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Research Article

Antiperiodic Boundary Value Problems for Finite Dimensional Differential Systems

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We study antiperiodic boundary value problems for semilinear differential and impulsive differential equations in finite dimensional spaces. Several new existence results are obtained.

1. Introduction

The study of antiperiodic solutions for nonlinear evolution equations is closely related to the study of periodic solutions, and it was initiated by Okochi [1]. During the past twenty years, antiperiodic problems have been extensively studied by many authors, see [1–31] and the references therein. For example, antiperiodic trigonometric polynomials are important in the study of interpolation problems [32, 33], and antiperiodic wavelets are discussed in [34]. Moreover, antiperiodic boundary conditions appear in physics in a variety of situations, see [35–40]. In Section 2 we consider the antiperiodic problem

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in R, \\ u(t) &= -u(t+T), \quad t \in R, \end{aligned} \tag{E 1.1}$$

where A is an $n \times n$ matrix, $f : R \times R^n \to R^n$ is continuous, and $f(t+T, x) = -f(t, x)$ for all $(t, x) \in R \times R^n$. Under certain conditions on the nondiagonal elements of A and f we prove an existence result for (E 1.1). In Section 3 we consider the antiperiodic boundary value problem

$$\begin{aligned} u'(t) &= Gu(t) + f(t, u(t)), \quad \text{a.e. } t \in J = [0, T],\ t \neq t_k, \\ u(0) &= -u(T), \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \dots, p, \end{aligned} \tag{E 1.2}$$

where $G : R^n \to R^n$ is a function satisfying $G0 = 0$, and $f : J \times R^n \to R^n$ is a Caratheodory function, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $I_k \in C(R^n, R^n)$. Under certain conditions on G, f, and $I_k(u)$ for $k = 1, 2, \ldots, p$, we prove an existence result for (E 1.2).

2. Antiperiodic Problem for Differential Equations in R^n

Let $|\cdot|$ be the norm in R^n. In this section we study

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in R,$$
$$u(t) = -u(t+T). \tag{E 2.1}$$

First, we have the following result.

Theorem 2.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix, where a_{ij} is the element of A in the ith row and jth column, $f : R \to R^n$ is continuous and $f(t+T) = -f(t)$ for $t \in R$. Suppose $(T/2)\Sigma_{1\le i<j\le n}|a_{ij} - a_{ji}| < 1$. Then the equation*

$$u'(t) = Au(t) + f(t), \quad t \in R,$$
$$u(t) = -u(t+T), \quad t \in R \tag{E 2.2}$$

has a unique solution.

Proof. Put $W_a = \{v(\cdot) \in C(R; R^n) : v(t) = -v(t+T)\}$. