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*Research Article*

# A Boundary Harnack Principle for Infinity-Laplacian and Some Related Results

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We prove a boundary comparison principle for positive infinity-harmonic functions for smooth boundaries. As consequences, we obtain (a) a doubling property for certain positive infinity-harmonic functions in smooth bounded domains and the half-space, and (b) the optimality of blowup rates of Aronsson's examples of singular solutions in cones.



## 1. Introduction

In this work, one of our main efforts is to prove a boundary Harnack principle for positive infinity-harmonic functions on domains with smooth boundaries. This will generalize the result in [1] proven for flat boundaries. In this connection, also see [2–5]. This result will also be applied to study some special positive infinity-harmonic functions defined on such domains. One could refer to these as infinity-harmonic measures, however, being solutions to a nonlinear equation, these are not true measures. We derive some properties of these functions and among these would be the doubling property. A decay rate and a halving property for such functions on the half-space will also be presented. Another application will be to show optimality of Aronsson's singular examples in cones, thus generalizing the result in [6, 7].

We now introduce notations for describing our results. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with boundary $\partial\Omega$. We say $u$ is infinity-harmonic in $\Omega$ if $u$ solves in the sense of viscosity

$$\Delta_\infty u = \sum_{i,j=1}^{n} D_i u(x) D_j u(x) D_{ij} u(x) = 0, \quad x \in \Omega. \tag{1.1}$$

For more discussion, see [8, 1, 9]. For a motivation for these problems, see [8, 10]. For

$r > 0$ and $x \in \mathbb{R}^n$, $B_r(x)$ will be the open ball centered at $x$ and has radius $r$. Let $\widehat{A}$ denote the closure of the set $A$ and let $\chi_A$ denote its characteristic function. Define $\Omega_r(x) = \Omega \cap B_r(x)$, $P_r(x) = \partial\Omega \cap B_r(x)$. We will assume throughout this work that $\partial\Omega \in C^2$. More precisely, we first define for every $x \in \partial\Omega$ $R_x$ to be the radius of the largest interior ball tangential to $\Omega$ at $x$. We will assume that $R_y > 0$ for every $y \in \partial\Omega$ and $R_x \geq R_y/2 > 0$, $x \in P_{\delta_y}(y)$, for some $\delta_y > 0$. For every $x \in \partial\Omega$, set $\nu_x$ to be the inner unit normal at $x$ and $x_s = x + s\nu_x$, $s > 0$. We will now state Theorem 1.1 which is the result about boundary Harnack principle [2, 1, 3, 4].

Theorem 1.1 (Boundary Harnack Principle). *Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega$ satisfying the interior ball condition as stated above. Let $u$ and $v$ be infinity-harmonic in $\Omega$. Suppose that $y \in \partial\Omega$, $0 < 4\delta \leq \inf(\delta_y, R_y/2)$, and $u, v > 0$ in $\Omega_{\delta_y}(y)$. Suppose that $u$, and $v$ vanish continuously on $P_{\delta_y}(y)$, then there exist positive constants $C$, $C_1$, $C_2$ independent of $u$, $v$, and $\delta$, such that for every $z \in \Omega_\delta(y)$,*

(i) $u(z) \leq cu(y_\delta)$,

(ii) $c_1 u(y_\delta)/v(y_\delta) \leq u(z)/v(z) \leq c_2 u(y_\delta)/v(y_\delta)$.

Inequality (i) is often referred to as the Carleson inequality. A proof is provided in Section 2. At this time, we are unable to determine if Theorem 1.1 also holds when $\Omega$ has Lipschitz continuous boundary. We will apply Theorem 1.1 to prove (a) the doubling property of solutions of (1.2), and (b) the optimality of blowup rates of the Aronsson singular functions in cones [6]. Let $\Omega$ be a bounded domain. Fix $y \in \partial\Omega$; for every $r > 0$, define $Q_r(y) = \partial\Omega \setminus \widehat{P}_r(y)$. Consider the problem

$$\Delta_\infty u(x) = 0, \quad x \in \Omega, \quad u(x) = 1, \quad x \in P_r(y), \quad u(x) = 0, \quad x \in Q_r(y). \tag{1.2}$$

By a solution $u$ of (1.2), we mean that (i) $u$ is infinity-harmonic, in the viscosity sense, in $\Omega$, and (ii) $u$ assumes the values 1 and 0 continuously on $P_r(y)$ and $Q_r(y)$. More precisely, if $x \in P_r(y)$ and $z \to x$, $z \in \Omega$, then $u(z) \to 1$, and analogously for $Q_r(y)$. We show the existence of bounded solutions of (1.2) in Lemma 3.1. One could refer to $u$ as the nonlinear infinity-harmonic measure in $\Omega$ (although we have not shown uniqueness). Clearly, it is not a true measure. Our motivation for studying such quantities arises from the works [2–5]. In the context of boundary behavior, for instance the Fatou theorem, the works [4, 5] have studied such solutions for the linearized version of the $p$-Laplacian for finite $p$. We will show that requiring boundedness implies the maximum principle and comparison, see Lemma 3.1. Let $H = \{x \in \mathbb{R}^n : x_n > 0\}$ denote the half-space in $\mathbb{R}^n$. Set $e_n$ to be the unit vector along the positive $x_n$-axis. Set $T = \{x \in \mathbb{R}^n : x_n = 0\}$; for $y \in T$, define $P_r(y) = T \cap B_r(y)$, $Q_r(y) = T \setminus \widehat{P}_r(y)$, and $M_y^u(\rho) = \sup_{\partial B_\rho(y) \cap H} u$. Define a solution $u$ of

$$\Delta_\infty u(z) = 0, \quad z \in H, \qquad u|_{P_r(y)} = 1, \qquad u|_{Q_r(y)} = 0, \tag{1.3}$$

to be infinity-harmonic in $\Omega$, in the sense of viscosity, $0 \leq u \leq 1$, continuous up to $P_r(y)$ and $Q_r(y)$, and $\limsup_{\rho\to\infty} M_y^u(\rho) = 0$. We will address the existence and uniqueness of such solutions in Lemma 3.4. We now state a result about the doubling property of solutions of (1.2) and (1.3). For $r > 0$, set $o_{3r} = 3re_n$.

THEOREM 1.2. (a) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. For* $y \in \partial\Omega$, *assume that* $P_r(y)$ *lies on a connected component of* $\partial\Omega$. *Let* $u^r$ *be a bounded solution of (1.2) in* $\Omega$ *and let* $r$ *be small. Then there are positive constants* $c$, $C$ *independent of* $r$, *such that* $u^r(y_r) \geq c$ *and*

$$u^{2r}(z) \leq Cu^r(z), \quad z \in \Omega \setminus B_{3r}(y). \tag{1.4}$$

(b) *Let* $H$ *be the half-space in* $\mathbb{R}^n$. *Let* $u_o^r$ *be the unique infinity-harmonic measure in* $H$. *Then there exist universal constants* $C_1 > 0$ *and* $0 < C_2 < 1$ *such that*

$$u_o^{2r}(z) \leq C_1 u_o^r(z), \quad z \in H \setminus B_{3r}(o), \qquad u_o^r(o_s) \leq C_2 u_o^{2r}(o_s), \quad s \geq 3r. \tag{1.5}$$

Estimates in Theorem 1.2 are well known for linear equations [3] and also for the linearized version for the $p$-Laplacian [4, 5]. While we are able to prove the doubling property for any $C^2$ domain (see Lemma 3.3), it is unclear how a halving property (i.e., $f(t) \leq cf(2t)$, $f$ positive, increasing, and $c < 1$) may be proven if true. In particular, it would be interesting to know if this is true when $\Omega$ is the unit ball. We now introduce notations for Theorem 1.3. For $\alpha > 0$, let $C_\alpha$ stand for the interior of the half-infinite cone in $H$, with apex at $o$, the $x_n$-axis as the axis of symmetry, and aperture $2\alpha$. For $r > 0$, let $M^u(r) = \sup_{z \in \partial B_r(o) \cap C_\alpha} u(z)$. We extend the result in [7] to show optimality of the Aronsson singular examples [6].

THEOREM 1.3. *For* $\alpha > 0$, *let* $C_\alpha$ *be as described above. Let* $u$, $v$ *be positive infinity-harmonic functions in* $C_\alpha$. *Assume that* (i) *both* $u$ *and* $v$ *vanish continuously on* $\partial C_\alpha \setminus \{o\}$, (ii) $\sup_{r>0} M^u(r) = \infty$, $\sup_{r>0} M^v(r) = \infty$, *and* (iii) $\lim_{r\to\infty} M^u(r) = \lim_{r\to\infty} M^v(r) = 0$. *Then there exists a constant* $C$, *depending on* $\alpha$, $u$, *and* $v$ *such that*

$$\frac{1}{C} \leq \frac{u(z)}{v(z)} \leq C, \quad z \in C_\alpha. \tag{1.6}$$

*Moreover, for every* $m = 1, 2, 3, \ldots$, *if* $\alpha = \pi/2m$ *and* $\omega$ *is a direction in* $C_{\pi/2m}$, *then for an appropriate* $\hat{C} = \hat{C}(\omega)$,

$$\frac{1}{\hat{C}|z|^{m^2/(2m+1)}} \leq u(z) \leq \frac{\hat{C}}{|z|^{m^2/(2m+1)}}, \quad z \in C_{\pi/2m} \text{ with } z = |z|\omega. \tag{1.7}$$

The last conclusion in Theorem 1.3 will follow from the works [6, 7]. While Theorem 1.3 applies to special situations, the main purpose is to understand better the blowup rates of singular solutions, and in some situations decay rates.

