GENERALIZED QUASILINEARIZATION METHOD AND HIGHER ORDER OF CONVERGENCE FOR SECOND-ORDER BOUNDARY VALUE PROBLEMS

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The method of generalized quasilinearization for second-order boundary value problems has been extended when the forcing function is the sum of 2-hyperconvex and 2-hyperconcave functions. We develop two sequences under suitable conditions which converge to the unique solution of the boundary value problem. Furthermore, the convergence is of order 3. Finally, we provide numerical examples to show the application of the generalized quasilinearization method developed here for second-order boundary value problems.

1. Introduction

The method of quasilinearization [1, 2] combined with the technique of upper and lower solutions is an effective and fruitful technique for solving a wide variety of nonlinear problems. It has been referred to as a generalized quasilinearization method. See [9] for details. The method is extremely useful in scientific computations due to its accelerated rate of convergence as in [10, 11].

In [4, 13], the authors have obtained a higher order of convergence (an order more than 2) for initial value problems. They have considered situations when the forcing function is either hyperconvex or hyperconcave. In [12], we have obtained the results of higher order of convergence for first order initial value problems when the forcing function is the sum of hyperconvex and hyperconcave functions with natural and coupled lower and upper solutions. In this paper we extend the result to the second-order boundary value problems when the forcing function is a sum of 2-hyperconvex and 2-hyperconcave functions. We have proved the existence of the unique solution of the nonlinear problem using natural upper and lower solutions. We demonstrate the iterates converge cubically to the unique solution of the nonlinear problem. We merely state the result related to coupled

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lower and upper solutions without proof due to monotony. Finally, we present two numerical applications of our theoretical results developed in our main result. We note that the monotone iterates may not converge linearly or quadratically in general. See [4, 8] for examples. However in our result we have provided sufficient conditions for cubic convergence. For real world applications see [5].

For this purpose, consider the following second-order boundary value problem (BVP for short):

$$-u'' = f(t,u)+g(t,u), \quad Bu(\mu) = b_\mu, \quad \mu = 0,1,\ t \in J \equiv [0,1], \tag{1.1}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_0,\tau_1 \geq 0$, $\tau_0+\tau_1 > 0$, $\nu_0,\nu_1 > 0$, $b_\mu \in R$ and $f,g \in C[J\times R,R]$.

Here we provide the definition of natural lower and upper solutions of (1.1). One can define coupled lower and upper solutions of the other types in the same manner. See for [14, 15] details.

Definition 1.1. The functions $\alpha_0,\beta_0 \in C^2[J,R]$ are said to be natural lower and upper solutions if

$$\begin{aligned} -\alpha_0'' &\leq f(t,\alpha_0)+g(t,\alpha_0), \qquad B\alpha_0(\mu) \leq b_\mu \quad \text{on } J, \\ -\beta_0'' &\geq f(t,\beta_0)+g(t,\beta_0), \qquad B\beta_0(\mu) \geq b_\mu \quad \text{on } J. \end{aligned} \tag{1.2}$$

In order to facilitate later explanations, we will need the following definition.

Definition 1.2. A function $h : A \to B$, $A,B \subset R$ is called m-hyperconvex, $m \geq 0$, if $h \in C^{m+1}[A,B]$ and $d^{m+1}h/du^{m+1} \geq 0$ for $u \in A$; h is called m-hyperconcave if the inequality is reversed.

In this paper, we use the maximum norm of u over J, that is,

$$\|u\| = \max_{t\in J} |u|. \tag{1.3}$$

Also throughout this paper we use the notation

$$\frac{\partial^k f(t,u)}{\partial u^k} = f^{(k)}(t,u) \tag{1.4}$$

for any function $f(t,u)$ and for $k = 0,1,2$.

In view of natural upper and lower solutions of (1.1), we will develop results when f is 2-hyperconvex and g is 2-hyperconcave. Furthermore, we show that these iterates converge uniformly and monotonically to the unique solution of (1.1), and the convergence is of order 3.

2. Preliminaries

In this section, we recall some well known theorems and corollaries which we need in our main results relative to the BVP

$$-u'' = f(t,u,u'), \quad Bu(\mu) = b_\mu, \quad \mu = 0,1,\ t \in J \equiv [0,1], \tag{2.1}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_0,\tau_1 \geq 0$, $\tau_0 + \tau_1 > 0$, $\nu_0,\nu_1 > 0$, $b_\mu \in R$ and $f \in C[J \times R \times R, R]$. For details see [3, 6, 7].

Theorem 2.1. *Assume that*

(i) $\alpha_0,\beta_0 \in C^2[J,R]$ *are lower and upper solutions of (2.1).*

(ii) f_u, $f_{u'}$ *exist, continuous,* $f_u < 0$ *and* $f_u \not\equiv 0$ *on* $\Omega = [(t,u,\overline{u}) : t \in [0,1],\ \beta_0 \leq u \leq \alpha_0]$ *and* $\overline{u} = \alpha_0'(t) = \beta_0'(t)$.

Then we have $\alpha_0(t) \leq \beta_0(t)$ *on* J.

Next we present a special case of the above theorem which is known as the maximum principle, when u' term is missing.

