Hindawi Publishing Corporation
Advances in Numerical Analysis
Volume 2012, Article ID 309871, 14 pages
doi:10.1155/2012/309871

Research Article

Two-Level Stabilized Finite Volume Methods for Stationary Navier-Stokes Equations

Anas Rachid,[1] Mohamed Bahaj,[2] and Noureddine Ayoub[2]

[1] *École Nationale Supérieure des Arts et Métiers-Casablanca, Université Hassan II, B.P. 150, Mohammedia, Morocco*
[2] *Department of Mathematics and Computing Sciences, Faculty of Sciences and Technology, University Hassan 1st, B.P. 577, Settat, Morocco*

Correspondence should be addressed to Anas Rachid, rachid.anas@gmail.com

Received 17 December 2011; Accepted 17 February 2012

Academic Editor: Weimin Han

We propose two algorithms of two-level methods for resolving the nonlinearity in the stabilized finite volume approximation of the Navier-Stokes equations describing the equilibrium flow of a viscous, incompressible fluid. A macroelement condition is introduced for constructing the local stabilized finite volume element formulation. Moreover the two-level methods consist of solving a small nonlinear system on the coarse mesh and then solving a linear system on the fine mesh. The error analysis shows that the two-level stabilized finite volume element method provides an approximate solution with the convergence rate of the same order as the usual stabilized finite volume element solution solving the Navier-Stokes equations on a fine mesh for a related choice of mesh widths.

1. Introduction

We consider a two-level method for the resolution of the nonlinear system arising from finite volume discretizations of the equilibrium, incompressible Navier-Stokes equations:

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{in } \partial\Omega, \tag{1.3}$$

where $u = (u_1(x), u_2(x))$ is the velocity vector, $p = p(x)$ is the pressure, $f = f(x)$ is the body force, $\nu > 0$ is the viscosity of the fluid, and $\Omega \subset \mathbb{R}^2$, the flow domain, is assumed to be bounded, to have a Lipschitz-continuous boundary $\partial\Omega$, and to satisfy a further condition stated in (H1).

Finite volume method is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer, and petroleum engineering. The method can be formulated in the finite difference framework or in the Petrov-Galerkin framework. Usually, the former one is called finite volume method [1, 2], MAC (marker and cell) method [3], or cell-centered method [4], and the latter one is called finite volume element method (FVE) [5–7], covolume method [8], or vertex-centered method [9, 10]. We refer to the monographs [11, 12] for general presentations of these methods. The most important property of FVE is that it can preserve the conservation laws (mass, momentum, and heat flux) on each control volume. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field.

On the other hand, the two-level finite element strategy based on two finite element spaces on one coarse and one fine mesh has been widely studied for steady semilinear elliptic equations [13, 14] and the Navier-Stokes equations [15–22]. For the finite volume element method, Bi and Ginting [23] have studied two-grid finite volume element method for linear and nonlinear elliptic problems; Chen et al. [24] have applied two-grid methods for solving a two-dimensional nonlinear parabolic equation using finite volume element method. Chen and Liu [25] have also studied this method for semilinear parabolic problems. However, to the best of our knowledge, there is no two-level finite volume convergence analysis for the Navier-Stokes equations in the literature.

In this paper we aim to combine FVE method based on $P_1 - P_0$ macroelement with two-level strategy to solve the two-dimensional Navier-Stokes (1.1)–(1.3). The heart of the analysis is the use of a transfer operator to connect finite volume and finite element estimations which will lead to more difficult term to estimate. We choose the two-grid spaces as two conforming finite element spaces X_H and X_h on one coarse grid with mesh size H and one fine grid with mesh size $h \ll H$. We propose two algorithms of two-level method for resolving the nonlinearity in the stabilized finite volume approximation of the problem (1.1)–(1.3): the simple and Newton algorithms. First we prove that the simple two-level stabilized finite volume solution (u^h, p^h) is the following error estimate:

$$\left\|u - u^h\right\|_1 + \left\|p - p^h\right\|_0 \leq C\left(h + H^2\right). \tag{1.4}$$

Second we prove that the Newton two-level stabilized finite volume solution (u^h, p^h) is the following error estimate:

$$\left\|u - u^h\right\|_1 + \left\|p - p^h\right\|_0 \leq C\left(h + H^3\left|\log h\right|^{1/2}\right), \tag{1.5}$$

where C denotes some generic constant which may stand for different values at its different occurrences.

Hence, the two-level algorithms achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^2)$ for the simple two-level stabilized finite volume solution and $h = O(H^3|\log h|^{1/2})$ for the Newton two-level stabilized finite volume solution. As a result, solving the nonlinear Navier-Stokes equations will not be much more difficult than solving one single linearized equation.

The rest of this paper is organized as follows. In the next section, we introduce some notations and construct a FVE scheme. In Section 3 we recall same preliminary estimates of the stabilized finite volume approximations. Finally the two-level FVE algorithms and the improved error estimates are presented and established in Section 4.