We now state some well-known results that will be used in this work. Let $u > 0$ be infinity-harmonic in a domain $\Omega$, suppose that $a, b \in \Omega$ such that the segment $ab$ is at least $\eta > 0$ away from $\partial\Omega$, then the following Harnack inequality holds:

$$u(a)e^{|a-b|/\eta} \geq u(b). \tag{1.8}$$

Let $B_r(a) \subset \Omega$, if $\omega$ is a unit vector and $0 \le t \le s < r$, then

$$\frac{u(a+t\omega)}{r-t} \le \frac{u(a+s\omega)}{r-s}, \qquad u(a+s\omega)(r-s) \le u(a+t\omega)(r-t). \tag{1.9}$$

We will refer to (1.9) as the monotonicity property of $u$. For (1.8) and (1.9), see [8, 1, 11, 7, 12, 13]. Moreover, $u$ is locally Lipschitz ($C^1$ if $n = 2$ [14]) and satisfies the comparison principle [15].

Finally, we mention that it is unclear if a boundary Holder continuity of the quotient of two infinity-harmonic functions holds for smooth domains. Such a result for general Lipschitz domains would undoubtedly be quite useful. For $p$-harmonic functions (finite $p$), we direct the reader to the recent work by John Lewis and Kaj Nystrom "Boundary Behaviour for $p$ Harmonic Functions in Lipschitz and Starlike Lipschitz Ring Domains." We thank John Lewis for sending us this work.

## 2. Proof of Theorem 1.1

Our proof is an adaptation of the methods developed in [2, 1, 3]. Since $\Delta_\infty$ is translation and rotation invariant, we may assume that the origin $o \in \partial\Omega$. Set $\operatorname{osc}_A u = \sup_{z\in A} u(z) - \inf_{z\in A} u(z)$ to be the oscillation function of $u$ on the set $A$. Recall that $\Omega_r(y) = \Omega \cap B_r(y), y \in \partial\Omega$.

*Step 1* (oscillation estimate near the boundary). Let $u > 0$ be infinity-harmonic in $\Omega$ and vanishing on a neighborhood of $o$, in $\partial\Omega$. Let $M^u(r) = \sup_{z\in\Omega_r(o)} u(z)$. By the maximum principle, $M^u(r) > 0$ and $u(z) \le M^u(r)$, $z \in \Omega_r(o)$. For $0 < \alpha \le \beta$, consider the function $w(z) = M^u(\alpha) + [M^u(\beta) - M^u(\alpha)](|z| - \alpha)/(\beta - \alpha)$, $z \in \Omega_\beta \setminus \Omega_\alpha$. Clearly, $u \le w$ on $\partial(\Omega_\beta \setminus \Omega_\alpha)$. Thus $u \le w$ in $\Omega_\beta \setminus \Omega_\alpha$. Thus

$$M^u(\gamma) \le M^u(\alpha) + \left[M^u(\beta) - M^u(\alpha)\right]\frac{\gamma-\alpha}{\beta-\alpha}, \quad \alpha \le \gamma \le \beta. \tag{2.1}$$

This implies that $\operatorname{osc}_{\Omega_r(o)} u = M^u(r)$ is convex in $r$. Since $u(o) = 0$, it follows that $2\operatorname{osc}_{\Omega_r(o)} u \le \operatorname{osc}_{\Omega_{2r}(o)} u$.

*Step 2* (Carleson inequality). We now use the interior ball condition. Since $\partial\Omega \in C^2$, $R_x \ge R_o/2$, $x \in P_{4\delta}(o)$, with $4\delta < \inf(\delta_y, R_o/2)$. For every $x \in \partial\Omega$, let $\nu_x$ denote the unit inner normal at $x$, and set $x_t = x + t\nu_x$, $0 \le t \le R_x$. We will prove that $u(z) \le Cu(o_\delta)$, $z \in \Omega_\delta(o)$. We will adapt a device, based on the Harnack inequality, from [3]. For $z \in \Omega_\delta$, define $x_z \in \partial\Omega$ to be the point nearest to $z$. Also set $d(z) = |x_z - z|$. Then $z = x_z + d(z)\nu_{x_z} = (x_z)_{d(z)}$; set $z^s = x_z + 2^{s-1}d(z)\nu_{x_z}$, $s = 1, 2, 3, \ldots$. By the Harnack inequality (1.8), for $z \in \Omega_{3\delta}(o)$,

$$u(z) \le \begin{cases} Mu(z^2) : 0 < d(z) < \dfrac{3\delta}{2}, \\[2ex] Mu(o_\delta) : \delta < d(z) < 3\delta. \end{cases} \tag{2.2}$$

We take $M = e^8$. We now make an observation which will be used repeatedly in what follows. If $d(z) \geq \delta/2^s$, then

$$u(z) \leq Mu(z^2) \leq \cdots \leq M^s u(z^s) \leq M^{s+1}u(o_\delta). \tag{2.3}$$

Suppose now that there is a $\xi_0 \in \Omega_\delta(o)$ such that $u(\xi_0) \geq M^{l+2}u(o_\delta)$, where $l = l(\delta)$ is large and will be determined later. Using the aforementioned observation, we obtain

$$\operatorname{dist}(\xi_0, \partial\Omega) \leq \frac{\delta}{2^l}. \tag{2.4}$$

Let $p_0 \in \partial\Omega$ be the nearest point to $\xi_0$. Clearly, $\xi_0 \in \Omega_{2^{-l}\delta}(p_0) \subset \Omega_{2\delta}(o)$. Thus, $\operatorname{osc}_{\Omega_{\delta 2^{-l}}(p_0)} u \geq u(\xi_0)$; thus by Step 1, for $m = 1, 2, 3 \ldots$,

$$\operatorname{osc}_{\Omega_{\delta 2^{-l+m}}(p_0)} u \geq 2^m \operatorname{osc}_{\Omega_{\delta 2^{-l}}(p_0)} u \geq 2^m u(\xi_0), \tag{2.5}$$

where $2^m \geq M^3 = e^{24}$. Select $m = 60$; thus $\operatorname{osc}_{\Omega_{\delta 2^{-l+m}}(p_0)} u \geq 2^m u(\xi_0) \geq M^{l+5}u(o_\delta)$. Thus there is a $\xi_1 \in \Omega_{\delta 2^{-l+m}}(p_0)$ such that $u(\xi_1) \geq M^{l+5}u(o_\delta)$. Arguing as done in (2.4), we see $\operatorname{dist}(\xi_1, \partial\Omega) \leq \delta 2^{-l-3}$. Letting $p_1 \in \partial\Omega$ to be closest to $\xi_1$, we see that $p_1 \in \Omega_{2\delta}(o)$. Repeating our previous argument,

$$\operatorname{osc}_{\Omega_{\delta 2^{-l-3+m}}(p_1)} u \geq 2^m \operatorname{osc}_{\Omega_{\delta 2^{-l-3}}(p_1)} u \geq 2^m u(\xi_1) \geq M^{l+8}u(o_\delta). \tag{2.6}$$

Thus we may find a $\xi_2 \in \Omega_{\delta 2^{-l-3+m}}(p_1)$ such that $u(\xi_2) \geq M^{l+8}u(o_\delta)$, and $\operatorname{dist}(\xi_2, \partial\Omega) \leq \delta 2^{-l-6}$. Thus we obtain a sequence of points $\xi_k \in \Omega$ and $p_k \in \partial\Omega$, $k = 1, 2, 3 \ldots$, such that

$$u(\xi_k) \geq M^{l+2+3k}u(o_\delta), \quad \operatorname{dist}(\xi_k, \partial\Omega) \leq \delta 2^{-l-3k}, \quad \xi_k \in \Omega_{\delta 2^{-l-3(k-1)+m}}(p_{k-1}). \tag{2.7}$$

Note that

$$|\xi_k - o| \leq \sum_{i=1}^{k-1} |\xi_{i+1} - \xi_i| + |\xi_0 - o| \leq \delta\left(1 + 2\sum_{i=0}^{k-1} 2^{-l-3i+m}\right). \tag{2.8}$$

Choose $l \geq 70$, then $|\xi_k - o| \leq 2\delta$. Noting that $u$ vanishes continuously on $\partial\Omega$ and letting $k \to \infty$ in (2.7) result in a contradiction. Thus the Carleson inequality in Theorem 1.1 follows.

*Step 3* (bounds near the boundary). We first derive a lower bound in terms of the distance to the boundary. For every $z \in \Omega_\delta(o)$, let $x_z$ and $d(z)$ be as in Step 2. Note that $d(z) \leq |z - o| \leq \delta$. Thus $x_z \in \Omega_{2\delta}(o)$. Call $\zeta_z = x_z + \delta\nu_{x_z}$, observe that $\zeta_z \in \Omega_{3\delta}(o)$. By monotonicity (1.9) and the interior ball condition, we have

$$\frac{u(z)}{d(z)} \geq \frac{u(\zeta_z)}{\delta} \geq e^{-6}\frac{u(o_\delta)}{\delta}, \tag{2.9}$$

since $|\zeta_z - o_\delta| \leq |x_z + \delta\nu_{x_z} - \delta\nu_o| \leq 4\delta$.

Let $z \in \Omega_\delta(o)$. As noted previously, $x_z \in \Omega_{2\delta}(o)$ and $\Omega_\delta(x_z) \subset \Omega_{3\delta}(o)$. Note that $z \in \Omega_\delta(x_z)$. Set $\mu_z = \sup_{\Omega_\delta(x_z)} u$, then by comparison $u(\xi) \leq \mu_z|\xi - x_z|/\delta$, $\xi \in \Omega_\delta(x_z)$. Thus

$u(z) \le \mu_z d(z)/\delta$. By the Carleson inequality, $\mu_z \le Cu(\zeta_z)$. Note that $|x_\delta - o_\delta| = |x_z + \delta\nu_x - \delta\nu_o| \le 4\delta$. By the Harnack inequality, $u(\zeta_z) \le e^4 u(o_\delta)$. Thus there is universal $\widehat{C}$, such that

$$\frac{u(z)}{d(z)} \le \widehat{C}\frac{u(o_\delta)}{\delta}, \quad z \in \Omega_\delta(o). \tag{2.10}$$

If $u$, $v$ are two positive infinity-harmonic functions in $\Omega_{4\delta}(o)$, then by (2.9) and (2.10), there exist universal constants $C_1$ and $C_2$ such that

$$C_1\frac{u(o_\delta)}{v(o_\delta)} \le \frac{u(z)}{v(z)} \le C_2\frac{u(o_\delta)}{v(o_\delta)}, \quad z \in \Omega_\delta(o). \tag{2.11}$$

This proves Theorem 1.1.

*Remark 2.1.* We comment that the distance function $d(z) = \operatorname{dist}(z, \partial\Omega)$, $z \in \Omega$, is $C^2$ and infinity-harmonic near $\partial\Omega$. Also the oscillation estimate in Step 1 continues to hold for Lipschitz boundaries. One could show a Carleson inequality by following the ideas in [2].

## 3. Proof of Theorem 1.2

In this section, we will assume that $\Omega$ is a bounded $C^2$ domain. For $y \in \partial\Omega$ and $r > 0$, recall the definitions of $P_r(y)$ and $Q_r(y)$. Note that both $P_r(y)$ and $Q_r(y)$ are relatively open in $\partial\Omega$. Let $u$ be a solution of (1.2). As in Section 2, for $x \in \partial\Omega$, $\nu_x$ and $x_t = x + t\nu_x$, $t > 0$, are as defined in Section 2. We will assume that $\Omega$ is bounded but we can extend our arguments to the case of the half-space $H$. We will always take $u$ to be bounded in this section. This will imply the maximum principle. At this time, it is not clear whether unbounded solutions to (1.2) exist. Let $C_y$ be the connected component of $\partial\Omega$ that contains $y$. In Lemma 3.1, we assume that $B_r(y) \cap \partial\Omega = B_r(y) \cap C_y$.

Lemma 3.1. *Let $\Omega \in C^2$ be a bounded domain. Let $y \in \partial\Omega$ and $r > 0$. The following holds.*

(i) *There exists a solution $u$ of the problem in (1.2) such that $0 < u < 1$ in $\Omega$.*
(ii) *If $v$ is any bounded solution of (1.2), then $0 < v < 1$ in $\Omega$.*
(iii) *There are a maximal solution $u_y^r$ and a minimal solution $\hat{u}_y^r$, in $\Omega$ such that if $v$ is any bounded solution of (1.2), then $\hat{u}_y^r \le v \le u_y^r$.*
(iv) *If $t < r$, then $u_y^t \le \lim_{\rho\uparrow r} u_y^\rho = \hat{u}_y^r \le u_y^r = \lim_{\hat{r}\downarrow r} u_y^{\hat{r}}$.*

*Moreover, $u_y^r$ satisfies the following comparison principle: if $\omega$, $w \in C(\widehat{\Omega})$ are infinity-harmonic, and $\omega \le u_y^r \le w$ on $\partial\Omega$, then $\omega \le u_y^r \le w$ in $\Omega$.*

*Proof.* Fix $y \in \partial\Omega$ and $r > 0$. We have broken up our proof into five steps. We first start with the existence of bounded solutions.

*Step 1* (existence). We use the existence results proven in [8, 15], for Lipschitz boundary data. Let $\eta > 0$ be small. Set $I_r(y) = \partial B_r(y) \cap \partial\Omega$, and for $t > 0$, set $S_t = P_r(y) \cup (\bigcup_{x \in I_r(y)} B_t(x) \cap \partial\Omega)$. The set $S_t$ is obtained by appending a $t$-band to $P_r(y)$. For $l = 1, 2, 3, \ldots$, let $f_l$ be such that

(i) $f_l \in C(\partial\Omega)$,
(ii) $f_l(x) = 1$, $x \in P_r(y)$,

(iii) $f_l(x) = 0, x \in \partial\Omega \setminus S_{\eta/l}$,
(iv) $f_l(x) = (\eta/l) - \text{dist}(x, P_r(y))/(\eta/l), x \in S_{\eta/l}$.

Now let $u_l \in C(\hat{\Omega})$ be the unique viscosity solution of the problem

$$\Delta_\infty u_l(z) = 0, \quad z \in \Omega, \quad u_l|_{\partial\Omega} = f_l. \tag{3.1}$$

Clearly, $0 < u_l < 1$ in $\Omega$. Since $f_l \geq f_{l+1}$, by comparison, there is a function $u_\eta$ such that $u_l \downarrow u_\eta$ in $\Omega$. We first show that if $x \in P_r(y)$ and $z \to x \in P_r(y)$, $z \in \Omega$, then $u_\eta(z) \to 1$. Consider the set $\Omega_\delta(x)$, where $\delta = \inf_{\xi \in Q_r(y)} |x - \xi|/2$. For $z \in \Omega_\delta(x)$, set $w(z) = 1 - |z - x|/\delta$. By comparison, for every $l$, $w \leq u_l \leq 1$ in $\Omega_\delta(x)$. Thus $1 - |z - x|/\delta \leq u_\eta(z) \leq 1$, $z \in \Omega_\delta(x)$. We see that $\lim_{z \to x} u_\eta(z) = 1$. For $x \in Q_r(y)$ and $\delta = \inf_{\xi \in P_r(y)} |\xi - x|/2$, it is clear that for $l$ large, $u_l(z) \leq |z - x|/\delta$, $z \in B_\delta(x)$. Thus $u_\eta(z) \to 0$ as $z \to x$. Moreover, the limit function $u_\eta$ does not depend on the width $\eta$ of the appended band $S_\eta$. An argument based on comparison shows easily that for any $\eta_1$, $\eta_2 > 0$, $u_{\eta_1} = u_{\eta_2}$. Set $u = u_\eta$. Our next step is to show that $u$ is a viscosity solution in $\Omega$. We first show that $u$ is locally Lipschitz in $\Omega$. To see this, take $x_1 \in \Omega$ and $t > 0$ such that $B_{4t}(x_1) \subset \Omega$. Select $x_2 \in B_t(x_1)$; set $\mu_l = sup_{B_{4t}(x_1)} u_l$. Applying monontonicity (1.9) in $B_t(x_1)$, we have for every $l$, $(\mu_l - u_l(x_1))/t \leq (\mu_l - u_l(x_2))/(t - |x_1 - x_2|)$. Rearranging terms (see [1, Lemma 3.6], also see [12]), noting that $u_l(x_1)$, $u_l(x_2) \geq 0$ and $\mu_l \leq \mu_1 \leq 1$, we obtain $|u_l(x_2) - u_l(x_1)|/|x_1 - x_2| \leq 1/t$. Fixing $x_1$, $x_2$ and letting $l \to \infty$, we obtain that $u$ is locally Lipschitz. Fix $\xi \in \Omega$ and for $0 \leq t < \text{dist}(\xi, p\Omega)$, set $M_l(t) = \sup_{B_t(\xi)} u_l$, $m_l = \inf_{B_t(\xi)} u_l$, $M(t) = \sup_{B_t(\xi)} u$, and $m(t) = \inf_{B_t(\xi)} u$. Using that (i) $u_k \leq u_j \leq u_l$ when $l < j < k$, (ii) $M_l$ is convex and $m_l$ is concave in $t$, it follows that for $a < c < b$ and $z \in \partial B_c(\xi)$,

$$\frac{b-c}{b-a} m_k(a) + \frac{c-a}{b-a} m_k(b) \leq m_k(c) \leq u_j(z) \leq M_l(c) \leq \frac{b-c}{b-a} M_l(a) + \frac{c-a}{b-a} M_l(b). \tag{3.2}$$

Now in (3.2) first letting $k \to \infty$, replacing $u_j(z)$ by $u(z)$, and then letting $l \to \infty$, we obtain that $M(t)$ is convex and $m(t)$ is concave. This implies that $u$ is a viscosity solution [8]. Part (i) now follows. A proof also could be worked by showing cone comparison.