Corollary 2.2. *Let* $q,r \in C[I,R]$ *with* $r(t) \geq 0$ *on* J. *Suppose further that* $p \in C^2[I,R]$ *and*

$$-p'' \leq -rp, \quad Bp(\mu) \leq 0. \tag{2.2}$$

Then $p(t) \leq 0$ *on* J. *If the inequalities are reversed, then* $p(t) \geq 0$ *on* J.

The next corollary is a special case of [9, Theorem 3.1.3].

Corollary 2.3. *Assume that* α_0, β_0 *are lower and upper solutions of (1.1) respectively such that* $\alpha_0(t) \leq \beta_0(t)$ *on* J. *Then there exists a solution* u *for the BVP (1.1) such that* $\alpha_0(t) \leq u(t) \leq \beta_0(t)$ *on* J.

3. Main results

In this section, we consider the BVP

$$-u'' = f(t,u) + g(t,u), \quad Bu(\mu) = b_\mu, \quad \mu = 0,1,\ t \in J \equiv [0,1], \tag{3.1}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_0,\tau_1 \geq 0$, $\tau_0 + \tau_1 > 0$, $\nu_0,\nu_1 > 0$, $b_\mu \in R$, $f,g \in C[\Omega,R]$, $\Omega = [(t,u) : \alpha_0(t) \leq u(t) \leq \beta_0(t),\ t \in J]$, and $\alpha_0,\beta_0 \in C^2[J,R]$ with $\alpha_0(t) \leq \beta_0(t)$ on J.

Here, we state the inequalities satisfied by $f(t,u)$ and $g(t,u)$ when $f(t,u)$ is 2-hyperconvex in u and $g(t,u)$ is 2-hyperconcave in u. We need these inequalities for our first main result.

Suppose that $f(t,u)$ is 2-hyperconvex in u, then we have the following inequalities,

$$f(t,\eta) \geq \sum_{i=0}^{2} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \quad \eta \geq \xi, \tag{3.2}$$

$$f(t,\eta) \leq \sum_{i=0}^{2} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \quad \eta \leq \xi. \tag{3.3}$$

Similarly, when $g(t,u)$ is 2-hyperconcave in u, we have the following inequalities:

$$g(t,\eta) \geq \sum_{i=0}^{1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2)}(t,\eta)(\eta-\xi)^2}{(2)!}, \quad \eta \geq \xi, \tag{3.4}$$

$$g(t,\eta) \leq \sum_{i=0}^{1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2)}(t,\eta)(\eta-\xi)^2}{(2)!}, \quad \eta \leq \xi. \tag{3.5}$$

Based on these inequalities, relative to the natural upper and lower solutions, we develop two monotone sequences which converge uniformly and monotonically to the unique solution of (3.1) and the order of convergence is 3.

Theorem 3.1. *Assume that*

(i) $\alpha_0,\beta_0 \in C^2[J,R]$ *are lower and upper solutions with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(ii) $f,g \in C^3[\Omega,R]$ *such that* $f(t,u)$ *is 2-hyperconvex in* u *on* J *[i.e.,* $f^{(3)}(t,u) \geq 0$ *for* $(t,u) \in \Omega$*],* $g(t,u)$ *is 2-hyperconcave in* u *on* J *[i.e.,* $g^{(3)}(t,u) \leq 0$ *for* $(t,u) \in \Omega$*],* $f(t,u)$ *is nondecreasing,* $g(t,u)$ *is nonincreasing and* $f_u + g_u < 0$ *on* Ω.

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$*,* $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order 3.*

Proof. The assumptions $f^{(3)}(t,u) \geq 0, g^{(3)}(t,u) \leq 0$ yield the inequalities (3.2), (3.3), (3.4), and (3.5) whenever $\alpha_0 \leq \eta,\ \xi \leq \beta_0$. Let us first consider the following BVPs:

$$\begin{aligned} -w'' &= \widetilde{F}(t,\alpha,\beta;w) \\ &= \sum_{i=0}^{2} \frac{f^{(i)}(t,\alpha)(w-\alpha)^i}{i!} + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha)(w-\alpha)^i}{i!} + \frac{g^{(2)}(t,\beta)(w-\alpha)^2}{2!}, \\ &\qquad Bw(\mu) = b_\mu \quad \text{on } J; \end{aligned} \tag{3.6}$$

$$\begin{aligned} -v'' &= \widetilde{G}(t,\alpha,\beta;v) \\ &= \sum_{i=0}^{2} \frac{f^{(i)}(t,\beta)(v-\beta)^i}{i!} + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta)(v-\beta)^i}{i!} + \frac{g^{(2)}(t,\alpha)(v-\beta)^2}{2!}, \\ &\qquad Bv(\mu) = b_\mu \quad \text{on } J. \end{aligned} \tag{3.7}$$