2. Finite Volume Scheme

2.1. Notations

We will use $\|\cdot\|_m$ and $|\cdot|_m$ to denote the norm and seminorm of the Sobolev space $(H^m(\Omega))^d$, $d=1,2$. Let $H_0^1(\Omega)$ be the standard Sobolev subspace of $H^1(\Omega)$ of functions vanishing on $\partial\Omega$. We introduce the following notations:

$$X=\left(H_0^1(\Omega)\right)^2,\qquad Y=L_0^2(\Omega)=\left\{q:q\in L^2(\Omega),\int_\Omega q=0\right\}. \tag{2.1}$$

The scalar product and norm in Y are denoted by the usual $L^2(\Omega)$ inner product $(\cdot,\cdot)$ and $\|\cdot\|_0$, respectively. As mentioned above, we need a further assumption on Ω.

H1

Assume that Ω is regular so that the unique solution $(v,q)\in X\times Y$ of the steady Stokes problem

$$-\Delta v+\nabla q=g,\qquad \nabla\cdot v=0;\quad v|_{\partial\Omega}=0 \tag{2.2}$$

for a prescribed $g\in(L^2(\Omega))^2$ exists and satisfies

$$\|v\|_2+\|q\|_1\le C\|g\|_0, \tag{2.3}$$

where $C>0$ is a constant depending on Ω.

The weak formulation of the problem (1.1)–(1.3) is to find $(u,p)\in X\times Y$ such that

$$\begin{gathered}a(u,v)-d(v,p)+b(u,u,v)=(f,v),\quad \forall v\in X,\\ d(u,q)=0,\quad \forall q\in Y,\end{gathered} \tag{2.4}$$

where the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and the trilinear form $b(\cdot,\cdot,\cdot)$ are given by

$$\begin{aligned}
a(u,v) &= \nu(\nabla u,\nabla v) = \nu\int_\Omega \nabla u : \nabla v\,dx, \quad \forall u,v\in X,\\
d(v,q) &= (\nabla\cdot v,q) = \int_\Omega q\nabla\cdot v\,dx, \quad \forall u,q\in X\times Y,\\
b(u,w,v) &= ((u\cdot\nabla w),v) + \frac{1}{2}((\nabla\cdot u)w,v)\\
&= \frac{1}{2}((u\cdot\nabla w),v) - \frac{1}{2}((u\cdot\nabla v),w) \quad \forall u,v,w\in X.
\end{aligned} \tag{2.5}$$

Introducing the generalized bilinear form on $(X\times Y)^2$ by

$$B\big((u,p);(v,q)\big) = a(u,v) - d(v,p) + d(u,q), \tag{2.6}$$

we can rewrite (2.4) in a compact form: find $(u,p)\in X\times Y$ such that

$$B\big((u,p);(v,q)\big) + b(u,u,v) = (f,v), \quad \forall(v,q)\in X\times Y. \tag{2.7}$$

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K\in\mathcal{T}_h$. We assume that the partition $\mathcal{T}_h$ has been obtained from a macrotriangular partition Λ_h by joining the sides of each element of Λ_h. Every element $K\in\mathcal{T}_h$ must lie in exactly one macroelement $\mathcal{K}$, which implies that macroelements do not overlap. For each $\mathcal{K}$, the set of interelement edges which are strictly in the interior of $\mathcal{K}$ will be denoted by $\Gamma_{\mathcal{K}}$, and the length of an edge $e\in\Gamma_{\mathcal{K}}$ is denoted by h_e.

Based on this triangulation, let X_h be the standard conforming finite element subspace of piecewise linear velocity,

$$X_h = \{v\in C(\Omega)\cap X : v|_K \text{ is linear}, \forall K\in\mathcal{T}_h; v|_{\partial\Omega} = 0\}, \tag{2.8}$$

and let Y_h be the piecewise constant pressure subspace

$$Y_h = \{q\in Y : q|_K \text{ is constant}, \forall K\in\mathcal{T}_h\}. \tag{2.9}$$

It is well known that the standard $P_1 - P_0$ element does not satisfy the inf-sup condition and cannot be applied to problem (1.1)–(1.3) directly. But a locally stabilized method based on the macroelement can be used to yield adequate approximations [6].

In order to describe the FVE method for solving problem (1.1)–(1.3), we will introduce a dual partition $\mathcal{T}_h^*$ based upon the original partition $\mathcal{T}_h$ whose elements are called control volumes. We construct the control volumes in the same way as in [5, 26]. Let z_K be the barycenter of $K\in\mathcal{T}_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z\in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z\in Z_h = \cup_{K\in\mathcal{T}_h} Z_h(K)$, we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z. Thus we finally obtain

