PARABOLIC INEQUALITIES WITH NONSTANDARD GROWTHS AND L^1 DATA

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We prove an existence result for solutions of nonlinear parabolic inequalities with L^1 data in Orlicz spaces.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Consider the following nonlinear parabolic problem:

$$\begin{gathered} \frac{\partial u}{\partial t} + A(u) = \chi \quad \text{in } Q, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined on $D(A) \subset W_0^{1,x}L_M(\Omega)$, with M is an N-function, and χ is a given data.

In the variational case (i.e., where $\chi \in W^{-1,x}E_{\overline{M}}(\Omega)$), it is well known that the solvability of (1.1) is done by Donaldson [2] and Robert [11] when the operator A is monotone, $t^2 \ll M(t)$, and $\overline{M}$ satisfies a Δ_2 condition, and by finally the recent work [3] for the general case.

In the L^1 case, an existence theorem is given in [4]. However, the techniques used in [4] do not allow us to adapt it for parabolic inequalities. It is our purpose in this paper to solve the obstacle problem associated to (1.1) in the case where $\chi \in L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$ and without assuming any growth restriction on M. The existence of solutions is proved via a sequence of penalized problems, with solutions u_n. A priori estimates of the truncation of u_n are obtained in some suitable Orlicz space. For the passage to the limit, the

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almost everywhere convergence of ∇u_n is proved via new techniques. As operators models, we can consider slow or fast growth:

$$A(u) = -\operatorname{div}\left((1+|u|)^2 \nabla u \frac{\log(1+|\nabla u|)}{|\nabla u|}\right),$$
$$A(u) = -\operatorname{div}(\nabla u \exp(|\nabla u|)). \tag{1.2}$$

For some classical and recent results in the setting of Orlicz spaces dealing with elliptic and parabolic equations, the reader can be referred to [8, 10, 12–14].

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, M is continous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \le t\}$ (see [1]).

The N-function is said to satisfy the Δ_2 condion if, for some $k > 0$,

$$M(2t) \le kM(t), \quad \forall t \ge 0, \tag{2.1}$$

when (2.1) holds only for $t \ge$ some $t_0 > 0$, then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon > 0$, $P(t)/Q(\epsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty}(Q^{-1}(t))/(P^{-1}(t)) = 0$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp., } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right). \tag{2.2}$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\} \tag{2.3}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M. \tag{2.4}$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \longrightarrow 0, \quad \forall |\alpha| \le 1. \tag{2.5}$$

This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [6, 7]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$, and set $Q = \Omega \times (0,T)$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivatives on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows:

$$\begin{aligned} W^{1,x}L_M(Q) &= \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q),\ \forall |\alpha| \le 1\}, \\ W^{1,x}E_M(Q) &= \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q),\ \forall |\alpha| \le 1\}. \end{aligned} \tag{2.6}$$

The latest space is a subset of the first one. They are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q}. \tag{2.7}$$

We can easily show that they form a complementary system when Ω satisfies the segment property.These spaces are considered as subspaces of the product spaces $\prod L_M(Q)$

which has $N+1$ copies. We will also consider the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$, then the function $t \to u(t) = u(\cdot,t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$, then $u(t)$ is $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore, the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0,T;W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^1L_M(\Omega)$. However, the scalar function $t \to \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \leq 1$.

2.6. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $D(Q)$. We can easily show as in [7] that when Ω has the segment property, then for all $u \in \overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})}$ there exist some $\lambda > 0$ and a sequence $(u_n) \subset D(Q)$ such that for all $|\alpha| \leq 1$, $\int_\Omega M((D_x^\alpha u_n - D_x^\alpha u)/\lambda)dx \to 0$ when $n \to \infty$. Consequently, $\overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})} = \overline{D(Q)}^{\sigma(\prod L_M, \prod L_{\overline{M}})}$, this space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \prod E_{\overline{M}}$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$, one has

$$\sum_{|\alpha| \leq 1} ||D_x^\alpha u||_{M,Q} \leq C \sum_{|\alpha| = 1} ||D_x^\alpha u||_{M,Q}, \tag{2.8}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system:

$$\left(\begin{array}{c|c} W_0^{1,x}L_M(Q) & F \\ \hline W_0^{1,x}E_M(Q) & F_0 \end{array} \right), \tag{2.9}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\prod L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-1,x}L_{\overline{M}}(Q) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \right\}. \tag{2.10}$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha| \leq 1} ||f_\alpha||_{\overline{M},Q}, \tag{2.11}$$

where the inf is taken on all possible decompositions $f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha$, $f_\alpha \in L_{\overline{M}}(Q)$. The space F_0 is then given by $F_0 = \{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

Defintion 2.1. We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D_x^\alpha u_n^\alpha + u_n^0, \qquad u = \sum_{|\alpha| \leq 1} D_x^\alpha u^\alpha + u^0 \tag{2.12}$$

with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence for all $|\alpha| \le 1$ and $u_n^0 \to u^0$ strongly in $L^1(Q)$.