We develop the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ using the above BVPs (3.6) and (3.7) respectively. Initially, we prove (α_0,β_0) are lower and upper solutions of (3.6) and (3.7) respectively. To begin, we will consider natural lower and upper solutions of the equation (3.1):

$$\begin{aligned} -\alpha_0'' &\leq f(t,\alpha_0) + g(t,\alpha_0), \quad B\alpha_0(\mu) \leq b_\mu, \\ -\beta_0'' &\geq f(t,\beta_0) + g(t,\beta_0), \quad B\beta_0(\mu) \geq b_\mu, \end{aligned} \tag{3.8}$$

where $\alpha_0(t) \le \beta_0(t)$. The inequalities (3.2) and (3.4), and (3.8) imply

$$\begin{aligned}
-\alpha_0'' &\le f(t,\alpha_0) + g(t,\alpha_0) \\
&= \widetilde{F}(t,\alpha_0,\beta_0;\alpha_0), \quad B\alpha_0(\mu) \le b_\mu, \\
-\beta_0'' &\ge f(t,\beta_0) + g(t,\beta_0) \\
&\ge \sum_{i=0}^{2} \frac{f^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!} + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!} + \frac{g^{(2)}(t,\beta_0)(\beta_0-\alpha_0)^2}{2!} \\
&= \widetilde{F}(t,\alpha_0,\beta_0;\beta_0), \quad B\beta_0(\mu) \ge b_\mu.
\end{aligned} \tag{3.9}$$

We can apply Corollary 2.3 together with (3.9) conclude that there exists a solution $\alpha_1(t)$ of (3.6) with $\alpha = \alpha_0$ and $\beta = \beta_0$ such that $\alpha_0 \le \alpha_1 \le \beta_0$ on J.

Using the inequalities (3.3), (3.5), and (3.8) on the same lines, we can get

$$-\beta_0'' \ge f(t,\beta_0) + g(t,\beta_0) = \widetilde{G}(t,\alpha_0,\beta_0;\beta_0), \quad B\beta_0(\mu) \ge b_\mu, \tag{3.10}$$

$$\begin{aligned}
-\alpha_0'' \le f(t,\alpha_0) + g(t,\alpha_0) &\le \sum_{i=0}^{2} \frac{f^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} + \frac{g^{(2)}(t,\alpha_0)(\alpha_0-\beta_0)^2}{2!} \\
&= \widetilde{G}(t,\alpha_0,\beta_0;\alpha_0), \quad B\alpha_0(\mu) \le b_\mu.
\end{aligned} \tag{3.11}$$

Hence α_0, β_0 are lower and upper solutions of (3.7) with $\alpha_0 \le \beta_0$. Applying Corollary 2.3, we obtain that there exists a solution $\beta_1(t)$ of (3.7) with $\alpha = \alpha_0$ and $\beta = \beta_0$ such that $\alpha_0 \le \beta_1 \le \beta_0$ on J.

Now we will prove that α_1 is a unique solution of (3.6). For this purpose we need to prove that $\partial\widetilde{F}(t,\alpha_0,\beta_0;\alpha_1)/\partial\alpha_1 < 0$. Since $f(t,u)$ is 2-hyperconvex in u and $g(t,u)$ is 2-hyperconcave in u on J with $f_u + g_u < 0$ on Ω, we get

$$\begin{aligned}
\frac{\partial\widetilde{F}(t,\alpha_0,\beta_0;\alpha_1)}{\partial\alpha_1} &= f^{(1)}(t,\alpha_1) + g^{(1)}(t,\alpha_1) - \frac{f^{(3)}(t,\xi_1)(\alpha_1-\alpha_0)^2}{(2)!} \\
&\quad + g^{(3)}(t,\eta_1)(\alpha_1-\alpha_0)(\beta_0-\xi_2) \\
&\le f^{(1)}(t,\alpha_1) + g^{(1)}(t,\alpha_1) < 0,
\end{aligned} \tag{3.12}$$

where $\alpha_0 \le \xi_1, \xi_2 \le \alpha_1$ and $\xi_2 \le \eta_1 \le \beta_0$. Hence by the special case of Theorem 2.1 with u'-term missing, we can conclude that α_1 is the unique solution of (3.6). Similarly we can prove that β_1 is the unique solution of (3.7).

Using the nonincreasing property of $g^{(2)}(t,u)$, (3.2), (3.3), (3.4), (3.5) with $\alpha_0 \le \alpha_1 \le \beta_0$, $\alpha_0 \le \beta_1 \le \beta_0$ we have

$$\begin{aligned}
-\alpha_1'' &= \widetilde{F}(t,\alpha_0,\beta_0;\alpha_1) \\
&= \sum_{i=0}^{2} \frac{f^{(i)}(t,\alpha_0)(\alpha_1-\alpha_0)^i}{i!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha_0)(\alpha_1-\alpha_0)^i}{i!} + \frac{g^{(2)}(t,\beta_0)(\alpha_1-\alpha_0)^2}{2!} \\
&\le f(t,\alpha_1)+g(t,\alpha_1), \quad B\alpha_1(\mu) \le b_\mu;
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
-\beta_1'' &= \widetilde{G}(t,\alpha_0,\beta_0;\beta_1) \\
&= \sum_{i=0}^{2} \frac{f^{(i)}(t,\beta_0)(\beta_1-\beta_0)^i}{i!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta_0)(\beta_1-\beta_0)^i}{i!} + \frac{g^{(2)}(t,\alpha_0)(\beta_1-\beta_0)^2}{2!} \\
&\ge f(t,\beta_1)+g(t,\beta_1), \quad B\beta_1(\mu) \ge b_\mu.
\end{aligned} \tag{3.14}$$

Since α_1, β_1 are lower and upper solutions of (3.1), we can apply the special case of Theorem 2.1 to obtain $\alpha_1 \le \beta_1$ on J. Thus we have $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$ on J.