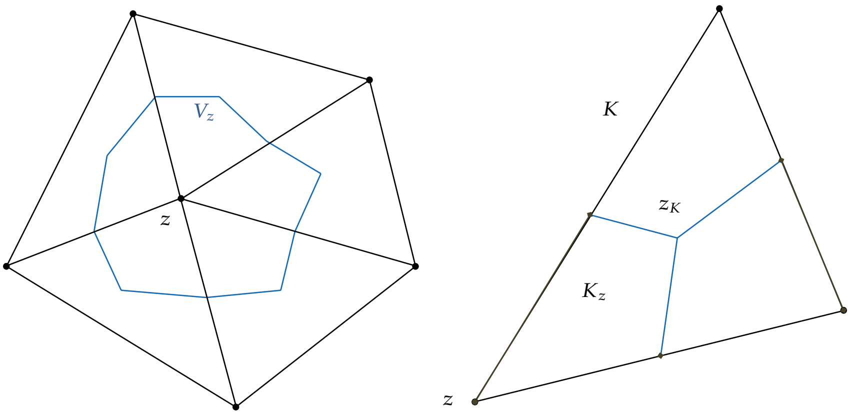

Figure 1: Left-hand side: a sample region with blue lines indicating the corresponding control volume V_z. Right-hand side: a triangle K partitioned into three subregions K_z.

a group of control volumes covering the domain Ω, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$. We denote by Z_h^0 the set of interior vertices.

We call the partition $\mathcal{T}_h^*$ regular or quasi-uniform if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \operatorname{meas}(V_z) \leq Ch^2, \quad V_z \in \mathcal{T}_h^*. \tag{2.10}$$

If the finite element triangulation $\mathcal{T}_h$ is quasi-uniform, then the dual partition $\mathcal{T}_h^*$ is also quasi-uniform [23].

2.2. Construction of the FVE Scheme

We formulate the FVE method for the problem (1.1)–(1.3) as follows: given a $z \in Z_h^0$ and $K \in \mathcal{T}_h$, integrating (1.1) over the associated control volume V_z and (1.1) over the element K and applying Green's formula, we obtain an integral conservation form

$$-\nu \int_{\partial V_z} \nabla u n \, ds + \int_{\partial V_z} p n \, ds + \int_{V_z} u \cdot \nabla u \, dx = \int_{V_z} f \, dx, \quad \forall z \in Z_h^0, \tag{2.11}$$

$$\int_K \nabla \cdot u \, dx = 0, \quad \forall K \in \mathcal{T}_h, \tag{2.12}$$

where n denotes the unit outer normal vector to ∂V_z (Figure 1).

Let $I_h^* : X_h \to X_h^*$ be the transfer operator defined by

$$I_h^* v = \sum_{z \in Z_h^0} v(z) \chi_z, \quad \forall v \in X_h, \tag{2.13}$$

where

$$X_h^* = \left\{ v = (v_1, v_2) \in \left(L^2(\Omega)\right)^2 : v_i|_{V_z} \text{ is constant}, i = 1,2\ \forall z \in Z_h^0 \right\}, \tag{2.14}$$

and χ_z is the characteristic function of the control volume V_z. The operator I_h^* satisfies [26]

$$\left\| I_h^* v \right\|_0 \le C \|v\|_0, \quad \forall v \in X_h. \tag{2.15}$$

Now for an arbitrary $I_h^* v$, we multiply (2.11) by $v(z)$ and sum over all $z \in Z_h^0$ to get

$$a_h\left(u, I_h^* v\right) - d_h\left(I_h^* v, p\right) + b_h\left(u, u, I_h^* v\right) = \left(f, I_h^* v\right), \quad \forall v \in X_h. \tag{2.16}$$

Here $a_h : X \times X_h \to \mathbb{R}$, $d_h : X_h \times Y \to \mathbb{R}$ and $b_h : X \times X \times X_h \to \mathbb{R}$ are defined by

$$\begin{aligned} a_h\left(u, I_h^* v\right) &= -\nu \sum_{z \in Z_h^0} v(z) \int_{\partial V_z} \nabla u n \, ds, \\ d_h\left(I_h^* v, p\right) &= \sum_{z \in Z_h^0} v(z) \int_{\partial V_z} p n \, ds, \\ b_h\left(u, u, I_h^* v\right) &= \sum_{z \in Z_h^0} v(z) \int_{V_z} (u \cdot \nabla) u \, dx. \end{aligned} \tag{2.17}$$

We also define the trilinear forms $\widetilde{b}(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ on $X \times X \times X_h$ by

$$\begin{aligned} \widetilde{b}\left(u, v, I_h^* w\right) &= \left((u \cdot \nabla) v, I_h^* w\right) + \frac{1}{2}\left((\nabla \cdot u) v, I_h^* w\right), \\ \overline{b}\left(u, v, w - I_h^* w\right) &= \left((u \cdot \nabla) v, w - I_h^* w\right) + \frac{1}{2}\left((\nabla \cdot u) v, w - I_h^* w\right). \end{aligned} \tag{2.18}$$

To formulate the discrete problem so as to eliminate any such potential difficulties, we rewrite (2.16) as follows:

$$a_h\left(u, I_h^* v\right) - d_h\left(I_h^* v, p\right) + \widetilde{b}\left(u, u, I_h^* v\right) = \left(f, I_h^* v\right), \quad \forall v \in X_h. \tag{2.19}$$

