Chatillon, France)," in *Comptes Rendus de l'Academie des Sciences*, C. Borde and P. Touboul, Eds., vol. 2 of *Serie 4*, p. 1249, 2001.

[87] G. Cocconi and E. E. Salpeter, "Upper limit for the anisotropy of ineria from the Mössbauer effect," *Physical Review Letters*, vol. 4, pp. 176–177, 1960.
[88] V. W. Hughes, H. G. Robinson, and V. Beltran-Lopez, "Upper limit for the anisotropy of inertial mass from nuclear resonance experiments," *Physical Review Letters*, vol. 4, pp. 342–344, 1960.
[89] S. A. Lewis, W. L. WiIliams, and V. W. Hughes, *Bulletin of the American Physical Society*, vol. 121, no. 11, 1966.
[90] R. W. P. Drever, "A search for anisotropy of inertial mass using a free precession technique," *Philosophical Magazine*, vol. 6, no. 65, pp. 683–687, 1961.
[91] J. D. Prestage, J. J. Bollinger, W. M. Itano, and D. J. Wineland, "Limits for spatial anisotropy by use of nuclear-spin-polarized $^9Be^+$ ions," *Physical Review Letters*, vol. 54, no. 22, pp. 2387–2390, 1985.
[92] B. Coll, in *Proceedings of the Spanish Relativistic Meeting*, EREs, Salamanca, Spain, 1998.
[93] B. Coll, J. Llosa, and D. Soler, "Three-dimensional metrics as deformations of a constant curvature metric," *General Relativity and Gravitation*, vol. 34, no. 2, pp. 269–282, 2002.
[94] J. Llosa and J. Carot, "Flat deformation theorem and symmetries in spacetime," *Classical and Quantum Gravity*, vol. 26, no. 5, p. 055013, 20, 2009.
[95] S. Capozziello and C. Stornaiolo, "Space-time deformations as extended conformal transformations," *International Journal of Geometric Methods in Modern Physics*, vol. 5, no. 2, pp. 185–195, 2008.
[96] G. Ter-Kazarian, "Two-step spacetime deformation-induced dynamical torsion," *Classical and Quantum Gravity*, vol. 28, no. 5, Article ID 055003, 2011.
[97] W.-T. Ni, *A Nonmetric Theory of Gravity*, Montana State University, Bozeman, Mont, USA, 1973, http://cgc.pmo.ac.cn/.
[98] W.-T. Ni, *Bulletin of the American Physical Society*, vol. 19, p. 655, 1974.
[99] W. T. Ni, "Equivalence principles and electromagnetism," *Physical Review Letters*, vol. 38, no. 7, pp. 301–304, 1977.
[100] W.-T. Ni, in *Proceedings of the International School and Symposium on Precision Measurement and Gravity Experiment*, p. 491, National Tsing Hua University, Taipei, Republic of China, 1983.
[101] A. B. Balakin and W.-T. Ni, "Non-minimal coupling of photons and axions," *Classical and Quantum Gravity*, vol. 27, no. 5, Article ID 055003, p. 23, 2010.
[102] S. M. Carroll, G. B. Field, and R. Jackiw, "Limits on a Lorentz- and parity-violating modification of electrodynamics," *Physical Review D*, vol. 41, no. 4, pp. 1231–1240, 1990.
[103] S. M. Carroll and G. B. Field, "Einstein equivalence principle and the polarization of radio galaxies," *Physical Review D*, vol. 43, no. 12, pp. 3789–3793, 1991.
[104] S. S. Chern and J. Simons, "Characteristic forms and geometric invariants," *Annals of Mathematics*, vol. 99, pp. 48–69, 1974.
[105] R. D. Peccei and H. R. Quinn, "CP conservation in the presence of pseudoparticles," *Physical Review Letters*, vol. 38, no. 25, pp. 1440–1443, 1977.
[106] S. Weinberg, *Physical Review Letters*, vol. 40, no. 4, p. 233, 1978.
[107] F. Wilczek, "Problem of strong P and T invariance in the presence of instantons," *Physical Review Letters*, vol. 40, no. 5, pp. 279–282, 1978.
[108] G. Ter-Kazarian, "Gravitation and inertia. A rearrangement of vacuum in gravity," *Astrophysics and Space Science*, vol. 327, no. 1, pp. 91–109, 2010.
[109] G. T. Kazarian, "Probing the origin of inertia behind spacetime deformation," Submitted, http://arxiv.org/abs/1105.5932.
[110] T. Eguchi, P. B. Gilkey, and A. J. Hanson, "Gravitation, gauge theories and differential geometry," *Physics Reports C*, vol. 66, no. 6, pp. 213–393, 1980.
[111] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol I*, Interscience Publishers, New York, NY, USA, 1963.
[112] J. Plebanski, "Forms and riemannian geometry," in *Proceedings of the International School of Cosmology and Gravitation*, Erice, Italy, 1972.
[113] A. Trautman, *Differential Geometry for Physicists*, vol. 2, Bibliopolis, Naples, Italy, 1984, Stony Brook lectures.