We will give the following approximation theorem which plays a crucial role when proving the existence result of solutions for parabolic inequalities.

THEOREM 2.2. *Let $\phi \in W_0^{1,x}E_M(Q) \cap L^\infty(Q)$ and consider the convex set $\mathcal{K}_\phi = \{v \in W_0^{1,x}L_M(Q) : v \ge \phi$ a.e. in $Q\}$. Then for every $u \in \mathcal{K}_\phi \cap L^\infty(Q)$ such that $\partial u/\partial t \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$, there exists $v_j \in \mathcal{K}_\phi \cap D(\overline{Q})$ such that*

$$\begin{aligned} v_j &\longrightarrow u \quad \text{in } W^{1,x}L_M(Q),\\ \frac{\partial v_j}{\partial t} &\longrightarrow \frac{\partial u}{\partial t} \quad \text{in } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q) \end{aligned} \tag{2.13}$$

for the modular convergence.

Proof. It is easily adapted from that given in [4, Theorem 3] and the approximation techniques of [9]. □

Remark 2.3. The result is still true for $\phi \in W^{1,x}E_M(Q) \cap L^\infty(Q)$, when Ω is more regular (see [9]).

In order to deal with the time derivative, we introduce a time mollification of a function $v \in L_M(Q)$. Thus, we define, for all $\mu > 0$ and all $(x,t) \in Q$,

$$v_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{v}(x,s) \exp\big(\mu(s-t)\big)\, ds, \tag{2.14}$$

where $\tilde{v}(x,s) = v(x,s)\chi_{(0,T)}(s)$ is the zero extension of v. The following proposition is fundamental in the sequel.

PROPOSITION 2.4 [5]. *If $v \in L_M(Q)$, then v_μ is measurable in Q, $\partial v_\mu/\partial t = \mu(v - v_\mu)$ and*

$$\int_Q M(v_\mu)\, dx\, dt \le \int_Q M(v)\, dx\, dt. \tag{2.15}$$

Recall now the following compactness result which is proved in [5].

PROPOSITION 2.5. *Assume that $(u_n)_n$ is a bounded sequence in $W_0^1 L_M(Q)$ such that $\partial u_n/\partial t$ is bounded in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$, then u_n is relatively compact in $L^1(Q)$.*

3. The main result

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$, with the segment property. Let P and M be two N-functions such that $P \ll M$. Consider now the operator $A : D(A) \subset W_0^{1,x}L_M(Q) \to W^{-1}L_{\overline{M}}(Q)$ in divergence form $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $\zeta, \zeta' \in \mathbb{R}^N$,

($\zeta \neq \zeta'$) and all $s,t \in \mathbb{R}$:

$$|a(x,t,s,\zeta)| \leq h(x,t) + k_1\overline{P}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\zeta|),$$
$$(a(x,t,s,\zeta) - a(x,t,s,\zeta'))(\zeta - \zeta') > 0, \tag{3.1}$$
$$a(x,t,s,\zeta)\zeta \geq \alpha M(|\zeta|) - d(x,t),$$

with $d \in L^1(Q)$, $\alpha, k_1, k_2, k_3, k_4 > 0$, and $h \in E_{\overline{M}}(Q)$. Let

$$\psi \in W_0^1 E_M(\Omega) \cap L^\infty(\Omega). \tag{3.2}$$

Finally, consider

$$f \in L^1(Q). \tag{3.3}$$

We define for all $t \in \mathbb{R}$, $k \geq 0$, $T_k(t) = \max(-k, \min(k,t))$, and $S_k(t) = \int_0^t T_k(\eta)d\eta$.

We will prove the following existence theorem.

Theorem 3.1. *Let $u_0 \in L^1(\Omega)$ such that $u_0 \geq 0$. Assume that (3.1)–(3.3) hold true. Then there exists at least one solution $u \in C([0,T];L^1(\Omega))$ such that $u(x,0) = u_0$ a.e. and for all $\tau \in]0,T]$,*

$$u \geq \psi \quad \text{a.e. in } Q,$$
$$T_k(u) \in W_0^{1,x}L_M(Q),$$
$$\int_\Omega S_k(u(\tau) - v(\tau))dx + \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v)dx\,dt$$
$$\leq \int_{Q_\tau} f T_k(u-v)dx\,dt + \int_\Omega S_k(u_0 - v(x,0))dx,$$
$$\forall k > 0 \text{ and } \forall v \in \mathcal{K}_\psi \cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q), \tag{p_ψ}$$

where $Q_\tau = \Omega \times]0,\tau[$.