We multiply (2.12) by $q \in Y_h$ and sum over all $K \in \mathcal{T}_h$: then, we obtain

$$b(u, q) = 0, \quad \forall q \in Y_h. \tag{2.20}$$

Now we rewrite (2.19) and (2.20) to a variational form similar to finite element problems. The locally stabilized FVE scheme is to find $(u_h, p_h) \in X_h \times Y_h$ such that

$$\begin{gathered} a_h(u_h, I_h^* v_h) - d_h(I_h^* v_h, p_h) + \tilde{b}(u_h, u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in X_h, \\ d(u_h, q_h) - \beta c_h(p_h, q_h) = 0, \quad \forall q_h \in Y_h, \end{gathered} \tag{2.21}$$

where

$$c_h(p, q) = \sum_{\mathcal{K} \in \Lambda_h} \sum_{e \in \Gamma_{\mathcal{K}}} h_e \int_e [p]_e [q]_e ds \tag{2.22}$$

is a stabilized form defined on $(H^1(\Omega) + Y_h)^2$, $[\cdot]_e$ is the jump operator across the edge e, and $\beta > 0$ is the local stabilization parameter. It is trivial that $c_h(p, q_h) = c_h(p_h, q) = c_h(p, q) = 0$, for all $p, q \in H^1(\Omega)$, for all $p_h, q_h \in Y_h$.

A general framework for analyzing the locally stabilized formulation (2.21) can be developed using the notion of equivalence class of macroelements. As in Stenberg [27] each equivalence class, denoted by $\mathcal{E}_{\widehat{\mathcal{K}}}$, contains macroelements which are topologically equivalent to a reference macroelement $\widehat{\mathcal{K}}$.

Let

$$\overline{B}_h((u_h, p_h); (I_h^* v_h, q_h)) = a_h(u_h, I_h^* v_h) - d_h(I_h^* v_h, p_h) + d(u_h, q_h), \tag{2.23}$$

$$B_h((u_h, p_h); (I_h^* v_h, q_h)) = \overline{B}_h((u_h, p_h); (I_h^* v_h, q_h)) + \beta c_h(p_h, q_h). \tag{2.24}$$

We can rewrite (2.21) in a compact form: find $(u_h, p_h) \in X_h \times Y_h$ such that

$$B_h((u_h, p_h); (I_h^* v_h, q_h)) + \tilde{b}(u_h, u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall (v_h, q_h) \in X_h \times Y_h. \tag{2.25}$$

3. Technical Preliminaries

This section considers preliminary estimates which will be very useful in the error estimates of two-level finite volume solution (u^h, p^h).

The following lemma gives the boundedness of the trilinear form $b(\cdot, \cdot, \cdot)$.

Lemma 3.1 (see [21]). *The following estimates hold:*

$$
\begin{gathered}
b(u,w,v) = -b(u,v,w), \\
|b(u,w,v)| \leq \frac{1}{2}C_0\|u\|_0^{1/2}|u|_1^{1/2}\left(|w|_1\|v\|_0^{1/2}|v|_1^{1/2} + \|w\|_0^{1/2}\|w\|_1^{1/2}\|v\|_1^{1/2}\right), \quad \forall u,v,w \in X, \\
|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leq C_1|u|_1\|v\|_2\|w\|_0, \\
\forall u \in X,\ v \in \left(H^2(\Omega)\right)^2 \cap X,\ w \in Y, \\
|b(u,v,w)| \leq C\left|\log h\right|^{1/2}|u|_1|v|_1\|w\|_0, \quad \forall u,v,w \in X_h.
\end{gathered} \tag{3.1}
$$

Here and after C_i, $i = 1,2$, and C are positive constants depending only on the data (ν, f, Ω).

The existence and uniqueness results of (2.7) can be found in [28, 29].

Theorem 3.2. *Assume that $\nu > 0$ and $f \in Y$ satisfy the following uniqueness condition:*

$$
1 - \frac{N_1}{\nu^2}\|f\|_{-1} > 0, \tag{3.2}
$$

where

$$
N_1 = \sup_{u,v,w \in X} \frac{b(u,v,w)}{|u|_1|v|_1|w|_1}. \tag{3.3}
$$

Then the problem (2.7) admits a unique solution $(u,p) \in (H_0^1(\Omega)^2 \cap X, H^1(\Omega) \cap Y)$ such that

$$
|u|_1 \leq \frac{1}{\nu}\|f\|_{-1}, \qquad \|u\|_2 + \|p\|_1 \leq C\|f\|_0. \tag{3.4}
$$

In [30] the following lemma was proved, which shows that the finite volume element bilinear forms $a_h(\cdot, I_h^*\cdot)$ and $d_h(I_h^*\cdot,\cdot)$ are equal to the finite element ones, respectively.

Lemma 3.3. *For any $u_h, v_h \in X_h$, and $q_h \in Y_h$, one has*

$$
\begin{gathered}
a_h\left(u_h, I_h^* v_h\right) = a\left(u_h, v_h\right), \\
d_h\left(I_h^* v_h, q_h\right) = d\left(v_h, q_h\right).
\end{gathered} \tag{3.5}
$$

The following theorem establishes the weak coercivity of (2.24) [6, 31].