Assume now that α_n and β_n are solutions of BVPs (3.6) and (3.7), respectively, with $\alpha = \alpha_{n-1}$ and $\beta = \beta_{n-1}$ such that $\alpha_{n-1} \le \alpha_n \le \beta_n \le \beta_{n-1}$ on J and

$$\begin{aligned}
-\alpha_n'' &\le f(t,\alpha_n)+g(t,\alpha_n), \quad B\alpha_n(\mu) \le b_\mu, \\
-\beta_n'' &\ge f(t,\beta_n)+g(t,\beta_n), \quad B\beta_n(\mu) \ge b_\mu,
\end{aligned} \tag{3.15}$$

Certainly this is true for $n = 1$.

We need to show that $\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n$ on J, where α_{n+1} and β_{n+1} are solutions of BVPs (3.6) and (3.7), respectively, with $\alpha = \alpha_n$ and $\beta = \beta_n$.

The inequalities (3.2) and (3.4), and (3.15) imply

$$\begin{aligned}
-\alpha_n'' &\le f(t,\alpha_n)+g(t,\alpha_n) \\
&= \widetilde{F}(t,\alpha_n,\beta_n;\alpha_n), \quad B\alpha_n(\mu) \le b_\mu, \\
-\beta_n'' &\ge f(t,\beta_n)+g(t,\beta_n) \\
&\ge \sum_{i=0}^{2} \frac{f^{(i)}(t,\alpha_n)(\beta_n-\alpha_n)^i}{i!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha_n)(\beta_n-\alpha_n)^i}{i!} + \frac{g^{(2)}(t,\beta_n)(\beta_n-\alpha_n)^2}{2!} \\
&= \widetilde{F}(t,\alpha_n,\beta_n;\beta_n), \quad B\beta_n(\mu) \ge b_\mu.
\end{aligned} \tag{3.16}$$

This proves that α_n, β_n are lower and upper solutions of (3.6) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Hence using (3.16) and Corollary 2.3 we can conclude that there exists a solution $\alpha_{n+1}(t)$ of (3.6) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \le \alpha_{n+1} \le \beta_n$ on J.

The inequalities (3.3) and (3.5), and (3.15) imply

$$-\beta_n'' \ge f(t,\beta_n) + g(t,\beta_n) = \widetilde{G}(t,\alpha_n,\beta_n;\beta_n), \quad B\beta_n(\mu) \ge b_\mu, \tag{3.17}$$

$$\begin{aligned} -\alpha_n'' &\le f(t,\alpha_n) + g(t,\alpha_n) \\ &\le \sum_{i=0}^{2} \frac{f^{(i)}(t,\beta_n)(\alpha_n - \beta_n)^i}{i!} \\ &\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta_n)(\alpha_n - \beta_n)^i}{i!} + \frac{g^{(2)}(t,\alpha_n)(\alpha_n - \beta_n)^2}{2!} \\ &= \widetilde{G}(t,\alpha_n,\beta_n;\alpha_n), \quad B\alpha_n(\mu) \le b_\mu. \end{aligned} \tag{3.18}$$

Hence α_n, β_n are lower and upper solutions of (3.7) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Applying Corollary 2.3 we can show that there exists a solution $\beta_{n+1}(t)$ of (3.7) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \le \beta_{n+1} \le \beta_n$ on J. In view of assumptions on f and g, α_{n+1}, β_{n+1} are unique by the special case of Theorem 2.1.

Furthermore, by (3.2), (3.3), (3.4), (3.5) with $\alpha_n \le \alpha_{n+1} \le \beta_n$, $\alpha_n \le \beta_{n+1} \le \beta_n$, and $g^{(2)}(t,u)$ nonincreasing u, we get

$$\begin{aligned} -\alpha_{n+1}'' &= \widetilde{F}(t,\alpha_n,\beta_n;\alpha_{n+1}) \\ &= \sum_{i=0}^{2} \frac{f^{(i)}(t,\alpha_n)(\alpha_{n+1} - \alpha_n)^i}{i!} \\ &\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha_n)(\alpha_{n+1} - \alpha_n)^i}{i!} + \frac{g^{(2)}(t,\beta_n)(\alpha_{n+1} - \alpha_n)^2}{2!} \\ &\le f(t,\alpha_{n+1}) + g(t,\alpha_{n+1}), \quad B\alpha_{n+1}(\mu) \le b_\mu; \\ -\beta_{n+1}'' &= \widetilde{G}(t,\alpha_n,\beta_n;\beta_{n+1}) \\ &= \sum_{i=0}^{2} \frac{f^{(i)}(t,\beta_n)(\beta_{n+1} - \beta_n)^i}{i!} \\ &\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta_n)(\beta_{n+1} - \beta_n)^i}{i!} + \frac{g^{(2)}(t,\alpha_n)(\beta_{n+1} - \beta_n)^2}{2!} \\ &\ge f(t,\beta_{n+1}) + g(t,\beta_{n+1}), \quad B\beta_{n+1}(\mu) \ge b_\mu. \end{aligned} \tag{3.19}$$

Since α_{n+1}, β_{n+1} are lower and upper solutions of (3.1) we can apply the special case of Theorem 2.1 and get $\alpha_{n+1} \le \beta_{n+1}$ on J. This proves $\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n$ on J. Hence by induction, we have