[114] V. De Sabbata and M. Gasperini, *Introduction to Gravitation*, World Scientific Publishing, Singapore, 1985.
[115] V. de Sabbata and E. Schmutzer, Eds., *Unified Field Theories of More Than Four Dimensions*, World Scientific, Singapore, 1983.
[116] V. De Sabbata and M. Gasperini, "On the Maxwell equations in a Riemann-Cartan space," *Physics Letters A*, vol. 77, no. 5, pp. 300–302, 1980.

[117] V. de Sabbata and C. Sivaram, *Spin and Torsion in Gravitation*, World Scientific, Singapore, 1994.
[118] N. J. Poplawski, "Spacetime and fields," Submitted, http://arxiv.org/abs/0911.0334.
[119] A. Trautman, "On the structure of the Einstein-Cartan equations," in *Differential Geometry, Symposia Mathematica 12*, pp. 139–162, Academic Press, London, UK, 1973.
[120] A. Trautman, *Encyclopedia of Mathematical Physics*, vol. 2, Elsevier, Oxford, UK, 2006.
[121] B. Mashhoon, "Vacuum electrodynamics of accelerated systems: nonlocal Maxwell's equations," *Annalen der Physik*, vol. 12, no. 10, pp. 586–598, 2003.
[122] B. Mashhoon, "Limitations of spacetime measurements," *Physics Letters A*, vol. 143, no. 4-5, pp. 176–182, 1990.
[123] B. Mashhoon, "The hypothesis of locality in relativistic physics," *Physics Letters A*, vol. 145, no. 4, pp. 147–153, 1990.
[124] B. Mashhoon, "Neutron interferometry in a rotating frame of reference," *Physical Review Letters*, vol. 61, no. 23, pp. 2639–2642, 1988.
[125] B. Mashhoon, "On the coupling of intrinsic spin with the rotation of the earth," *Physics Letters A*, vol. 198, no. 1, pp. 9–13, 1995.
[126] F. Gronwald and F. W. Hehl, in *Proceedings of the 14th Course of the School of Cosmology and Gravitation on Quantum Gravity*, P. G. Bergmann, V. de Sabbata, and H.-J. Treder, Eds., World Scientific, Erice, Italy, 1996.
[127] F. Gronwald and F. W. Hehl, "On the Gauge aspects of gravity," Submitted, http://arxiv.org/abs/gr-qc/9602013.
[128] S. Weinberg, *Gravitation and Cosmology*, J. W. and Sons, New York, NY, USA, 1972.
[129] C. M. Will, *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge, UK, 1993.
[130] C. M. Will and K. Nordtvedt, Jr., "Conservation laws and preferred frames in relativistic gravity. I. Preferred-frame theories and an extended PPN formalism," *The Astrophysical Journal*, vol. 177, pp. 757–774, 1972.
[131] K. Nordtvedt, Jr. and C. M. Will, "Conservation laws and preferred frames in relativistic gravity. II. Experimental evidence to rule out preferred-frame theories of gravity," *The Astrophysical Journal*, vol. 177, pp. 775–792, 1972.
[132] C. M. Will, "Relativistic gravity in the solar system. III. Experimental disproof of a class of linear theories of gravitation," *The Astrophysical Journal*, vol. 185, pp. 31–42, 1973.
[133] P. von der Heyde, "The equivalence principle in the U_4 theory of gravitation," *Lettere al Nuovo Cimento*, vol. 14, no. 7, pp. 250–252, 1975.
[134] F. W. Hehl, "Four lectures on Poincare gauge theory," in *Proceedings of the 6th Course of the School of Cosmology and Gravitation on Spin, Torsion, Rotation, and Supergravity*, P. G. Bergmann and V. de Sabbata, Eds., vol. 5, Plenum, Erice, Italy, May 1979.
[135] G. Modanese and M. Toller, "Radial gauge in Poincaré Gauge field theories," *Journal of Mathematical Physics*, vol. 31, no. 2, pp. 452–458, 1990.
[136] B. Z. Iliev, *Preprints: E5-92-507, E5-92-508, E5-92-543*, Joint Institute for Nuclear Research, Dubna, Russia, 1992.
[137] D. Hartley, "Normal frames for non-Riemannian connections," *Classical and Quantum Gravity*, vol. 12, no. 11, pp. L103–L105, 1995.
[138] M. Blagojević, *Gravitation and Gauge Symmetries*, Series in High Energy Physics, Cosmology and Gravitation, IOP Publishing, Bristol, UK, 2002.
[139] A. Papapetrou, "Einstein's theory of gravitation and flat space," *Proceedings of the Royal Irish Academy*, vol. 52, pp. 11–23, 1948.
[140] A. Papapetrou, *Proceedings of the Royal Society A*, vol. 202, p. 248, 1951.
[141] A. Papapetrou, *Lectures on General Relativity*, Reidel D, 1974.
[142] P. G. Bergmann and R. Thompson, "Spin and angular momentum in general relativity," *Physical Review*, vol. 89, pp. 400–407, 1953.
[143] C. Møller, *Annals of Physics*, vol. 4, p. 347, 1958.