Theorem 3.4. *Given a a stabilization parameter $\beta \geq \beta_0$, suppose that every macroelement $\mathcal{K} \in \Lambda_h$ belongs to one of the equivalence classes $\mathcal{E}_{\widehat{\mathcal{K}}}$ and that the following macroelement connectivity* *condition is valid: for any two neighboring macroelements $\mathcal{K}_1$ and $\mathcal{K}_2$ with $\int_{\mathcal{K}_1\cap\mathcal{K}_2} \neq 0$, there exists $v \in X_h$ such that*

$$\text{supp } v \subset \mathcal{K}_1 \cap \mathcal{K}_2 \int_{\mathcal{K}_1\cap\mathcal{K}_2} v \cdot n ds \neq 0. \tag{3.6}$$

Then

$$\alpha_1\left(|u_h|_1 + \|p_h\|_0\right) \leq \sup_{(v_h,q_h)\in X_h\times Y_h} \frac{B_h\left((u,p);(I_h^* v_h, q_h)\right)}{|v_h|_1 + \|q_h\|_0}, \tag{3.7}$$

for all $(u_h, p_h) \in X_h \times Y_h$, and

$$\left|B_h\left((u,p);(I_h^* v_h, q_h)\right)\right| \leq \alpha_2\left(|u_h|_1 + \|p_h\|_0\right)\left(|v_h|_1 + \|q_h\|_0\right) \quad (v_h, q_h) \in X_h \times Y_h, \tag{3.8}$$

where $\alpha_1, \alpha_2 > 0$ are constants independent of h and β, β_0 is some fixed positive constant, and n is the out-normal vector.

Next, we establish the existence and the uniqueness of FVE scheme (2.25), by the fixed-point theorem, in the following.

Theorem 3.5 (see [6]). *Suppose the assumptions of Theorems 3.2 and 3.4 hold, and the body force f satisfies the following uniqueness condition*

$$1 - \frac{4N}{\nu^2}\|f\|_{-1} > 0. \tag{3.9}$$

Then the variation problem (2.25) *admits a unique solution $(u_h, p_h) \in (X_h \times Y_h)$ such that*

$$|u_h|_1 \leq \frac{1}{\nu}\|f\|_{-1}, \qquad \|p_h\|_0 \leq \frac{\alpha_2}{\alpha_1}\|f\|_{-1} + \frac{4\alpha_2 N}{\alpha_1 \nu^2}\|f\|_{-1}, \tag{3.10}$$

where

$$N = \max\{CN_1, N_2\}, \qquad N_2 = \sup_{u,v,w\in X} \frac{\widetilde{b}(u,v,I_h^* w)}{|u|_1 |v|_1 |w|_1}. \tag{3.11}$$

For the error estimate, we introduce the Galerkin projection $(R_h, Q_h) : X \times Y \to X_h \times Y_h$ defined by

$$B_h\left(\left(R_h(v,q), Q_h(v,q)\right);\left(I_h^* v_h, q_h\right)\right) = \overline{B}_h\left((v,q);\left(I_h^* v_h, q_h\right)\right) \tag{3.12}$$

for each $(v,q) \in X \times Y$ and all $(v_h, q_h) \in X_h \times Y_h$. We obtain the following results by using the standard Galerkin finite element [6, 17].

Theorem 3.6. *Under the assumptions of Theorems 3.2 and 3.4, the projection* (R_h, Q_h) *satisfies*

$$\left|v - R_h(v,q)\right|_1 + \left\|q - Q_h(v,q)\right\|_0 \le C\left(|v|_1 + \|q\|_0\right), \tag{3.13}$$

for all $(v,q) \in X \times Y$ *and*

$$\left\|v - R_h(v,q)\right\|_0 + h\left(\left|v - R_h(v,q)\right|_1 + \left\|q - Q_h(v,q)\right\|_0\right) \le Ch^2\left(\|v\|_2 + \|q\|_1\right), \tag{3.14}$$

for all $(v,q) \in (D(A) \cap X) \times (H^1(\Omega) \cap Y)$.

Then the optimal error estimates can be obtained as follows.

Theorem 3.7 (see [6, 32]). *Under the assumptions of Theorems 3.2, 3.4, 3.5, and 3.6, the solution* (u_h, p_h) *of* (2.25) *satisfies*

$$\|u - u_h\|_0 + h\left(|u - u_h|_1 + \left\|p - p_h\right\|_0\right) \le Ch^2. \tag{3.15}$$

4. Two-Level FVE Algorithms and Its Error Analysis

In this section, we will present two-level stabilized finite volume element algorithm for (1.1)–(1.3) and derive some optimal bounds for errors. The idea of the two-level method is to reduce the nonlinear problem on a fine mesh into a linear system on a fine mesh by solving a nonlinear problem on a coarse mesh. The basic mechanisms are two quasi-uniform triangulations of Ω, $\mathcal{T}_H$, and $\mathcal{T}_h$, with two different mesh sizes H and $h(h \ll H)$, and the corresponding solutions spaces (X_H, Y_H) and (X_h, Y_h), which satisfy $(X_H, Y_H) \subset (X_h, Y_h)$ and will be called the coarse and the fine spaces, respectively. Now find (u^h, p^h) as follows.