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0. \tag{3.20}$$

By the fact that α_n, β_n are lower and upper solutions of (3.1) with $\alpha_n \le \beta_n$ and Corollary 2.3 we can conclude that there exists a solution $u(t)$ of (3.1) such that $\alpha_n \le u \le \beta_n$ on J. From this we can obtain that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le u \le \beta_n \le \cdots \le \beta_1 \le \beta_0. \tag{3.21}$$

Using Green's function, we can write $\alpha_n(t)$ and $\beta_n(t)$ as follows:

$$\begin{aligned} \alpha_n(t) &= \int_0^1 K(t,s)\widetilde{F}(s,\alpha_{n-1}(s),\beta_{n-1}(s);\alpha_n(s))\,ds, \\ \beta_n(t) &= \int_0^1 K(t,s)\widetilde{G}(s,\alpha_{n-1}(s),\beta_{n-1}(s);\beta_n(s))\,ds. \end{aligned} \tag{3.22}$$

Here $K(t,s)$ is the Green's function given by

$$K(t,s) = \begin{cases} \dfrac{1}{c}x(s)y(t), & 0 \le s \le t \le 1, \\ \dfrac{1}{c}x(t)y(s), & 0 \le t \le s \le 1, \end{cases} \tag{3.23}$$

where $x(t) = (\tau_0/\nu_0)t + 1$, $y(t) = (\tau_1/\nu_1)(1-t) + 1$ are two linearly independent solutions of $-u'' = 0$ and $c = x(t)y'(t) - x'(t)y(t)$. We can prove that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are equicontinuous and uniformly bounded. Now applying Ascoli-Arzela's theorem, we can show that there exist subsequences $\{\alpha_{n,j}(t)\}$, $\{\beta_{n,j}(t)\}$ such that $\alpha_{n,j}(t) \to \rho(t)$ and $\beta_{n,j}(t) \to r(t)$ with $\rho(t) \le u \le r(t)$ on J. Since the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ are monotone, we have $\alpha_n(t) \to \rho(t)$ and $\beta_n(t) \to r(t)$. Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \le u \le r(t) = \lim_{n\to\infty} \beta_n(t). \tag{3.24}$$

Next we show that $\rho(t) \ge r(t)$. From BVPs (3.6) and (3.7) we get

$$\begin{aligned} -\rho''(t) &= f(t,\rho) + g(t,\rho), \qquad B\rho(\mu) = b(\mu), \\ -r''(t) &= f(t,r) + g(t,r), \qquad Br(\mu) = b(\mu). \end{aligned} \tag{3.25}$$

Set $p(t) = r - \rho$ and note that $Bp(\mu) = 0$. We have

$$\begin{aligned} -p'' &= -r''(t) - (-\rho''(t)) = f(t,r) + g(t,r) - f(t,\rho) - g(t,\rho) \\ &= f_u(t,\xi)(r-\rho) + g_u(t,\eta)(r-\rho) = \big(f_u(t,\xi) + g_u(t,\eta)\big)p, \end{aligned} \tag{3.26}$$

where ξ, η are between ρ and r. This implies that $-p'' \le -kp$, where $f_u + g_u \le -k < 0$. Now applying Corollary 2.2 we get $p \le 0$ or $r(t) \le \rho(t)$ on J. This proves $r(t) = \rho(t) = u(t)$. Hence $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of (3.1).

Let us consider the order of convergence of $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to the unique solution $u(t)$ of (3.1). To do this, set

$$\begin{aligned} p_n(t) &= u(t) - \alpha_n(t) \geq 0, \\ q_n(t) &= \beta_n(t) - u(t) \geq 0, \end{aligned} \tag{3.27}$$

for $t \in J$ with $Bp_n(\mu) = Bq_n(\mu) = 0$.

Therefore we can write

$$p_{n+1} = \int_0^1 K(t,s)\left[f(s,u) + g(s,u) - \widetilde{F}(s,\alpha_n,\beta_n;\alpha_{n+1})\right]ds, \tag{3.28}$$

where $K(t,s)$ is the Green's function given by (3.23).

Now using the Taylor series expansion with Lagrange remainder, and the mean value theorem together with (ii) of the hypothesis, we obtain

$$\begin{aligned}
0 &\leq p_{n+1} \\
&= \int_0^1 K(t,s)\Bigg\{f(s,u) + g(s,u) \\
&\qquad - \Bigg[\sum_{i=0}^{2} \frac{f^{(i)}(s,\alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\
&\qquad\quad + \sum_{i=0}^{1} \frac{g^{(i)}(s,\alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} + \frac{g^{(2)}(s,\beta_n)(\alpha_{n+1}-\alpha_n)^2}{2!}\Bigg]\Bigg\}ds \\
&= \int_0^1 K(t,s)\Bigg\{f(s,u) + g(s,u) \\
&\qquad - \Bigg[f(s,\alpha_{n+1}) - \frac{f^{(3)}(s,\xi_1)(\alpha_{n+1}-\alpha_n)^3}{(3)!} + g(s,\alpha_{n+1}) \\
&\qquad\quad - \frac{g^{(2)}(s,\xi_2)(\alpha_{n+1}-\alpha_n)^2}{2!} + \frac{g^{(2)}(s,\beta_n)(\alpha_{n+1}-\alpha_n)^2}{2!}\Bigg]\Bigg\}ds \\
&\leq \int_0^1 K(t,s)\Bigg[f_u(s,\eta_1)(u-\alpha_{n+1}) + g_u(s,\eta_2)(u-\alpha_{n+1}) \\
&\qquad + \frac{f^{(3)}(s,\xi_1)(u-\alpha_n)^3}{(3)!} - \frac{g^{(3)}(s,\eta_3)(\beta_n-\xi_2)(u-\alpha_n)^2}{2}\Bigg]ds \\
&= \int_0^1 K(t,s)\Bigg\{\left[f_u(s,\eta_1) + g_u(s,\eta_2)\right]p_{n+1} \\
&\qquad + \frac{f^{(3)}(s,\xi_1)p_n^3}{(3)!} - \frac{g^{(3)}(s,\eta_2)p_n^2(q_n+p_n)}{2}\Bigg\}ds,
\end{aligned} \tag{3.29}$$