Remark 3.2. Since $\{v \in \mathcal{K}_\psi \cap L^\infty(Q) : \partial v/\partial t \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)\} \subset C([0,T], L^1(\Omega))$, (see [4]), the first and the latest terms of problem (p_ψ) are well defined.

Proof

Step 1. A priori estimates.

For the sake of simplicity, we assume that $d(x,t) = 0$.

Consider the approximate equations

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) - nT_n(u_n - \psi)^- = f_n,$$
$$u_n \in W_0^{1,x}L_M(Q), \quad u_n(x,0) = u_0^n, \tag{P_n}$$

where $f_n \to f$ strongly in $L^1(Q)$ and $u_0^n \to u_0$ strongly in $L^1(\Omega)$. Thanks to [3, Theorem 3.1], there exists at least one solution u_n of problem (P_n). By choosing $T_k(u_n - T_h(u_n)), h \geq \|\psi\|_\infty$ as test function in (P_n), we get

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle + \int_{h\leq|u_n|\leq h+k} a(u_n, \nabla u_n)\nabla u_n \, dx\, dt \\ - \int_Q nT_n(u_n-\psi)^- T_k(u_n - T_h(u_n))\, dx\, dt = \int_Q f_n T_k(u_n - T_h(u_n))\, dx\, dt. \tag{3.4}$$

On the one hand, we have

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle = \int_\Omega S_k^h(u_n(T))\, dx - \int_\Omega S_k^h(u_n^0)\, dx, \tag{3.5}$$

where $S_k^h(s) = \int_0^s T_k(q - T_h(q))\, dq$, and by using the fact that $\int_\Omega S_k^h(u_n(T))\, dx \geq 0$ and $|\int_\Omega S_k^h (u_n^0)| \leq k\|u_n^0\|_1$, we get

$$\alpha \int_{h\leq|u_n|\leq h+k} M(|\nabla u_n|)\, dx\, dt - \int_Q nT_n(u_n-\psi)^- T_k(u_n - T_h(u_n))\, dx\, dt \leq Ck, \quad \forall n \in \mathbb{N}, \tag{3.6}$$

so that

$$-\int_Q nT_n(u_n-\psi)^- \frac{T_k(u_n - T_h(u_n))}{k}\, dx\, dt \leq C. \tag{3.7}$$

Since $-nT_n(u_n-\psi)^- T_k(u_n - T_h(u_n)) \geq 0$, for every $h \geq \|\psi\|_\infty$, we deduce by Fatou's lemma as $k \to 0$ that

$$\int_Q nT_n(u_n-\psi)^- \leq C. \tag{3.8}$$

Using in (P_n) the test function $T_k(u_n)\chi(0,\tau)$, we get for every $\tau \in (0,T)$,

$$\int_\Omega S_k(u_n(\tau))\, dx + \int_{Q_\tau} a(x,t,T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\, dx\, dt \\ + \int_{Q_\tau} nT_n((u_n-\psi)^-) T_k(u_n)\, dx\, dt \leq Ck \tag{3.9}$$

which gives thanks to (3.8)

$$\int_\Omega S_k(u_n(\tau))\, dx + \int_{Q_\tau} a(x,t,T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\, dx\, dt \leq Ck, \tag{3.10}$$

$$\int_Q M(|\nabla T_k(u_n)|)\, dx\, dt \leq Ck. \tag{3.11}$$

On the other hand, by using [6, Lemma 5.7], there exist two positive constants μ_1 and μ_2 such that

$$\int_Q M\left(\frac{T_k(u_n)}{\mu_1}\right)dx\,dt \le \mu_2 \int_Q M(|\nabla T_k(u_n)|)\,dx\,dt \tag{3.12}$$

which implies, by using (3.11), that

$$\text{meas}\{|u_n| > k\} \le \frac{\mu_2 Ck}{M(k/\mu_1)} \tag{3.13}$$

so that

$$\lim_{k\to\infty} \text{meas}\{|u_n| > k\} = 0 \quad \text{uniformly with respect to } n. \tag{3.14}$$