Algorithm 4.1 (Simple two-level stabilized FVE approximation). We have the following steps:

Step 1. On the coarse mesh $\mathcal{T}_H$, solve the stabilized Navier-Stokes problem.
Find $(u_H, p_H) \in X_H \times Y_H$ such that, for all $(v_H, q_H) \in X_H \times Y_H$,

$$B_H\left((u_H, p_H); (I_H^* v_H, q_H)\right) + \tilde{b}\left(u_H, u_H, I_H^* v_H\right) = \left(f, I_H^* v_H\right). \tag{4.1}$$

Step 2. On the fine mesh $\mathcal{T}_h$, solve the stabilized linear Stokes problem.
Find $(u^h, p^h) \in X_h \times Y_h$ such that, for all $(v_h, q_h) \in X_h \times Y_h$,

$$B_h\left(\left(u^h, p^h\right); (I_h^* v_h, q_h)\right) + \tilde{b}\left(u_H, u_H, I_h^* v_h\right) = \left(f, I_h^* v_h\right). \tag{4.2}$$

Next, we study the convergence of (u^h, p^h) to (u, p) in some norms. For convenience, we set $e = R_h(u,p) - u^h$ and $\eta = Q_h(u,p) - p^h$.

Theorem 4.2. *Under the assumptions of Theorems 3.2, 3.4, 3.5, and 3.6 for H and h, the simple two-level stabilized FVE solution* (u^h, p^h) *satisfies the following error estimates:*

$$\left|u - u^h\right|_1 + \left\|p - p^h\right\|_0 \le C\left(h + H^2\right). \tag{4.3}$$

Proof. Subtracting (4.2) from (2.25) and using the Galerkin projection (3.12), it is easy to see that

$$\begin{aligned}
&B_h\big((e,\eta);(I_h^* v_h,q_h)\big) + b(u,u-u_H,v_h) + b(u-u_H,u,v_h) - b(u-u_H,u-u_H,v_h) \\
&\quad + \overline{b}(u-u_H,u-u_H,v_h - I_h^* v_h) - \overline{b}(u-u_H,u,v_h - I_h^* v_h) \\
&\quad - \overline{b}(u,u-u_H,v_h - I_h^* v_h) + \overline{b}(u,u,v_h - I_h^* v_h) = (f, v_h - I_h^* v_h),
\end{aligned} \tag{4.4}$$

for all $(v_h,q_h) \in X_h \times Y_h$. Due to (3.7), Lemma 3.1, (3.15), and (4.4), we have

$$\begin{aligned}
\alpha_1(|e|_1 + \|\eta\|_0) &\le \sup_{(v_h,q_h)\in X_h\times Y_h} \frac{B_h((u,p);(I_h^* v_h,q_h))}{|v_h|_1 + \|q_h\|_0} \\
&\le C\|u\|_2\|u-u_H\|_0 + C|u-u_H|_1^2 + Ch\|u\|_2|u-u_H|_1 + Ch \\
&\le C\Big(H^2 + h\Big),
\end{aligned} \tag{4.5}$$

which, along with (3.14), yields

$$\begin{aligned}
\Big|u-u^h\Big|_1 + \Big\|p-p^h\Big\|_0 &\le |u - R_h(u,p)|_1 + \|p - Q_h(u,p)\|_0 + |e|_1 + \|\eta\|_0 \\
&\le C\Big(h + H^2\Big).
\end{aligned} \tag{4.6}$$

□

Algorithm 4.3 (The Newton two-level stabilized FVE approximation). We have the following steps:

Step 1. On the coarse mesh $\mathcal{T}_H$, solve the stabilized Navier-Stokes problem. Find $(u_H,p_H) \in X_H \times Y_H$ by (4.1).

Step 2. On the fine mesh $\mathcal{T}_h$, solve the stabilized linear Stokes problem. Find $(u^h,p^h) \in X_h \times Y_h$ such that, for all $(v_h,q_h) \in X_h \times Y_h$,

$$B_h\Big(\Big(u^h,p^h\Big);(I_h^* v_h,q_h)\Big) + b_h\Big(u^h,u_H,I_h^* v_h\Big) + b_h\Big(u_H,u^h,I_h^* v_h\Big) = (f,I_h^* v_h) + b_h(u_H,u_H,I_h^* v_h). \tag{4.7}$$

Now, we will study the convergence of the Newton two-level stabilized finite element solution (u^h,p^h) to (u,p) in some norms. To do this, let us set $e = R_h(u,p) - u^h$, $E = u - R_h(u,p)$, and $\eta = Q_h(u,p) - p^h$.