Throughout the rest of the proof, $u$ will stand for the solution constructed in Step 1.

*Step 2* (comparison). We now prove an easy comparison result for $u$. Let $f \in C(\partial\Omega)$ and let $u_f \in C(\hat{\Omega})$ be the unique infinity-harmonic function with boundary values $f$. Let $f \leq \chi_{P_r(y)}$. Using comparison, we see that for every $l$, $u_f \leq u_l$ in $\Omega$. Thus $u_f \leq u$ in $\Omega$. Now let $f \geq \chi_{P_r(y)}$, set $\varepsilon > 0$. Since $f \geq 1$ in $P_r(y)$, there exists a $\delta > 0$ such that $f + \varepsilon \geq 1$ in $B_\delta(x) \cap \partial\Omega$, for every $x \in \partial\Omega \cap \partial B_r(y)$. Take $l$ large so that $\eta/l \leq \delta/2$. By comparison, $u_l \leq u_f + \varepsilon$ in $\Omega$. Thus we have $u \leq u_f$ in $\Omega$.

*Step 3* (maximum principle). We now prove part (ii). Let $v$ be any bounded solution of (1.2). We will adapt an argument used in [11]. We observe that there is an $R_0 > 0$ such that for $x \in P_{2r}(y)$, $R_x \geq R_0$, and consequently, $\bigcup_{x \in P_{2r}(y)} B_{R_0/4}(x_{R_0/4}) \subset \Omega$. In what follows we take the quantities $\sigma$, $\eta < R_0/10$. We exploit the special geometry of $P_r(y)$ to achieve our proof.

Set $I_r(y) = \partial\Omega \cap \partial B_r(y)$; for every $x \in I_r(y)$ and $\sigma > 0$, define $m_x(\sigma) = \inf_{\partial B_\sigma(x) \cap \Omega} v$ and $M_x(\sigma) = \sup_{\partial B_\sigma(x) \cap \Omega} v$. Clearly, $m_x(\sigma) \leq 0$ and $M_x(\sigma) \geq 1$. We claim that $M_x$ is convex and $m_x$ is concave in $\sigma$. To see this, take $z \in \Omega$ with $0 < a \le |z-x| \le b$. Set $w(z) = m_x(a) + [m_x(b) - m_x(a)](|z-x| - a)/(b-a)$. Clearly, $w \le 0$. By comparison, $w \le v$ in $(B_b(x) \setminus B_a(x)) \cap \Omega$. Thus $m_x(\sigma)$ is concave in $\sigma$, and one can show analogously that $M_x(\sigma)$ is convex. Define $m^y(\sigma) = \inf_{x \in I_r(y)} m_x(\sigma)$ and $M^y(\sigma) = \sup_{x \in I_r(y)} M_x(\sigma)$, then for $\sigma > 0$,

$$\begin{aligned}
&\text{(i) } M^y(\sigma) \ge 1 \text{ is convex,} \qquad m^y(\sigma) \le 0 \text{ is concave in } \sigma,\\
&\text{(ii) } m^y(\sigma) \le v(z) \le M^y(\sigma), \quad z \in \Omega \setminus \cup_{x \in I_r(y)} B_\sigma(x),\\
&\text{(iii) } M^y(\sigma) \uparrow, \quad m^y(\sigma) \downarrow \quad \text{as } \sigma \downarrow 0.
\end{aligned} \tag{3.3}$$

Note that $v = 0$ or $1$ on $\partial\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$. Thus (3.3)(i) follows easily. Now using (3.3)(i) and comparison in the set $\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$ yields (3.3)(ii). Clearly, $M^y(\sigma)(m^y(\sigma))$ is the supremum (infimum) of $v$ on $\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$. The conclusion in (3.3)(iii) follows by observing that $\bigcup_{x \in I_r(y)} B_{\sigma_1}(x) \subset \bigcup_{x \in I_r(y)} B_{\sigma_2}(x)$, when $\sigma_1 > \sigma_2$. By (3.3), the quantities $M(0) = \lim_{\sigma \to 0} M^y(\sigma)$ and $m(0) = \lim_{\sigma \to 0} m^y(\sigma)$ exist. By our assumptions, $-\infty < m(0) \le v \le M(0) < \infty$. We show that $m(0) = 0$. Assume instead that $m(0) < 0$. Recall that $v$ is continuous up to $Q_r(y)$ and $P_r(y)$. For $x \in \partial\Omega$, let $\rho(x) = \operatorname{dist}(x, P_r(y))$ and $\widehat{\rho}(x) = \operatorname{dist}(x, Q_r(y))$. For $x \in Q_r(y)$, define $w_x(z) = m(0)|z-x|/\rho(x)$ in the set $\Omega_{\rho(x)}(x)$. By comparison $w_x \le v$ in $\Omega_{\rho(x)}(x)$, and $v \ge m(0)/2$, in $\Omega_{\rho(x)/2}(x)$. For $x \in P_r(y)$, define $\omega_x(z) = 1 + (m(0)-1)|z-x|/\widehat{\rho}(x)$ in $\Omega_{\widehat{\rho}(x)}(x)$. Then $v \ge \omega_x$ in $\Omega_{\widehat{\rho}(x)}(x)$ and $v \ge m(0)/2$, in $\Omega_{\widehat{\rho}(x)/2}(x)$. Let $\eta > 0$ be small. Set $A_\eta = \{x \in \partial\Omega : \rho(x) \ge \eta\}$ and $B_\eta = \{x \in P_r(y) : \widehat{\rho}(x) \ge \eta\}$. We now apply the above observations to obtain

$$v(z) \ge \begin{cases} \dfrac{m(0)}{2} : z \in \Omega_{\rho(x)/2}(x), & x \in A_\eta,\\[2ex] \dfrac{m(0)}{2} : z \in \Omega_{\widehat{\rho}(x)/2}(x), & x \in B_\eta. \end{cases} \tag{3.4}$$

Set $S = \bigcup_{\eta > 0} \bigcup_{x \in A_\eta} \Omega_{\rho(x)/2}(x)$ and $T = \bigcup_{\eta > 0} \bigcup_{x \in B_\eta} \Omega_{\widehat{\rho}(x)/2}(x)$, and call $G_y = \Omega \setminus (S \cup T)$. For $l = 1, 2, 3 \ldots$, let $z_l \in \Omega$ be such that $v(z_l) \le 7m(0)/8$ and $v(z_l) \to m(0)$, as $l \to \infty$. By (3.4), $z_l \in \Omega \setminus G_y$, and by the maximum principle, $\operatorname{dist}(z_l, I_r(y)) \to 0$.

In the discussion that follows, we will assume that $n > 2$. Recalling that $I_r(y) = \partial B_r(y) \cap \partial\Omega$, it follows that $I_r(y)$ is smooth. For every $l$, let $x_l \in I_r(y)$ be the closest point to $z_l$ and $d_l = |x_l - z_l|$. Note that the segment $x_l z_l$ is normal to $I_r(y)$. Since $x_l \in \partial B_r(y)$, $yx_l \perp \partial B_r(y)$, and so $yx_l \perp I_r(y)$. Let $T_l$ be the hyperplane tangential to $\partial\Omega$ at $x_l$, and let $\Pi_l$ be the 2-dimensional plane containing the segments $yx_l$ and $yz_l$. Thus $\Pi_l \perp I_r(y)$ at $x_l$ and $\nu_{x_l}$ lies in $\Pi_l$. Note that $\Pi_l \perp T_l$ and $I_r(y)$ is tangential to $T_l$ at $x_l$. Call $J_l = \partial\Omega \cap \Pi_l$, observe that the curve $J_l \perp I_r(y)$ at $x_l$. It is easy to see that if $x \in J_l$ is close to $x_l$, then (i) $\rho(x) = |x - x_l|$ if $x \in P_r(y)$, and (ii) $\widehat{\rho}(x) = |x - x_l|$ if $x \in Q_r(y)$. Now consider the set $C_l = \Pi_l \cap \partial B_{d_l}(x_l) \setminus G_y$. As noted above $z_l \in C_l$, moreover one can find $\alpha_l \in C_l$ such that $v(\alpha_l) = 3m(0)/4$. We will apply the Harnack inequality in $C_l$ to obtain a contradiction. In (3.4), take $\eta = d_l$ and we observe the following. Since $\partial\Omega \in C^2$ and $x_l$'s lie in a compact set, it follows that for $q \in C_l$, $\operatorname{dist}(q, \partial\Omega) \approx \operatorname{dist}(q, T_l) = O(d_l)$, as $d_l \to 0$. In other words, $\operatorname{dist}(q, \partial\Omega)$ has a lower bound of the order of $d_l$. We show this as follows. First note that since $\partial\Omega \in C^2$, it permits a local parametrization near $x_l$, where $x_n = \nu_{x_l}$, $x_n = 0$ is $T_l$, and $x_n = \phi(x_1, \ldots, x_{n-1})$ describes $\partial\Omega$. Clearly, $\operatorname{dist}(q, \partial\Omega) \le |q - x_l| = d_l$. We will show that (a) $\operatorname{dist}(q,\partial\Omega) \geq \operatorname{dist}(q,T_l) + O(d_l^2)$ and (b) $\operatorname{dist}(q,T_l) \approx O(d_l)$, uniformly in $l$. (a) Let (i) $q_{\partial\Omega}$ be the point on $\partial\Omega$ closest to $q$, (ii) let $q_{T_l}$ be the point on $T_l$ closest to $q$, (iii) $q_{\text{int}}$ the point of intersection of the line, containing the segment $qq_{\partial\Omega}$, and $T_l$, and (iv) let $q_{\partial\Omega}^{T_l}$ be the point on $T_L$ closest to $q_{\partial\Omega}$. Clearly, $|q-q_{T_l}| \leq |q-x_l| = d_l$ and $q_{\partial\Omega} \in B_{2d_l}(x_l)$. Since $\partial\Omega \in C^2$, $|q_{\partial\Omega} - q_{\partial\Omega}^{T_l}| = O(d_l^2)$. If $|q-q_{\partial\Omega}| \geq |q-q_{\text{int}}|$, then $|q-q_{\partial\Omega}| \geq |q-q_{\text{int}}| + \operatorname{dist}(q_{\partial\Omega},T_l) = |q-q_{\text{int}}| + O(d_l^2) \geq |q-q_{T_l}| + O(d_l^2)$. Let $|q-q_{\partial\Omega}| < |q-q_{\text{int}}|$. If $|q-q_{\partial\Omega}| \geq |q-q_{T_l}|$, then we are done. Otherwise, $|q-q_{\partial\Omega}| + |q_{\partial\Omega} - q_{\partial\Omega}^{T_l}| \geq |q-q_{\partial\Omega}^{T_l}| \geq |q-q_{T_l}|$. Thus $|q-q_{\partial\Omega}| \geq |q-q_{T_l}| + O(d_l^2)$.