Take now a nondecreasing function $\theta_k \in C^2(\mathbb{R})$ such that $\theta_k(s) = s$ for $|s| \le k/2$ and $\theta_k(s) = k\,\text{sign}(s)$ for $|s| > k$. By multiplying the approximate equation by $\theta_k'(u_n)$, we get

$$\begin{aligned}\frac{\partial\theta_k(u_n)}{\partial t} &- \text{div}\,(a(x,t,u_n,\nabla u_n)\theta'(u_n)) + a(x,t,u_n,\nabla u_n)\nabla u_n\theta''(u_n)\\ &- nT_n(u_n-\psi)^-\theta_k'(u_n) = f_n\theta_k'(u_n),\end{aligned} \tag{3.15}$$

which implies that $\partial\theta_k(u_n)/\partial t$ is bounded in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$. Since $\theta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, we have by Proposition 2.5 that $\theta_k(u_n)$ is relatively compact in $L^1(Q)$ and so that $u_n \to u$ a.e. in Q, and from (3.8) by using Fatou's lemma, we get $u \ge \psi$ a.e. in Q. Consequently,

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{weakly in } W_0^{1,x}L_M(Q) \tag{3.16}$$

for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.

Step 2. Almost everywhere convergence of the gradients.

Since $T_k(u) \in W_0^{1,x}L_M(Q)$, then there exists a sequence $(\alpha_j^k) \subset D(Q)$ such that $\alpha_j^k \to T_k(u)$ for the modular convergence in $W_0^{1,x}L_M(Q)$. In the sequel and throughout the paper, $\chi_{j,s}$ and χ_s will denote, respectively, the characteristic functions of the sets $Q^{j,s} = \{(x,t) \in \Omega : |\nabla T_k(\alpha_j^k)| \le s\}$ and $Q^s = \{(x,t) \in \Omega : |\nabla T_k(u)| \le s\}$. For the sake of simplicity, we will write only $\epsilon(n,j,\mu,s)$ to mean all quantities (possibly different) such that $\lim_{s\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,s) = 0$.

Taking now $T_\eta(u_n - T_k(\alpha_j^k)_\mu)$, $\eta > 0$ as test function in (P_n), we get

$$\begin{aligned}\left\langle \frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle &+ \int_Q a(x,u_n,\nabla u_n)\nabla T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\\ &- \int_Q nT_n\left((u_n-\psi)^-\right)T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)dx\,dt \le C\eta,\end{aligned} \tag{3.17}$$

and by using (3.8), we get

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle + \int_Q a(u_n,\nabla u_n)\nabla T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right) \le C\eta. \tag{3.18}$$

The first term of the left-hand side of the last equality reads as

$$\begin{aligned}\left\langle \frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle &= \left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle \\ &\quad + \left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle.\end{aligned} \tag{3.19}$$

The second term of the last equality can be written as

$$\begin{aligned}&\left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle \\ &\quad = \int_\Omega S_\eta\left(u_n(T) - T_k(\alpha_j^k)_\mu(T)\right)dx - \int_\Omega S_\eta(u_0^n)\,dx \geq -\eta\int_\Omega |u_0^n|\,dx \geq -\eta C,\end{aligned} \tag{3.20}$$

the third term can be written as

$$\left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right\rangle = \mu\int_Q \left(T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu\right)\left(T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right), \tag{3.21}$$

thus by letting $n, j \to \infty$ and since $\alpha_j^k \to T_k(u)$ a.e. in Q and by using Lebesgue theorem,

$$\begin{aligned}&\int_Q \left(T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu\right)\left(T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)\right)dx\,dt \\ &\quad = \int_Q (T_k(u) - T_k(u)_\mu)(T_\eta(u - T_k(u)_\mu))\,dx\,dt + \epsilon(n,j).\end{aligned} \tag{3.22}$$

Consequently,

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta\left(T_k\left(u_n - T_k(\alpha_j^k)_\mu\right)\right)\right\rangle \geq \epsilon(n,j) - \eta C. \tag{3.23}$$

On the other hand,

$$\begin{aligned}&\int_Q a(u_n, \nabla u_n)\nabla T_\eta\left(u_n - T_k(\alpha_j^k)_\mu\right)dx\,dt \\ &\quad = \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}\,dx\,dt \\ &\qquad + \int_{\{k < |u_n|\}\cap\{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n)\nabla u_n\,dx\,dt \\ &\qquad - \int_{\{k < |u_n|\}\cap\{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n)\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}}\,dx\,dt\end{aligned} \tag{3.24}$$

which implies, by using the fact that $\int_{\{k<|u_n|\}\cap\{|u_n-T_k(\alpha_j^k)_\mu|<\eta\}} a(u_n,\nabla u_n)\nabla u_n dx\,dt \geq 0$, that

$$\begin{aligned}\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} & a(u_n,\nabla u_n)\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx\,dt \\ & \leq C\eta + \int_{\{k<|u_n|\}\cap\{|u_n-T_k(\alpha_j^k)_\mu|<\eta\}} a(u_n,\nabla u_n)\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx\,dt.\end{aligned} \tag{3.25}$$