Theorem 4.4. *Under the assumptions of Theorems 3.2, 3.4, 3.5, and 3.6 for H and h, the Newton two-level stabilized FVE solution (u^h,p^h) satisfies the following error estimates:*

$$\Big|u-u^h\Big|_1 + \Big\|p-p^h\Big\|_0 \le C\Big(h + |\log h|^{1/2} H^3\Big). \tag{4.8}$$

Proof. Subtracting (4.7) from (2.25), using the Galerkin projection (3.12) and taking $(v_h, q_h) = (e, \eta)$, we get

$$\begin{aligned}
&B_h\left((e,\eta);\left(I_h^* e,\eta\right)\right) + b(E,u,e) + b(R_h - u_H, u - u_H, e) + b(u_H, E, e) \\
&\quad - \overline{b}\left(R_h - u_H, R_h - u_H, e - I_h^* e\right) - \overline{b}\left(R_h, R_h - u, e - I_h^* e\right) - \overline{b}\left(R_h, u, e - I_h^* e\right) \\
&\quad - b(e, u_H, e) + \overline{b}\left(e, u_H, e - I_h^* e\right) + \overline{b}\left(u_H, e, e - I_h^* e\right) = \left(f, e - I_h^* e\right).
\end{aligned} \tag{4.9}$$

Using Lemma 3.1 and Theorems 3.2, 3.5, and 3.7, we obtain

$$\begin{aligned}
\left|b(E,u,e) + b(u_H,E,e) + \left(f, e - I_h^* e\right)\right| &\leq C(|u|_1 + |u_H|_1)|E|_1|e|_1 + \|f\|_0 \left\|e - I_h^* e\right\|_0 \\
&\leq Ch|e|_1,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
|b(R_h - u_H, u - u_H, e)| &= |b(R_h - u_H, e, u - u_H)| \\
&\leq C\left|\log h\right|^{1/2} |R_h - u_H|_1 |e|_1 |u - u_H|_0 \\
&\leq C\left|\log h\right|^{1/2} |R_h - u|_1 + |u - u_H|_1 |e|_1 |u - u_H|_0 \\
&\leq C\left|\log h\right|^{1/2} H^3 |e|_1.
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\left|\overline{b}\left(R_h - u_H, R_h - u_H, e - I_h^* e\right)\right| &\leq C\left|\log h\right|^{1/2} |R_h - u_H|_1 |R_h - u_H|_1 \left\|e - I_h^* e\right\|_0 \\
&\leq C\left|\log h\right|^{1/2} H^3 |e|_1.
\end{aligned} \tag{4.12}$$

$$\left|\overline{b}\left(R_h, R_h - u, e - I_h^* e\right)\right| \leq N|R_h|_1 |R_h - u|_1 |e|_1 \leq Ch|e|_1, \tag{4.13}$$

$$\left|\overline{b}\left(R_h - u, e - I_h^* e\right)\right| \leq C|R_h|_1 \|u\|_2 \left\|e - I_h^* e\right\|_0 \leq Ch|e|_1 \leq Ch|e|_1, \tag{4.14}$$

$$\begin{aligned}
&\nu|e|_1^2 - |b(e, u_H, e)| - \left|\overline{b}\left(e, u_H, e - I_h^* e\right)\right| - \left|\overline{b}\left(u_H, e, e - I_h^* e\right)\right| \\
&\quad \geq \nu|e|_1^2 - 3N|u_H|_1 |e|_1^2 \\
&\quad \geq \nu\left(1 - \frac{3N}{\nu^2}\|f\|_{-1}\right)|e|_1^2.
\end{aligned} \tag{4.15}$$

Combining (4.10)–(4.15) with (4.9) yields

$$|e|_1 \leq C\left(h + \left|\log h\right|^{1/2} H^3\right). \tag{4.16}$$

Thanks to (3.7), (4.9), Theorems 3.2 and 3.5, and estimates (4.10)–(4.14) and (4.16), we have

$$\begin{aligned}\|\eta\|_0 &\leq (\alpha_1)^{-1} \sup_{(v_h,q_h)\in X_h\times Y_h} \frac{B_h((e,\eta);(I_h^* v_h, q_h))}{|v_h|_1 + \|q_h\|_0} \\ &\leq C\Big(h + \log|h|^{1/2} H^3 + |e|_1\Big) \\ &\leq C\Big(h + \log|h|^{1/2} H^3\Big).\end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17) with (3.14) and Theorem 3.2 yields (2.13). □

References

[1] Z. Q. Cai, "On the finite volume element method," *Numerische Mathematik*, vol. 58, no. 7, pp. 713–735, 1991.

[2] R. E. Ewing, T. Lin, and Y. Lin, "On the accuracy of the finite volume element method based on piecewise linear polynomials," *SIAM Journal on Numerical Analysis*, vol. 39, no. 6, pp. 1865–1888, 2002.

[3] S. H. Chou and D. Y. Kwak, "Analysis and convergence of a MAC-like scheme for the generalized Stokes problem," *Numerical Methods for Partial Differential Equations*, vol. 13, no. 2, pp. 147–162, 1997.

[4] P. Chatzipantelidies, "A finite volume method based on the Crouziex Raviart element for elliptic PDEs in two dimension," *Numerical Mathematics*, vol. 82, pp. 409–432, 1999.

[5] R. Ewing, R. Lazarov, and Y. Lin, "Finite volume element approximations of nonlocal reactive flows in porous media," *Numerical Methods for Partial Differential Equations*, vol. 16, no. 3, pp. 285–311, 2000.

[6] H. Guoliang and H. Yinnian, "The finite volume method based on stabilized finite element for the stationary Navier-Stokes problem," *Journal of Computational and Applied Mathematics*, vol. 205, no. 1, pp. 651–665, 2007.

[7] X. Ye, "A discontinuous finite volume method for the Stokes problems," *SIAM Journal on Numerical Analysis*, vol. 44, no. 1, pp. 183–198, 2006.