(b) We now estimate $|q-q_{T_l}|$. Let $p_l = J_l \cap \partial B_{d_l}(x_l)$, then $|q-p_l| \geq d_l/2$. See the paragraph preceding proof of (a). Note that $\operatorname{dist}(p_l,T_l) = O(d_l^2)$, since $\partial\Omega \in C^2$. If $\langle p_l - x_l, \nu_{x_l}\rangle \geq 0$, then $\operatorname{dist}(q,T_l) \geq d_l/3$. If $\langle p_l - x_l, \nu_{x_l}\rangle < 0$, it again follows that $\operatorname{dist}(q,T_l) \geq d_l/3$.

We now apply the Harnack inequality, employing the above estimate for $(q,\partial\Omega)$, to see that for some $c > 0$ independent of $d_l$,

$$(v(z_l) - m(0)) \geq e^{-c}(v(\alpha_l) - m(0)) \geq \frac{e^{-c}|m(0)|}{4}. \tag{3.5}$$

Letting $l \to \infty$, we get $0 \geq |m(0)|/4$. Thus $m(0) = 0$. To show that $M(0) = 1$, we work with function $1-u$ and in place of $m(0)$, we take $1 - M(0)$. Arguing analogously, one may now show that $M(0) = 1$. When $n = 2$, $I_r(y)$ reduces to two points and one may again adapt the above argument to obtain part (ii).

*Step 4* (maximal solution $u_y^r$). Our next goal is to show that $u \geq v$, where $v$ is any bounded solution of (1.2). Recall that for $x \in \partial\Omega$, $\nu_x$ is the unit inner normal to $\partial\Omega$ at $x$ and $x_s = x + s\nu_x$. Since $\partial\Omega \in C^2$ and is bounded, there exists a $\delta > 0$ such that for every $x \in \partial\Omega$, $R_x \geq \delta$. Let $\varepsilon > 0$, small, with $\varepsilon \leq \min(1/10^4, \delta^2/10^4, r^2/10^4)$. For every $x \in \partial\Omega$, set $\Omega^\varepsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq \varepsilon\}$. Then $\partial\Omega^\varepsilon = \{x_\varepsilon : x \in \partial\Omega\}$. We will estimate $u$ and $v$ on $\partial\Omega^\varepsilon$. To this end, set $P_\varepsilon = \{x_\varepsilon : x \in P_r(y)\}$ and $Q_\varepsilon = \{x_\varepsilon : x \in Q_r(y)\}$. Note that

$$Q_\varepsilon = \partial\Omega^\varepsilon \setminus \widehat{P}_\varepsilon, \qquad \operatorname{dist}(\partial\Omega,\partial\Omega^\varepsilon) = \varepsilon, \qquad \Omega^\varepsilon \uparrow \Omega, \quad \text{as } \varepsilon \downarrow 0. \tag{3.6}$$

For $z \in \partial\Omega^\varepsilon$, let $z^\varepsilon$ be the nearest point on $\partial\Omega$. Clearly, $z = (z^\varepsilon)_\varepsilon$. If $z \in Q_\varepsilon$, then $u(z^\varepsilon) = 0$, and if $z \in P_\varepsilon$, then $u(z^\varepsilon) = 1$. Set

$$N_\varepsilon = \{z \in Q_\varepsilon : \operatorname{dist}(z^\varepsilon, P_r(y)) \geq \sqrt{\varepsilon}\}, \qquad O_\varepsilon = \{z \in P_\varepsilon : \operatorname{dist}(z^\varepsilon, Q_r(y)) \geq \sqrt{\varepsilon}\}. \tag{3.7}$$

For $v$, we use comparison as follows. For $x \in Q_r(y)$ with $\operatorname{dist}(x,P_r(y)) \geq \sqrt{\varepsilon}$,

$$v(\xi) \leq \frac{|\xi - x|}{\sqrt{\varepsilon}}, \quad \xi \in B_{\sqrt{\varepsilon}}(x) \cap \Omega, \text{ implying that } 0 < v(x_\varepsilon) \leq \sqrt{\varepsilon}. \tag{3.8}$$

Similarly, for $x \in P_r(y)$ with $\operatorname{dist}(x,Q_r(y)) \geq \sqrt{\varepsilon}$,

$$1 - v(\xi) \leq \frac{|\xi - x|}{\sqrt{\varepsilon}}, \quad \xi \in B_{\sqrt{\varepsilon}}(x) \cap \Omega, \text{ implying that } 1 - \sqrt{\varepsilon} \leq v(x_\varepsilon) \leq 1. \tag{3.9}$$

Thus from (3.8) and (3.9), we obtain that

$$0 < v \leq \sqrt{\varepsilon} \quad \text{on } N_\varepsilon, \qquad 1 - \sqrt{\varepsilon} \leq v < 1 \quad \text{on } O_\varepsilon. \tag{3.10}$$

Note that (3.10) is satisfied by any solution of (1.2), and in particular holds also for $u$. However, we will work with $u_l$ instead. Fix $\eta > 0$, and recall from Step 1 that for $l = 1, 2, 3, \ldots$,

$$f_l(x) = \frac{(\eta/l) - \operatorname{dist}(x, P_r(y))}{(\eta/l)}, \quad x \in S_{\eta/l}, \; f_l(x) = 0, \; x \in \partial\Omega \setminus S_{\eta/l}. \tag{3.11}$$

For ease of presentation, set $j = 4l/\eta$. We will work with $l$'s such that $j\sqrt{\varepsilon} < 1$. For $x \in \partial\Omega$ with $\operatorname{dist}(x, P_r(y)) \leq 3\sqrt{\varepsilon}$, we see that

$$u_l(x) = 1, \quad x \in P_r(y), \qquad u_l(x) \geq \frac{(\eta/l) - 3\sqrt{\varepsilon}}{(\eta/l)} \geq 1 - j\sqrt{\varepsilon}, \quad x \notin P_r(y). \tag{3.12}$$

We now use comparison in $B_{\sqrt{\varepsilon}}(x) \cap \Omega$, with $\operatorname{dist}(x, P_r(y)) \leq 2\sqrt{\varepsilon}$. Set $w_x(z) = j\sqrt{\varepsilon} + (1 - j\sqrt{\varepsilon})|z - x|/\sqrt{\varepsilon}$. Clearly, $w_x \geq 1 - u_l$ in $B_{\sqrt{\varepsilon}}(x) \cap \Omega$. Using (3.9) and noting that $u_l \geq u$, we have for $x \in \partial\Omega$,

(i) $u_l(x_\varepsilon) \geq 1 - \sqrt{\varepsilon}$, $x \in P_r(y)$, with $\operatorname{dist}(x, Q_r(y)) \geq \sqrt{\varepsilon}$,

(ii) $u_l(x_\varepsilon) \geq (1 - j\sqrt{\varepsilon})(1 - \sqrt{\varepsilon})$, with $\operatorname{dist}(x, P_r(y)) \leq 2\sqrt{\varepsilon}$.

Call $J_\varepsilon = \{x_\varepsilon : x \in \partial\Omega \text{ and } \operatorname{dist}(x, P_r(y)) \leq 2\sqrt{\varepsilon}\}$. From (3.7), $J_\varepsilon \supset O_\varepsilon$, $J_\varepsilon \cap N_\varepsilon \neq \varnothing$, and $u_l(x_\varepsilon) \geq (1 - j\sqrt{\varepsilon})(1 - \sqrt{\varepsilon})$, $x \in J_\varepsilon$. Using (3.8) and (3.9), we see that $u_l + 2j\sqrt{\varepsilon} \geq v$ on $\partial\Omega^\varepsilon$. By comparison, $u_l + 2j\sqrt{\varepsilon} \geq v$ in $\Omega^\varepsilon$. Letting $\varepsilon \to 0$, we obtain $u_l \geq v$ in $\Omega$. Now letting $l \to \infty$, we see that $u \geq v$ in $\Omega$.