Since $a(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(Q))^N$, there exists some $h_{k+\eta}\in(L_{\overline{M}}(Q))^N$ such that

$$a(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \rightharpoonup h_{k+\eta} \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\Big(\prod L_{\overline{M}},\prod E_M\Big). \tag{3.26}$$

Consequently,

$$\begin{aligned}\int_{\{k<|u_n|\}\cap\{|u_n-T_k(\alpha_j^k)_\mu|<\eta\}} & a(u_n,\nabla u_n)\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx\,dt \\ & = \int_{\{k<|u|\}\cap\{|u-T_k(\alpha_j^k)_\mu|<\eta\}} h_{k+\eta}\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx\,dt + \epsilon(n),\end{aligned} \tag{3.27}$$

where we have used the fact that $\nabla T_k(\alpha_j^k)_\mu \chi_{\{k<|u_n|\}\cap\{|u_n-T_k(\alpha_j^k)_\mu|<\eta\}}$ tends strongly to $\nabla T_k(\alpha_j^k)_\mu \chi_{\{k<|u|\}\cap\{|u-T_k(\alpha_j^k)_\mu|<\eta\}}$ in $(E_M(Q))^N$. Letting $j\to\infty$, we obtain

$$\begin{aligned}\int_{\{k<|u_n|\}\cap\{|u_n-T_k(\alpha_j^k)_\mu|<\eta\}} & a(u_n,\nabla u_n)\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx\,dt \\ & = \int_{\{k<|u|\}\cap\{|u-T_k(u)_\mu|<\eta\}} h_{k+\eta}\nabla T_k(u)_\mu \chi_{\{|\nabla T_k(u)|>s\}} dx\,dt + \epsilon(n,j).\end{aligned} \tag{3.28}$$

Thanks to Proposition 2.4, one easily has

$$\begin{aligned}\int_{\{k<|u|\}\cap\{|u-T_k(u)_\mu|<\eta\}} & h_{k+\eta}\nabla T_k(u)_\mu \chi_{\{|\nabla T_k(u)|>s\}} dx\,dt \\ & = \int_{\{k<|u|\}\cap\{|u-T_k(u)|<\eta\}} h_{k+\eta}\nabla T_k(u) \chi_{\{|\nabla T_k(u)|>s\}} dx\,dt + \epsilon(\mu) = \epsilon(\mu,s).\end{aligned} \tag{3.29}$$

Hence

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx\,dt \leq C\eta + \epsilon(n,j,\mu,s). \tag{3.30}$$

On the other hand, note that

$$\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx\,dt \\
&\quad= \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s} dx\,dt \\
&\qquad+ \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\Big[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}\Big] dx\,dt.
\end{aligned} \tag{3.31}$$

The latest integral tends to 0 as n and j go to ∞. Indeed, we have that

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\Big[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}\Big] dx\,dt \tag{3.32}$$

tends to

$$\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k\Big[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}\Big] dx\,dt \tag{3.33}$$

as $n \to \infty$, since

$$a(T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\Big(\prod L_{\overline{M}}, \prod E_M\Big) \tag{3.34}$$

while $\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} \in (E_{\overline{M}}(Q))^N$. It is obvious that

$$\int_{\{T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k\Big[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}\Big] dx\,dt \tag{3.35}$$

goes to 0 as $j \to \infty$ by using Lebesgue theorem. We deduce then that

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s} dx\,dt \le C\eta + \epsilon(n,j,\mu,s). \tag{3.36}$$

Let now $0 < \delta < 1$. We have

$$\begin{aligned}
&\int_{Q^r} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]^\delta dx\,dt \\
&\quad\le C\text{meas}\Big\{\Big|T_k(u_n) - T_k(\alpha_j^k)_\mu\Big| > \eta\Big\}^\delta \\
&\qquad + C\Bigg[\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}\cap Q_r} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)] dx\,dt\Bigg]^\delta.
\end{aligned} \tag{3.37}$$