where $\alpha_n \le \xi_1$, $\xi_2 \le \alpha_{n+1} \le \eta_1$, $\eta_2 \le u$, and $\xi_2 \le \eta_3 \le \beta_n$. Let $|K(t,s)| \le A_1$, $|f_u(t,u) + g_u(t,v)| \le A_2$, $|f^{(3)}(t,u)/3!| \le A_3$, and $|g^{(3)}(t,u)/2| \le A_4$. Then we have

$$||p_{n+1}|| \le k_1||p_n||^3 + k_2||p_n||^2(||q_n|| + ||p_n||), \tag{3.30}$$

where $k_1 = A_1A_3/(1-A_1A_2)$ and $k_2 = A_1A_4/(1-A_1A_2)$.

Similarly, we can write

$$q_{n+1} = \int_0^1 K(t,s)[\widetilde{G}(s,\alpha_n,\beta_n;\beta_{n+1}) - f(s,u) - g(s,u)]ds, \tag{3.31}$$

where $K(t,s)$ is the Green's function given by (3.23).

Using the Taylor series expansion with Lagrange remainder, and the mean value theorem together with (ii), we can show

$$||q_{n+1}|| \le k_1||q_n||^3 + k_2||q_n||^2(||q_n|| + ||p_n||), \tag{3.32}$$

where $k_1 = A_1A_3/(1-A_1A_2)$ and $k_2 = A_1A_4/(1-A_1A_2)$.

Hence combining (3.30) and (3.32) we obtain

$$\begin{aligned}\max_{t\in J}|u(t)-\alpha_{n+1}(t)| &+ \max_{t\in J}|\beta_{n+1}(t)-u(t)| \\ &\le C\Big[\max_{t\in J}|u(t)-\alpha_n(t)| + \max_{t\in J}|\beta_n(t)-u(t)|\Big]^3,\end{aligned} \tag{3.33}$$

where C is an appropriate positive constant.

This completes the proof. □

We note that the unique solution we have obtained is the unique solution of (3.1) in the sector determined by the lower and upper solutions.

Next we merely state a result without proof using coupled lower and upper solutions of (3.1). However, in order to show the existence of the unique solution of the iterates, we use the existence result [7, Theorem 2.4.1]. for systems and a special case of the comparison theorem of [7].

Theorem 3.2. *Assume that*

(i) $\alpha_0, \beta_0 \in C^2[J,R]$ *are coupled lower and upper solutions of (3.1) with* $\alpha_0(t) \le \beta_0(t)$ *on* J *such that*

$$\begin{aligned}-\alpha_0'' &\le f(t,\beta_0) + g(t,\alpha_0), \qquad B\alpha_0(\mu) \le b_\mu \quad \text{on } J,\\ -\beta_0'' &\ge f(t,\alpha_0) + g(t,\beta_0), \qquad B\beta_0(\mu) \ge b_\mu \quad \text{on } J;\end{aligned} \tag{3.34}$$

(ii) $f, g \in C^3[\Omega, R]$ *such that* $f(t,u)$ *is 2-hyperconvex in* u *on* J *[i.e.,* $f^{(3)}(t,u) \geq 0$ *for* $(t,u) \in \Omega$*],* $g(t,u)$ *is 2-hyperconcave in* u *on* J *[i.e.,* $g^{(3)}(t,u) \leq 0$ *for* $(t,u) \in \Omega$*],* $f(t,u)$*,* $g(t,u)$ *are nonincreasing with* $f_u - g_u > 0$ *and*

$$f_u(t,u) \leq -\max_{\Omega}\left[f^{(3)}(t,u)\right](\beta_0 - \alpha_0)^2 \leq 0 \quad \text{on } \Omega. \tag{3.35}$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$*,* $n \geq 0$ *such that*

$$\begin{aligned}
-\alpha_n'' &= \sum_{i=0}^{1} \frac{f^{(i)}(t,\beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} + \frac{f^{(2)}(t,\alpha_{n-1})(\beta_n - \beta_{n-1})^2}{(2)!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} + \frac{g^{(2)}(t,\beta_{n-1})(\alpha_n - \alpha_{n-1})^2}{(2)!}, \\
&\qquad B\alpha_n(\mu) = b_\mu \quad \text{on } J; \\
-\beta_n'' &= \sum_{i=0}^{1} \frac{f^{(i)}(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} + \frac{f^{(2)}(t,\beta_{n-1})(\alpha_n - \alpha_{n-1})^2}{(2)!} \\
&\quad + \sum_{i=0}^{1} \frac{g^{(i)}(t,\beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} + \frac{g^{(2)}(t,\alpha_{n-1})(\beta_n - \beta_{n-1})^2}{(2)!}, \\
&\qquad B\beta_n(\mu) = b_\mu \quad \text{on } J,
\end{aligned} \tag{3.36}$$

which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order 3.