From here on, we call $u_y^r = u$ and refer to it as the maximal solution of (1.2); clearly, $v \leq u_y^r$.

*Step 5* (minimal solution $\hat{u}_y^r$). It is clear from Step 1 that for $r_1 < r_2$, $u_y^{r_1} \leq u_y^{r_2}$ (working with the corresponding $u_l$'s). Note that $u_y^r$ is locally Lipschitz but uniformly so in $r$. Set $\hat{u}_y^r = \sup_{t<r} u_y^t$. Using Step 1, $\hat{u}_y^r$ is a solution of (1.2) and $\hat{u}_y^r \leq u_y^r$. The comparison principle in Step 2 also holds. We now show that $\hat{u}_y^r \leq v$, where $v$ is any solution of (1.2). We do this by showing that $u_y^t \leq v$ whenever $t < r$. Fix $t < r$; we proceed as in Step 4. Let $\delta > 0$ be as in Step 4. Let $d > 0$, small, such that $0 < d \leq \min(\delta^2/10^4, r^2/10^4, (r-t)^2/100)$; set $\Omega^d = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq d\}$. As in Step 4, define $P_{d,s} = \{x_d : x \in P_s(y)\}$, where $s$ is either $r$ or $t$. Now define $Q_{d,s}$ analogously. Then (3.6) holds. For $z \in \partial\Omega^d$, recall that $z^d \in \partial\Omega$ is such that $|z - z^d| = d$. Now for each $s = r, t$, define the sets $N_{d,s}$ and $O_{d,s}$, both subsets of $\Omega^d$, along the lines of (3.7). Using (3.8), (3.9), and (3.10), we obtain

(i) $0 < u_y^t(\xi) \leq \sqrt{d}$, $\xi \in N_{d,t}$,

(ii) $1 - \sqrt{d} \leq u_y^t(\xi) \leq 1$, $\xi \in O_{d,t}$,

(iii) $0 < v(\xi) \leq \sqrt{d}$, $\xi \in N_{d,r}$,

(iv) $1 - \sqrt{d} \leq v(x_d) \leq 1$, $x \in O_{d,r}$.

Clearly, $O_{d,r} \supset O_{d,t}$, $N_{d,t} \supset N_{d,r}$, and for small $d$, $N_{d,t} \cap O_{d,r} \neq \varnothing$. Thus $v + 2\sqrt{d} \geq u_y^t$ on $\partial\Omega^d$. Using comparison in $\Omega^d$ and taking $d \to 0$, we obtain $v \geq u_y^t$. The claim now follows. Call $\hat{u}_y^r$ the minimal solution. By Step 4 and arguments presented here, we have the first part of part (iii), namely,

$$u_y^t \le \hat{u}_y^r \le v \le u_y^r, \quad \text{whenever } t < r. \tag{3.13}$$

Since $u_y^{\tilde{r}} \ge u_y^r$, $\tilde{r} > r$. Setting $\tilde{u} = \lim_{\tilde{r}\downarrow r} u_y^{\tilde{r}} \ge u_y^r$, imitating Step 1, one can show that $\tilde{u}$ solves (1.2). Thus by Step 4, $\lim_{\tilde{r}\downarrow r} u_y^{\tilde{r}} = u_y^r$. Part (iv) is also proven. □

*Remark 3.2.* If one could prove that $\lim_{t\uparrow r} u_y^t = u_y^r$, then uniqueness would follow, however, equality here would be a stronger result. A proof of this is shown for the half-space in Lemma 3.4.

Let $v_y^r > 0$ satisfy the following:

$$\Delta_\infty v_y^r(z) = 0, \quad z \in \Omega, \qquad v_y^r|_{Q_r(y)} = 0, \qquad v \in C(\hat{\Omega} \setminus \Omega_r(y)). \tag{3.14}$$

LEMMA 3.3. *For $y \in \partial\Omega$, $r > 0$, let $u^r$ be a bounded solution of (1.2) and let $v_y^r$ be as in (3.14). Assume that r is small. Then $u^r(y_r) \ge c$, for some universal constant $c > 0$. Moreover, there exist universal constants $C > 0$ and $\overline{C} > 0$ such that*

(i) $u^{2r}(z) \le Cu^r(z)$, $z \in \Omega \setminus \Omega_{3r}(y)$,

(ii) $u^r(z)/\overline{C} \le v_y^r(z)/v_y^r(y_r) \le \overline{C}u^r(z)$, $z \in \Omega \setminus \Omega_{2r}(y)$.

*Proof.* By the maximum principle, $0 < u^r < 1$ in $\Omega$. Let $w(z) = (r - |z-y|)/r$, $z \in \Omega_r(y)$. Then $u^r \ge w$ on $\partial B_r(y) \cap \Omega$, and $u^r \ge w$ on $P_r(y)$. By comparison, $u^r \ge w$ in $\Omega_r(y)$. Thus $u^r(y_{r/2}) \ge w(y_{r/2}) = 1/2$. We may now use the Harnack inequality to conclude that

$$u^r(y_r) \ge e^{-2}u^r(y_{r/2}) \ge \frac{1}{2e^2}. \tag{3.15}$$

We now prove part (i), the "doubling" property of $u^r$ in $\Omega \setminus \Omega_{3r}(y)$. We will use the boundary Harnack inequality and comparison. Note that $u^{2r} = u^r = 0$, $x \in \partial\Omega \setminus P_{5r/2}(y)$. We consider $\Omega_{r/4}(x)$, $x \in \partial B_{5r/2}(y) \cap \partial\Omega$. By Theorem 1.1(ii),

$$\frac{u^r(z)}{u^{2r}(z)} \ge C_1 \frac{u^r(x_{r/4})}{u^{2r}(x_{r/4})}, \quad z \in \partial B_{5r/2}(y) \cap \Omega_{r/4}(x). \tag{3.16}$$

We now use the Harnack inequality and (3.15) to conclude that there are universal constants $C_2$, $C_3$, and $C_4$ such that

$$\frac{u^r(z)}{u^{2r}(z)} \ge C_2 \frac{u^r(y_{5r/2})}{u^{2r}(y_{5r/2})} \ge C_3 \frac{u^r(y_r)}{u^{2r}(y_r)} \ge C_4, \quad z \in \partial B_{5r/2}(y) \cap \Omega. \tag{3.17}$$

We now use comparison in $\Omega \setminus \Omega_{5r/2}(y)$ to conclude part (i). We now prove part (ii). For every $x \in \partial\Omega \cap \partial B_{2r}(y)$, we have by Theorem 1.1(ii) that

$$C_1 \frac{u^r(x_{r/2})}{v_y^r(x_{r/2})} \le \frac{u^r(z)}{v_y^r(z)} \le C_2 \frac{u^r(x_{r/2})}{v_y^r(x_{r/2})}, \quad z \in \partial B_{2r}(y) \cap \Omega_{r/2}(x). \tag{3.18}$$

As done before, we may use the Harnack inequality to conclude that

$$C_3 \frac{u^r(y_r)}{v_y^r(y_r)} \le \frac{u^r(z)}{v_y^r(z)} \le C_4 \frac{u^r(y_r)}{v^r(y_r)}, \quad z \in \partial B_{2r}(y) \cap \Omega. \tag{3.19}$$

Thus using (3.15), we obtain

$$\frac{v_y^r(y_r) u^r(z)}{C} \le v_y^r(z) \le C v_y^r(y_r) u^r(z), \quad z \in \partial B_{2r}(y) \cap \Omega. \tag{3.20}$$

The claim follows by comparison. □

We now look at the case of the half-space $H = \{x \in \mathbb{R}^n : x_n > 0\}$. Set $T = \{x \in \mathbb{R}^n : x_n = 0\}$; for $y \in T$, define $P_r(y) = H \cap B_r(y)$, $Q_r(y) = T \setminus \widehat{P}_r(y)$, and $M_y^u(\rho) = \sup_{\partial B_\rho(y) \cap H} u$. Define a solution $u$ of

$$\Delta_\infty u(z) = 0, \quad z \in H, \ u|_{P_r(y)} = 1, \ u|_{Q_r(y)} = 0, \tag{3.21}$$

to satisfy the equation in the sense of viscosity, $0 \le u \le 1$, $u$ is continuous up to $P_r(y)$ and $Q_r(y)$, and $\limsup_{\rho \to \infty} M_y^u(\rho) = 0$. Set $L_y = \{y + se_n : s \in \mathbb{R}\}$. In the proof of Lemma 3.4, we make use of an example of a positive singular infinity-harmonic function in the half-space [6, 11]. We utilize the definition in Step 2 of Theorem 1.3 as appears in Section 4. For Lemma 3.4, we define $\phi(x) = f(\theta)/|x|^{1/3}$, where $\theta$ is the conical angle at $y$ and $f(\theta)$ is the function $f_m(\theta)$ when $m = 1$. Then $\phi(x)$ blows up at $y$, vanishes elsewhere on $T$, and decays to zero at inifinity. In what follows, we make frequent use of the results in [7].