On the other hand, we have for every $s \geq r$, $r > 0$,

$$
\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\cap Q^r\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \\
&\quad \leq \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\,dt \\
&\quad \leq \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s}]\,dx\,dt \\
&\qquad + \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u_n))[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(u)\chi_s]\,dx\,dt \\
&\qquad + \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} [a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s}) - a(T_k(u_n),\nabla T_k(u)\chi_s)]\nabla T_k(u_n)\,dx\,dt \\
&\qquad - \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})\nabla T_k(\alpha_j^k)\chi_{j,s}\,dx\,dt \\
&\qquad + \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s\,dx\,dt \\
&\quad \leq I_1(n,j,\mu,s) + I_2(n,j,\mu,s) + I_3(n,j,\mu,s) + I_4(n,j,\mu,s) + I_5(n,j,\mu,s).
\end{aligned}
\tag{3.38}
$$

We will go to the limit as n, j, μ, and $s \to \infty$ in the last fifth integrals of the last side. Starting with I_1, we have

$$
\begin{aligned}
I_1(n,j,\mu,s) &\leq C\eta + \epsilon(n,j,\mu,s) \\
&\qquad - \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s}\,dx\,dt
\end{aligned}
\tag{3.39}
$$

since

$$
\begin{aligned}
&a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})\chi_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} \\
&\qquad \longrightarrow a(T_k(u),\nabla T_k(\alpha_j^k)\chi_{j,s})\chi_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} \quad \text{in } (E_{\overline{M}}(Q))^N,
\end{aligned}
\tag{3.40}
$$

while

$$
\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N. \tag{3.41}
$$

We deduce then that

$$\begin{aligned}&\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)\chi_{j,s}\,dx\,dt\\&\qquad=\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u),\nabla T_k(\alpha_j^k)\chi_{j,s})\nabla T_k(u)-\nabla T_k(\alpha_j^k)\chi_{j,s}\,dx\,dt+\epsilon(n)\end{aligned}\tag{3.42}$$

which gives by letting $j\to\infty$ and using the modular convergence of $\nabla T_k(\alpha_j^k)$, that

$$\begin{aligned}&\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} a(T_k(u),\nabla T_k(\alpha_j^k)\chi_{j,s})\nabla T_k(u)-\nabla T_k(\alpha_j^k)\chi_{j,s}\,dx\,dt\\&\qquad=\int_Q a(T_k(u),\nabla T_k(u)\chi_s)\nabla T_k(u)-\nabla T_k(u)\chi_s\,dx\,dt+\epsilon(j)=\epsilon(j).\end{aligned}\tag{3.43}$$

Finally,

$$I_1(n,j,\mu,s)\le C\eta+\epsilon(n,j,\mu,s)+\epsilon(n,j)=\epsilon(n,j,\mu,s,\eta).\tag{3.44}$$

For what concerns I_2, by letting $n\to\infty$, we have

$$I_2(n,j,\mu,s)=\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k[\nabla T_k(\alpha_j^k)\chi_{j,s}-\nabla T_k(u)\chi_s]\,dx\,dt+\epsilon(n)\tag{3.45}$$

since

$$a(T_k(u_n),\nabla T_k(u_n))\chi_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\rightharpoonup h_k\quad\text{weakly in }(L_{\overline{M}}(Q))^N\text{ for }\sigma\Big(\prod L_{\overline{M}},\prod E_{\overline{M}}\Big),\tag{3.46}$$

while

$$\chi_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}[\nabla T_k(\alpha_j^k)\chi_{j,s}-\nabla T_k(u)\chi_s]\longrightarrow\chi_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}}\nabla T_k(\alpha_j^k)\chi_{j,s}-\nabla T_k(u)\chi_s\tag{3.47}$$

strongly in $(E_M(Q))^N$. By letting now $j\to\infty$, and using Lebesgue theorem, we deduce then that

$$I_2(n,j,\mu,s)=\epsilon(n,j).\tag{3.48}$$

Similar tools as above give

$$\begin{aligned}I_3(n,j,\mu,s)&=\epsilon(n,j),\\I_4(n,j,\mu,s)&=\int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)+\epsilon(n,j,\mu,s),\\I_5(n,j,\mu,s)&=\int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)+\epsilon(n,j,\mu,s).\end{aligned}\tag{3.49}$$

Combining (3.37)–(3.48) and (3.49), we get

$$\begin{aligned} &\int_{Q_r} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx\,dt \\ &\qquad \leq C\text{meas}\left\{\left|T_k(u_n) - T_k(\alpha_j^k)_\mu\right| < \eta\right\}^\delta + C(\epsilon(n,j,s,\mu,\eta))^{1-\delta}, \end{aligned} \tag{3.50}$$

and by passing to the limit sup over n, j, μ, s, and, η

$$\lim_{n\to\infty}\int_{Q^r} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx\,dt = 0, \tag{3.51}$$

and thus there exists a subsequence also denoted by (u_n) such that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q^r, \tag{3.52}$$

and since r is arbitrary, we obtain

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q. \tag{3.53}$$

Step 3. Passage to the limit.