Remark 3.3. Similar results can be obtained for the other two coupled upper and lower solutions of (3.1) and the numerical applications of these results can be demonstrated.

4. Numerical results

Next we will provide an example which satisfies all the hypotheses of Theorem 3.1 which demonstrates the application of Theorem 3.1.

Example 4.1. Let us consider the following BVP:

$$\begin{aligned}
-u'' &= u^3 - 2u^4 - 0.1u + 0.4, \\
u(0) &= 0, \qquad u(1) = 1.
\end{aligned} \tag{4.1}$$

It is easy to check that $\alpha_0(t) \equiv 0$ and $\beta_0(t) \equiv 1$ are natural lower and upper solutions for (4.1), respectively. Let $H(t,u)$ denote the right-hand side of (4.1) and split it into nonincreasing and nondecreasing functions as $H(t,u) = f(t,u) + g(t,u)$ where

$$\begin{aligned}
f(t,u) &= u^3, \\
g(t,u) &= -2u^4 - 0.1u + 0.4.
\end{aligned} \tag{4.2}$$

Table 4.1. Table of three α,β-iterates of (4.1).

t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$
0.1	0.071613	0.105795	0.114155	0.115155	0.121882	0.211998
0.2	0.139816	0.207722	0.224408	0.226396	0.239650	0.372556
0.3	0.206071	0.305997	0.330820	0.333780	0.352693	0.497196
0.4	0.272568	0.401158	0.433435	0.437334	0.460118	0.596776
0.5	0.342305	0.494293	0.532364	0.537104	0.561279	0.679076
0.6	0.419460	0.587139	0.627930	0.633249	0.656144	0.749881
0.7	0.510237	0.681965	0.720845	0.726212	0.745445	0.813727
0.8	0.624444	0.781200	0.812382	0.816952	0.830723	0.874419
0.9	0.778369	0.886852	0.904514	0.907236	0.914404	0.935401

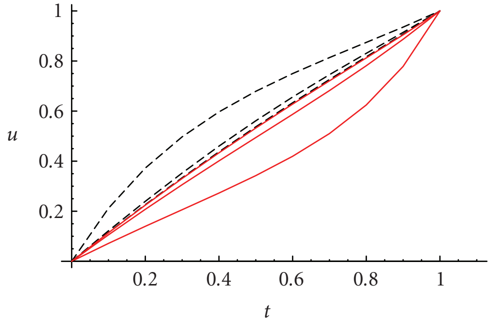

Figure 4.1

It is easy to show that

$$\begin{aligned} f_{uuu} &= 6 > 0, \\ g_{uuu} &= -48u \le 0 \end{aligned} \tag{4.3}$$

for $0 \le u \le 1$. Hence f is a 2-hyperconvex function and g is a 2-hyperconcave function. Now we need to check the following conditions in order to use Theorem 3.1:

$$\begin{aligned} f_u(t,u) &= 3u^2 \ge 0, \\ g_u(t,u) &= -8u^3 - 0.1 \le 0, \\ f_u(t,u) + g_u(t,u) &= 3u^2 - 8u^3 - 0.1 < 0, \end{aligned} \tag{4.4}$$

whenever $0 \le u \le 1$. Hence we can apply the iterates of Theorem 3.1. Using the nonlinear finite-difference methods for BVPs and Mathematica we can find the α,β-iterates as given in Table 4.1.

The α-iterates (with broken line) and the β-iterates (with unbroken line) can be seen on Figure 4.1.

Given the specific finite difference scheme, we can apply it to obtain lower and upper solutions. Then, we can make the difference between upper and lower solutions arbitrarily small. The obtained numerical solution however, will be close to the actual solution of the nonlinear problem (4.1) only within the truncation error of the finite difference scheme chosen.

Now we will provide a numerical example to show the usefulness of Theorem 3.2.

Example 4.2. Let us discuss the following second-order BVP:

$$-u'' = 3\cos u - 27e^{u/3} + 25.5, \qquad u(0.1) = 0.1, \qquad u(0.5) = 0.5. \tag{4.5}$$

Denote the right-hand side of (4.5) by $H(t,u)$. We can split the forcing function into two functions as $H(t,u) = f(t,u) + g(t,u)$ where

$$\begin{aligned} f(t,u) &= 3\cos u, \\ g(t,u) &= -27e^{u/3} + 25.5. \end{aligned} \tag{4.6}$$

If we choose $\alpha_0(t) \equiv 0.1$, $\beta_0(t) \equiv 0.5$, and $0.1 \leq t \leq 0.5$ we get

$$\begin{gathered} 0 \leq 3\cos 0.5 - 27e^{0.1/3} + 25.5 = 0.21758, \\ 0 \geq 3\cos 0.1 - 27e^{0.5/3} + 25.5 = -3.41172, \\ 0.1 \leq 0.1, \quad 0.5 \geq 0.5. \end{gathered} \tag{4.7}$$

Thus $\alpha_0(t) \equiv 0.1$ and $\beta_0(t) \equiv 0.5$ are coupled lower and upper solutions for (4.5) of the type defined in Theorem 3.2.