[8] S. H. Chou, "Analysis and convergence of a covolume method for the generalized Stokes problem," *Mathematics of Computation*, vol. 66, no. 217, pp. 85–104, 1997.

[9] M. Berggren, "A vertex-centered, dual discontinuous Galerkin method," *Journal of Computational and Applied Mathematics*, vol. 192, no. 1, pp. 175–181, 2006.

[10] S.-H. Chou and D. Y. Kwak, "Multigrid algorithms for a vertex-centered covolume method for elliptic problems," *Numerische Mathematik*, vol. 90, no. 3, pp. 441–458, 2002.

[11] R. Eymard, T. Gallouët, and R. Herbin, "Finite volume methods," in *Handbook of Numerical Analysis, Vol. VII*, pp. 713–1020, North-Holland, Amsterdam, The Netherlands, 2000.

[12] V. R. Voller, *Basic Control Volume Finite Element Methods for Fluids and Solids*, vol. 1, World Scientific, Hackensack, NJ, USA, 2009.

[13] J. Xu, "A novel two-grid method for semilinear elliptic equations," *SIAM Journal on Scientific Computing*, vol. 15, no. 1, pp. 231–237, 1994.

[14] J. Xu, "Two-grid discretization techniques for linear and nonlinear PDEs," *SIAM Journal on Numerical Analysis*, vol. 33, no. 5, pp. 1759–1777, 1996.

[15] V. Ervin, W. Layton, and J. Maubach, "A posteriori error estimators for a two-level finite element method for the Navier-Stokes equations," *Numerical Methods for Partial Differential Equations*, vol. 12, no. 3, pp. 333–346, 1996.

[16] V. Girault and J.-L. Lions, "Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra," *Portugaliae Mathematica*, vol. 58, no. 1, pp. 25–57, 2001.

[17] Y. He, "Two-level method based on finite element and Crank-Nicolson extrapolation for the time-dependent Navier-Stokes equations," *SIAM Journal on Numerical Analysis*, vol. 41, no. 4, pp. 1263–1285, 2003.

[18] J. Li, "Investigations on two kinds of two-level stabilized finite element methods for the stationary Navier-Stokes equations," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1470–1481, 2006.

[19] W. Layton, "A two-level discretization method for the Navier-Stokes equations," *Computers & Mathematics with Applications*, vol. 26, no. 2, pp. 33–38, 1993.

[20] W. Layton and W. Lenferink, "Two-level Picard, defect correction for the Navier-Stokes equations," *Applied Mathematics and Computation*, vol. 80, pp. 1–12, 1995.

[21] W. Layton and L. Tobiska, "A two-level method with backtracking for the Navier-Stokes equations," *SIAM Journal on Numerical Analysis*, vol. 35, no. 5, pp. 2035–2054, 1998.

[22] L. Zhu and Y. He, "Two-level Galerkin-Lagrange multipliers method for the stationary Navier-Stokes equations," *Journal of Computational and Applied Mathematics*, vol. 230, no. 2, pp. 504–512, 2009.

[23] C. Bi and V. Ginting, "Two-grid finite volume element method for linear and nonlinear elliptic problems," *Numerische Mathematik*, vol. 108, no. 2, pp. 177–198, 2007.

[24] C. Chen, M. Yang, and C. Bi, "Two-grid methods for finite volume element approximations of nonlinear parabolic equations," *Journal of Computational and Applied Mathematics*, vol. 228, no. 1, pp. 123–132, 2009.

[25] C. Chen and W. Liu, "Two-grid finite volume element methods for semilinear parabolic problems," *Applied Numerical Mathematics*, vol. 60, no. 1-2, pp. 10–18, 2010.

[26] P. Chatzipantelidis, R. D. Lazarov, and V. Thomée, "Error estimates for a finite volume element method for parabolic equations in convex polygonal domains," *Numerical Methods for Partial Differential Equations*, vol. 20, no. 5, pp. 650–674, 2004.

[27] R. Stenberg, "Analysis of mixed finite elements methods for the Stokes problem: a unified approach," *Mathematics of Computation*, vol. 42, no. 165, pp. 9–23, 1984.

[28] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, vol. 5 of *Theory and Algorithms*, Springer, Berlin, Germany, 1986.

[29] R. Temam, *Navier-Stokes Equations*, vol. 2 of *Theory and Numerical Analysis*, North-Holland, Amsterdam, The Netherlands, 3rd edition, 1984.

[30] X. Ye, "On the relationship between finite volume and finite element methods applied to the Stokes equations," *Numerical Methods for Partial Differential Equations*, vol. 17, no. 5, pp. 440–453, 2001.

[31] N. Kechkar and D. Silvester, "Analysis of locally stabilized mixed finite element methods for the Stokes problem," *Mathematics of Computation*, vol. 58, no. 197, pp. 1–10, 1992.

[32] G. He, Y. He, and X. Feng, "Finite volume method based on stabilized finite elements for the nonstationary Navier-Stokes problem," *Numerical Methods for Partial Differential Equations*, vol. 23, no. 5, pp. 1167–1191, 2007.