Let $\phi \in \mathcal{K}_\psi \cap D(\overline{Q})$. Choosing now $T_k(u_n - \phi)\chi_{(0,\tau)}$ as test function in (P_n), we get

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \phi)\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \phi)\,dx\,dt \\ &\qquad - \int_{Q_\tau} nT_n(u_n - \psi)^- T_k(u_n - \phi)\,dx\,dt = \int_{Q_\tau} f_n T_k(u_n - \phi)\,dx\,dt \end{aligned} \tag{3.54}$$

which gives, by $-\int_{Q_\tau} nT_n(u_n - \psi)^- T_k(u_n - \phi)\,dx\,dt \geq 0$,

$$\begin{aligned} &\int_\Omega S_k(u_n(\tau) - \phi(\tau))\,dx + \left\langle \frac{\partial \phi}{\partial t}, T_k(u_n - \phi)\right\rangle_{Q_\tau} \\ &\qquad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \phi)\,dx\,dt \\ &\qquad \leq \int_{Q_\tau} f_n T_k(u_n - \phi)\,dx\,dt + \int_\Omega S_k(u_n(0) - \phi(0))\,dx. \end{aligned} \tag{3.55}$$

We will show that

$$u_n \longrightarrow u \quad \text{in } C([0,T],L^1(\Omega)). \tag{3.56}$$

Since $T_k(u) \in \mathcal{K}_\psi$, for every $k \geq \|\psi\|_\infty$, there exists a sequence (w_j) in $D(\overline{Q}) \cap \mathcal{K}_\phi$ such that

$$w_j \longrightarrow T_k(u) \quad \text{in } W_0^{1,x}L_M(Q) \tag{3.57}$$

for the modular convergence. Choosing now $\Phi_{j,\mu}^{i,l} = T_l(w_j)_\mu + e^{-\mu t}T_l(\eta_i)$, with $\eta_i \geq 0$ converges to u_0 in $L^1(\Omega)$, as test function in (3.55),

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt \\ &\quad - \int_{Q_\tau} nT_n(u_n-\psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt = \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt. \end{aligned} \tag{3.58}$$

On the one hand, we have

$$\left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} = \mu\int_{Q_\tau}(T_l(w_j) - \Phi_{j,\mu}^{i,l})T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt \geq \epsilon(n,j,\mu,l); \tag{3.59}$$

on the other hand, by using the monotonicity of a and the fact that $-\int_{Q_\tau} nT_n(u_n - \psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt \geq 0$, we deduce that

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_n,\nabla \Phi_{j,\mu}^{i,l})\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt \\ &\quad \leq \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l})\,dx\,dt. \end{aligned} \tag{3.60}$$

Since, for every $\epsilon > 0$,

$$\begin{aligned} &\left|\chi_{Q_\tau} a(x,t,u_n,\nabla\Phi_{j,\mu}^{i,l})\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})\right| \\ &\quad \leq \epsilon\overline{M}(a(x,t,T_{k+\|l\|_\infty}(u_n),\nabla\Phi_{j,\mu}^{i,l})) + M\left(\frac{|\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})|}{\epsilon}\right), \end{aligned} \tag{3.61}$$

we have by using Vitali's theorem

$$\limsup_{l\to\infty}\limsup_{i\to\infty}\limsup_{\mu\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} \leq 0 \tag{3.62}$$

uniformly on τ. Therefore, by writing

$$\begin{aligned} \int_\Omega S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l})\,dx &= \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} - \left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l})\right\rangle_{Q_\tau} \\ &\quad + \int_\Omega S_k(u_0 - T_l(\eta_i))\,dx \end{aligned} \tag{3.63}$$

and using (3.55) and (3.59), we see that

$$\int_\Omega S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l})\,dx \leq \epsilon(n,j,\mu,i,l) \tag{3.64}$$

which implies, by writing

$$\int_\Omega S_k\left(\frac{u_n(\tau)-u_m(\tau)}{2}\right)dx \le \frac{1}{2}\left(\int_\Omega S_k(u_n(\tau)-\Phi_{j,\mu}^{i,l})\,dx + \int_\Omega S_k(u_m(\tau)-\Phi_{j,\mu}^{i,l})\,dx\right), \tag{3.65}$$

that

$$\int_\Omega S_k\left(\frac{u_n(\tau)-u_m(\tau)}{2}\right)dx \le \epsilon_1(n,m), \tag{3.66}$$