Next we can show that

$$\begin{aligned} f_{uuu} &= 3\sin u > 0, \\ g_{uuu} &= -e^{u/3} < 0 \end{aligned} \tag{4.8}$$

for $0.1 \leq u \leq 0.5$. Hence f is 2-hyperconvex function and g is 2-hyperconcave function. Now we need to check the following conditions in order to apply Theorem 3.2:

$$\begin{gathered} f_u(t,u) = -3\sin u < 0, \\ g_u(t,u) = -9e^{u/3} < 0, \\ f_u(t,u) - g_u(t,u) = e^u + u^2 > 0, \\ -3\sin 0.1 \leq -3\sin 0.5(0.5-0.1)^2 \leq 0, \end{gathered} \tag{4.9}$$

whenever $0.1 \leq u \leq 0.5$. Hence all the hypotheses of Theorem 3.2 are satisfied and we can apply the given iterates. Now using the nonlinear finite-difference methods for BVPs and Mathematica we can derive the α,β-iterates in Table 4.2.

The graph on Figure 4.2 shows α-iterates (with broken line) and the β-iterates (with unbroken line).

Table 4.2. Table of three α,β-iterates of (4.5).

t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$
0.10	0.100000	0.100000	0.100000	0.100000	0.100000	0.100000
0.15	0.141017	0.141551	0.141573	0.141646	0.142080	0.145352
0.20	0.182354	0.182909	0.182942	0.183077	0.183878	0.189605
0.25	0.225015	0.225334	0.225365	0.225541	0.226590	0.233909
0.30	0.270074	0.270116	0.270138	0.270329	0.271473	0.279436
0.35	0.318505	0.318582	0.318595	0.318773	0.319842	0.327418
0.40	0.372012	0.372103	0.372110	0.372247	0.373077	0.379177
0.45	0.432077	0.432091	0.432095	0.432169	0.432624	0.436164
0.50	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000

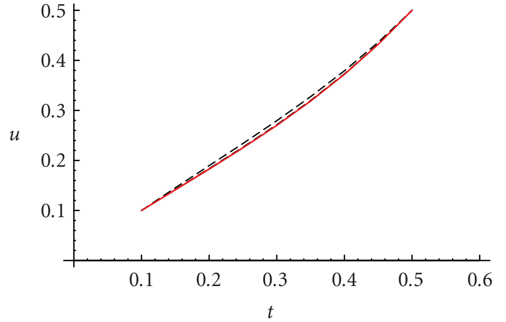

Figure 4.2

Remark 4.3. Note that the interval in the above example is due to the fact that f and g satisfies the hypothesis of Theorem 3.2 on the specific interval chosen.

5. Conclusion

We have used iterates of nonlinearity of order 2 when the forcing function is the sum of 2-hyperconvex and 2-hyperconcave. We develop two sequences depending on the type of the lower and upper solutions, which converge rapidly (order 3) to the unique solution of (3.1). We demonstrate the application of the results with numerical applications.

References

[1] R. E. Bellman, *Methods of Nonlinear Analysis. Vol. 1*, Mathematics in Science and Engineering, vol. 61-I, Academic Press, New York, 1970.

[2] R. E. Bellman and R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-Value Problems*, Modern Analytic and Computional Methods in Science and Mathematics, vol. 3, American Elsevier, New York, 1965.

[3] S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Mathematics in Science and Engineering, vol. 109, Academic Press, New York, 1974.

[4] A. Cabada and J. J. Nieto, *Rapid convergence of the iterative technique for first order initial value problems*, Applied Mathematics and Computation **87** (1997), no. 2-3, 217–226.

[5] ———, *Quasilinearization and rate of convergence for higher-order nonlinear periodic boundary-value problems*, Journal of Optimization Theory and Applications **108** (2001), no. 1, 97–107.

[6] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 181, Marcel Dekker, New York, 1994.

[7] G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics, vol. 27, Pitman, Massachusetts, 1985.

[8] V. Lakshmikantham and J. J. Nieto, *Generalized quasilinearization iterative method for initial value problems*, Nonlinear Studies **2** (1995), 1–9.

[9] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Mathematics and Its Applications, vol. 440, Kluwer Academic, Dordrecht, 1998.

[10] V. B. Mandelzweig, *Quasilinearization method and its verification on exactly solvable models in quantum mechanics*, Journal of Mathematical Physics **40** (1999), no. 12, 6266–6291.

[11] V. B. Mandelzweig and F. Tabakin, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Computer Physics Communications **141** (2001), no. 2, 268–281.

[12] T. Melton and A. S. Vatsala, *Generalized quasilinearization and higher order of convergence for first order initial value problems*, to appear in Dynamic Systems & Applications.

[13] R. N. Mohapatra, K. Vajravelu, and Y. Yin, *Extension of the method of quasilinearization and rapid convergence*, Journal of Optimization Theory and Applications **96** (1998), no. 3, 667–682.

[14] M. Sokol and A. S. Vatsala, *A unified exhaustive study of monotone iterative method for initial value problems*, Nonlinear Studies **8** (2001), no. 4, 429–438.

[15] I. H. West and A. S. Vatsala, *Generalized monotone iterative method for initial value problems*, Applied Mathematics Letters **17** (2004), no. 11, 1231–1237.

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