Lemma 3.4 (Half-Space). *Let $H = \{x \in \mathbb{R}^n : x_n > 0\}$. Then there exists a unique solution $u_y^r$ of the problem in (3.21) such that $0 < u_y^r < 1$. Moreover, if $\sigma^2 = \sum_{i=1}^{n-1}(z_i - y_i)^2$, then $u_y^r(z)$ is symmetric about the line $L_y$, and $u_y^r(z) = u_y^r(\sigma, z_n)$ is decreasing in $\sigma$.*

*Proof.* Let $0 < r < \rho$ and set $H_\rho = H \cap B_\rho(y)$. For $l = 1, 2, 3, \ldots$, and $\eta > 0$, define $u_l^{r,\rho}$ to be the unique solution of the problem

$$\Delta_\infty u_l^{r,\rho}(z) = 0, \quad z \in H \cap B_\rho(y), \qquad u_l^{\rho,r}|_T = f_l, \tag{3.22}$$

where $f_l$ is the function as defined in Step 1 of Lemma 3.1. Clearly, $0 < u_l^{r,\rho} < 1$ and by comparison, $u_l^{r,\rho} \uparrow u_l^r$ as $\rho \uparrow \infty$. Arguing as in Step 1 of Lemma 3.1, we may show that $u_l^r$ is infinity-harmonic in $H$ and $u_l^r|_T = f_l$. Set $M_l(s) = \sup_{\partial B_s(y) \cap H} u_l^r$; we now show that $M_l(s) \to 0$ as $s \to \infty$. The following are true: (i) $0 < M_l(s) \le 1$, $s > 0$, (ii) $M_l(s) = 1$, $0 < s \le r$, and (iii) $M_l(s) < 1$ and is convex in $s$, whenever $s > r + (\eta/l)$. Thus $M_l(s)$ is nonincreasing. Let $\phi > 0$ be the Aronsson singular solution on $H$ as described above. Adapting the argument in Step 2 of Theorem 1.2 (this follows below), one may show that for some $C > 0$ depending only on $\phi(y_{2r})$, $u_l^{r,\rho}(z) \le C\phi(z)$, $z \in \partial B_{2r}(y) \cap H$, and $0 = u_l^{r,\rho} < \phi$ on $\partial B_\rho(y) \cap H$. Thus $u_l^{r,\rho}(z) \le C\phi(z)$, $z \in H_\rho \setminus H_{2r}$. Clearly, $u_l^r \le C\phi$, in $H$, and $M_l(s) \to 0$ as $s \to \infty$. Hence $u_l^r$ solves (3.21) with modified boundary data $f_l$. Since $u_l^r$ decreases in $l$, it follows that $u_l^r \to u_y^r$, where now $u_y^r$ solves (3.21). Let $v$ be any solution of (3.21). We show that $u_y^r \ge v$ in $H$. Let $\varepsilon > 0$ be small. We adapt Step 4 in Lemma 3.1 and use the comparison lemma [7, Lemma 2.2]. We work with $u_l^r$ and estimate both $u_l^r$ and $v$ on the set $\{x \in H : x_n = \varepsilon\}$. Let $r_0 > 100\sqrt{\varepsilon}$ be such that $0 \le \sup(M_y^v(\rho), M_l^r) \le \varepsilon$, $\rho \ge r_0$. We work in $S_\varepsilon^{r_0} = \{x \in H : x_n \ge \varepsilon, \text{ and } |x| \le r_0\}$. By letting $r_0 \to \infty$ and, as done in Step 4 of Lemma 3.1, then letting $\varepsilon \to 0$, one can show that $u_l^r \ge v$ in $H$ for any $l$. Thus $u_y^r \ge v$. Also we may show by adapting Step 5 that $u_y^t \le v \le u_y^r$, $t < r$. Call $u_y^r$ the maximal solution. Note that $u_y^r$ is symmetric about $L_y$, since by reflection, $u_l^{r,\rho}$ is symmetric about $L_y$. Writing $u_l^{r,\rho}(z) = u_l^{r,\rho}(\sigma, z_n)$ and using reflection and comparison (see [7, Lemma 2.6]), we see that $u_l^{r,\rho}(\sigma, z_n)$ decreases in $\sigma$. Thus the same holds for $u_y^r$.

We now scale as follows. For $\theta > 0$, set $z^\theta = y + \theta(z - y)$, $z \in H$, then $\mu^\theta(z) = u_y^r(z^\theta)$ solves (3.21) with $P_r(y)$ replaced by $P_{r/\theta}(y)$. Clearly, by the maximality of $u_y^{r/\theta}$,

$$\mu^\theta(z) \le u_y^{r/\theta}(z), \quad \text{implying that } u_y^r(z^\theta) \le u_y^{r/\theta}(z),\ z \in H. \tag{3.23}$$

Using $\theta > 1$ and that $u_y^r$ is maximal, we see that if $v$ is any solution of (3.21), then (3.23) implies that $u_y^r(z^\theta) \le u_y^{r/\theta}(z) \le v(z) \le u_y^r(z)$, $z \in H$. Letting $\theta \uparrow 1$ and using continuity, we obtain uniqueness of $u_y^r$. □

*Proof of Theorem 1.2.* Set $T_s = \{x_n = s\}$, $s > 0$ and $M(s) = \sup_{T_s} u$. Let $u^r = u_o^r$ solve (3.21). We will assume that $\lim_{\rho\to\infty} \sup_{\partial B_\rho(o)\cap H} u^r = 0$. Our proof will use and adapt results from [7].

*Step 1.* By Lemma 3.4, $u^r$ is unique. To show the doubling property of $u^r$, we use Lemma 3.3(ii), the comparison result in [7, Lemma 2.2], and (3.15), that is, $u^r(o_r) \ge c > 0$. We now focus on the halving property. By Lemma 3.4, $u^r$ is unique and $u^{2r}(x) = u^r(x/2)$, $x \in H$. Thus our goal is to show that

$$u^r(o_{4r}) \le \alpha u^{2r}(o_{4r}) = \alpha u^r(o_{2r}), \tag{3.24}$$

for some universal $0 < \alpha < 1$. We now make an observation. By Lemma 3.4, for $t > 1$, $u^r(x) \le u^{rt}(x) = u^r(x/t)$. If $v$ is a unit vector with $\langle v, e_n\rangle \ge 0$, then $u^r(sv)$, $s > 0$, is a decreasing function of $s$. In particular, writing a point on the $x_n$-axis as $(0, x_n)$, $u^r(0, x_n)$ decreases in $x_n$. By Lemma 3.4, $u(0,s) = M(s)$, $s > 0$, and $M(s)$ is decreasing. To see that $M(s)$ is convex in $s$, for $0 < s < t$, consider the set $H_{s,t} = \{x \in H : s < x_n < t\}$. The function $w(x) = M(s) + [M(t) - M(s)][x_n - s]/(t - s)$ is infinity-harmonic in $H_{s,t}$, and by comparison, $u^r \le w$ in $H_{s,t}$. The claim follows. Since $u^r(o_{2r}) = M(2r)$, $u(o_{4r}) = M(4r)$, and $\lim_{s\to\infty} M(s) = 0$, by convexity it is clear that $u(o_{4r})/u(o_{2r}) = \alpha < 1$. Our goal is to show that $\alpha$ is independent of $r$.

*Step 2* (decay estimate). We show that $u^r(x)$ decays like $|x|^{-1/3}$. We use the work [7]. Let $v(x) = f(\theta)|x|^{-1/3}$, where $\theta = \theta(x) = \cos^{-1} x_n/|x|$, be the Aronsson example of a singular solution in the half-space $H$ (see Section 4). Consider the set $A_t = H \cap \partial B_t(o)$, $t > 0$. Employing Theorem 1.1(ii), the Harnack inequality, and following the proof of Theorem 1.1 in [7], we see that there are universal constants $C_1$, $C_2$ such that

$$C_1 \frac{u^r(x)}{v(x)} \le \frac{u^r(o_t)}{v(o_t)} \le C_2 \frac{u^r(x)}{v(x)}, \quad x \in A_t,\ t \ge 2r. \tag{3.25}$$

Set $\Gamma(t) = \sup_{A_t} u^r/v$, $\gamma(t) = \inf_{A_t} u^r/v$, $t \geq 2r$. We now proceed as in [7, Corollary 2.3] to see that there is a universal constant $C_3$ such that for $t \geq 2r$ and $2r \leq t_1 \leq t_2$,

$$\gamma(t) \leq \Gamma(t), \qquad \Gamma(t) \leq C_3\gamma(t), \qquad \Gamma(t_2) \leq \Gamma(t_1), \qquad \gamma(t_2) \geq \gamma(t_1). \tag{3.26}$$

For generality, let $\lambda \geq 2$ and $v(x) = v_{\lambda r}(x) = f(\theta)(\lambda r/|x|)^{1/3}u^r(o_{\lambda r})/f(0)$. Then $v(o_{\lambda r}) = u^r(o_{\lambda r})$. Note that $f(0) = (16)^{-1/3}$, and by (3.15) and the Harnack inequality, $u^r(o_{\lambda r}) \geq e^{1-2\lambda}u^r(o_{r/2}) \geq e^{1-2\lambda}/2$. Using (3.25) and (3.26), we see that there are universal constants $C_4$ and $C_5$ such that

$$C_5 \leq \gamma(\lambda r) \leq \gamma(t) \leq \frac{u^r(x)}{v_{\lambda r}(x)} \leq \Gamma(t) \leq \Gamma(\lambda r) \leq C_4, \quad x \in A_t,\ t \geq \lambda r. \tag{3.27}$$