we deduce then that

$$\int_\Omega |u_n(\tau)-u_m(\tau)|\,dx \le \epsilon_2(n,m), \quad \text{not depending on } \tau, \tag{3.67}$$

and thus (u_n) is a Cauchy sequence in $C([0,T],L^1(\Omega))$, and since $u_n \to u$, a.e. in Q, we deduce that

$$u_n \longrightarrow u \quad \text{in } C([0,T],L^1(\Omega)). \tag{3.68}$$

Go back now to (3.48) and pass to the limit to obtain

$$\begin{aligned}\int_\Omega S_k(u(\tau)-\phi(\tau))\,dx &+ \left\langle \frac{\partial \phi}{\partial t}, T_k(u-\phi)\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\phi)\,dx\,dt \\ &\le \int_{Q_\tau} f\,T_k(u-\phi)\,dx\,dt + \int_\Omega S_k(u(0)-\phi(0))\,dx\end{aligned} \tag{3.69}$$

since for every $v \in \mathcal{K}_\psi \cap L^\infty(Q)$, there exists $v_j \in \mathcal{K}_\psi \cap D(\overline{Q})$ such that

$$\begin{aligned} &v_j \longrightarrow v \quad \text{for the modular convergence in } W_0^{1,x}L_M(Q), \\ &\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial v}{\partial t} \quad \text{for the modular in } W^{-1,x}L_{\overline{M}}(Q)+L^1(Q), \end{aligned} \tag{3.70}$$

we deduce then that

$$\begin{aligned}\int_\Omega S_k(u(\tau)-v(\tau))\,dx &+ \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\ &\le \int_{Q_\tau} f\,T_k(u-v)\,dx\,dt + \int_\Omega S_k(u(0)-v(0))\,dx\end{aligned} \tag{3.71}$$

which completes the proof. □

Remark 3.3. A similar result can be proved when dealing with the right-hand side in $L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$ or replacing the assumption (3.1) by the general one:

$$|a(x,t,s,\zeta)| \leq b(|s|)\left(h(x,t) + \overline{M}^{-1}M(k_4|\zeta|)\right), \tag{3.72}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function. Indeed, we consider the following approximate problems:

$$\begin{gathered} \frac{\partial u_n}{\partial t} - \operatorname{div}\left(a(x,t,T_n(u_n),\nabla u_n)\right) - nT_n(u_n - \psi)^- = f_n, \\ u_n \in W_0^{1,x}L_M(Q), \quad u_n(x,0) = u_0^n, \end{gathered} \tag{P_n}$$

and we conclude by adapting the same steps.

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References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York, 1975.

[2] T. Donaldson, *Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems*, Journal of Differential Equations **16** (1974), no. 2, 201–256.

[3] A. Elmahi and D. Meskine, *Parabolic equations in Orlicz spaces*, Journal of the London Mathematical Society. Second Series **72** (2005), no. 2, 410–428.

[4] ———, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces*, Portugaliae Mathematica. Nova Série **62** (2005), no. 2, 143–183.

[5] ———, *Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces*, Nonlinear Analysis. Theory, Methods & Applications **60** (2005), no. 1, 1–35.

[6] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Transactions of the American Mathematical Society **190** (1974), 163–205.

[7] ———, *Some approximation properties in Orlicz-Sobolev spaces*, Studia Mathematica **74** (1982), no. 1, 17–24.

[8] ———, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Nonlinear Functional Analysis and Its Applications, Part 1 (Berkeley, Calif, 1983), Proc. Sympos. Pure Math., vol. 45, American Mathematical Society, Rhode Island, 1986, pp. 455–462.

[9] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Analysis. Theory, Methods & Applications **11** (1987), no. 3, 379–392.

[10] V. K. Le and K. Schmitt, *Quasilinear elliptic equations and inequalities with rapidly growing coefficients*, Journal of the London Mathematical Society. Second Series **62** (2000), no. 3, 852–872.

[11] J. Robert, *Inéquations variationnelles paraboliques fortement non linéaires*, Journal de Mathématiques Pures et Appliquées. Neuvième Série **53** (1974), 299–320.

[12] M. Rudd, *Nonlinear constrained evolution in Banach spaces*, Ph.D. thesis, University of Utah, Utah, 2003.

[13] ———, *Weak and strong solvability of parabolic variational inequalities in Banach spaces*, Journal of Evolution Equations **4** (2004), no. 4, 497–517.
[14] M. Rudd and K. Schmitt, *Variational inequalities of elliptic and parabolic type*, Taiwanese Journal of Mathematics **6** (2002), no. 3, 287–322.

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