Thus $C_5 v_{\lambda r}(x) \leq u^r(x) \leq C_4 v_{\lambda r}(x)$, $x \in H \setminus B_t(o)$. Thus

$$C_5\frac{f(\theta)}{f(0)}\left(\frac{\lambda r}{|x|}\right)^{1/3} \leq \frac{u^r(x)}{u^r(o_{\lambda r})} \leq C_4\frac{f(\theta)}{f(0)}\left(\frac{\lambda r}{|x|}\right)^{1/3}, \quad |x| \geq \lambda r. \tag{3.28}$$

*Step 3.* Using (3.28) and Step 1, it follows that for $\kappa > 1$ and $|x| = \kappa\lambda r$, there are universal constants $C_6$ and $C_7$ such that

$$\frac{C_6}{\kappa^{1/3}} \leq \frac{u(o_{\kappa\lambda r})}{u(o_{2r})} \leq \frac{C_7}{\kappa^{1/3}} \Longrightarrow \frac{C_6}{\kappa^{1/3}} \leq \frac{M(\kappa\lambda r)}{M(\lambda r)} \leq \frac{C_7}{\kappa^{1/3}}. \tag{3.29}$$

Choose $l = \sup(3, 3/C_7)$ and set $\kappa = (lC_7)^3 > 3$.

$$M(2\lambda r) \leq \frac{\kappa-2}{\kappa-1}M(\lambda r) + \frac{1}{\kappa-1}M(\kappa\lambda r) \leq \frac{1}{\kappa-1}\left(\kappa - 2 + \frac{1}{l}\right)M(\lambda r) \leq \frac{\kappa-5/3}{\kappa-1}M(\lambda r). \tag{3.30}$$

Clearly, $\alpha < 1$ in (3.24) and is universal. □

## 4. Proof of Theorem 1.3

In this section, we will present another application of Theorem 1.1. We show that any two positive singular infinity-harmonic singular functions, defined in a cone, are comparable. As a consequence, we will show the optimality of the blowup rates of the Aronsson examples [6]. This will extend the results in [7]. First we prove a version of monotonicity that holds in a cone.

Lemma 4.1 (Monotonicity). *For $0 < \alpha \leq \pi/2$, let $C_\alpha$ denote the interior of the cone $\{x : x_n > 0,\ x_n = \|x\|\cos\alpha\}$. Let $u > 0$ be $\infty$-harmonic in $C_\alpha$. Suppose that $v$ is a unit vector that lies in $C_\alpha$, that is, $\langle v, e_n\rangle \geq \cos\alpha$ and let $\theta = \alpha - \cos^{-1}\langle e_n, v\rangle$, then for $0 < t < s$, $u(tv)/t^{\sin\theta} \geq u(sv)/s^{\sin\theta}$, and $u(tv)t^{\sin\theta} \leq u(sv)s^{\sin\theta}$.*

*Proof.* We use the version of the Harnack inequality proved in [7, Lemma 2.1]. Then $\sigma(\tau) = (t+\tau(s-t))\nu$, $0 \le \tau \le 1$, while $d(\tau) = \sigma(\tau)\sin\theta$. Thus

$$\begin{aligned} u(t\nu) &\ge u(s\nu)\exp\Big(-\int_0^1 \frac{s-t}{\sin\theta(t+\tau(s-t))}d\tau\Big) \\ &= u(s\nu)\exp\Big(-\frac{\log(s/t)}{\sin\theta}\Big) = u(s\nu)\Big(\frac{t}{s}\Big)^{1/\sin\theta}. \end{aligned} \tag{4.1}$$

Thus $u(t\nu)/t^{1/\sin\theta} \ge u(s\nu)/s^{1/\sin\theta}$. Switching $t\nu$ by $s\nu$ yields the second inequality. □

*Proof of Theorem 1.3.* Our proof will be an adaptation of the proof of Theorem 1.1 in [7]. First note that $M^u(r)$ is convex. By using comparison, we see that $u(x) \le M^u(t) + [M^u(s) - M^u(t)](|x|-t)/(s-t)$ in the annulus $C_\alpha \cap (B_s(o) \setminus B_t(o))$, $0 < t < s$. Thus $M^u(r)$ is decreasing, $\lim_{r\to 0} M^u(r) = \infty$, and $\lim_{r\to\infty} M^u(r) = 0$.

*Step 1.* We will prove that any two positive solutions $u$ and $v$ are comparable in $C_\alpha$. Now consider the set $C_{\alpha,r} = C_\alpha \cap B_r(o)$. Then (i) for $x \in \partial C_\alpha \cap \partial B_r(o)$, $R_x = r\tan\alpha$, and (ii) for $y \in \partial C_\alpha \cap B_{r/4}(x)$, $R_y \ge (3r/4)\tan\alpha$. In Theorem 1.1, we may take $\delta = (r/8)\tan\alpha$. Thus there are universal constants $C_1$ and $C_2$ such that

$$C_1\frac{u(z)}{v(z)} \le \frac{u(x_\delta)}{v(x_\delta)} \le C_2\frac{u(z)}{v(z)}, \quad z \in C_\alpha \cap B_{(r\tan\alpha)/4}(x). \tag{4.2}$$

Set $p_r = re_n$; let $S_{r,x}$ be the great circle centered at $o$, has radius $r$, and passing through $p_r$ and $x$. Using the Harnack inequality, we may conclude that for $\xi \in (C_\alpha \cap S_{r,x}) \setminus B_{(r\tan\alpha)/4}(x)$, there are constants $A_1 = A_1(\alpha)$ and $A_2 = A_2(\alpha)$ such that $A_1u(p_r) \le u(\xi) \le A_2u(p_r)$. This holds for every $x \in \partial C_\alpha \cap \partial B_r(o)$. Using (4.2), we obtain that there are constants $C_3 = C_3(\alpha)$ and $C_4(\alpha)$ such that

$$C_3\frac{u(z)}{v(z)} \le \frac{u(p_r)}{v(p_r)} \le C_4\frac{u(z)}{v(z)}, \quad z \in C_{\alpha,r}. \tag{4.3}$$

It is clear that (4.3) holds for every $r > 0$. Define $\Gamma(r) = \sup_{z\in C_{\alpha,r}} u(z)/v(z)$ and $\gamma(r) = \inf_{z\in C_{\alpha,r}} u(z)/v(z)$. Thus from (4.3), there is a constant $C_5 = C_5(\alpha)$ such that $\gamma(r) \le \Gamma(r) \le C_5\gamma(r)$, $r > 0$. By comparison, $\Gamma(r)$ is decreasing and $\gamma(r)$ is increasing in $r$, see [7, Lemma 2.2 and Corollary 2.3]. Thus using the above inequality, we see that $0 < \gamma(0) \le \gamma(r) \le \Gamma(r) \le \Gamma(0) \le C_5\gamma(0) < \infty$, $r > 0$. Thus there is a constant $C$ such that $v(x)/C \le u(x) \le Cv(x)$, $x \in C_\alpha$.

*Step 2.* We now show the optimality of the Aronsson examples. We first observe that by arguing as in [7, Lemmas 2.5 and 2.6], any solution $u$ is axially symmetric in $C_\alpha$. If we set $\rho^2 = \sum_{i=1}^n x_i^2$ and $\theta = \cos^{-1}\langle x, e_n\rangle/|x|$, then $u(x) = u(\rho,\theta)$, $x \in C_\alpha$, and

$$\Delta_\infty u = u_\rho^2 u_{\rho\rho} + \frac{2u_\rho u_\theta u_{\rho\theta}}{\rho^2} + \frac{u_\theta^2 u_{\theta\theta}}{\rho^4} - \frac{u_\rho u_{\theta\theta}^2}{\rho^3} = 0. \tag{4.4}$$

Note that there is no explicit dependence on the dimension $n$. For each $m = 1,2,3,\ldots$, set

$\alpha = \pi/2m$. The Aronsson example in the planar cone $C_{\pi/2m}$ is given by $w_m(x) = w_m(|x|, \theta) = f_m(\theta)/|x|^{m^2/(2m+1)}$, where

$$f_m(\theta) = \left|1 - \frac{\cos^2 t}{k}\right|^{(k-1)/2} \cos t, \quad \theta = \int_0^t \frac{\sin^2 s}{k - \cos^2 s} ds, \; k = -\frac{m^2}{2m+1}. \tag{4.5}$$

Note that $\theta = t - (1 + 1/m)\arctan(m \tan t/(m+1))$. From above $w_m$ is symmetric in $\theta$ and reinterpreting the polar angle $\theta$ to be the conical angle, we obtain an example in higher dimensions. This continues to be a viscosity solution in $C_{\pi/2m}$, see the appendix in [11, 7]. Note that $w_m(|x|, \theta) > 0$, $-\pi/2m \leq \theta \leq \pi/2m$, and $w_m(\pm\pi/2m) = 0$. We now have the desired conclusion by using Step 1. □

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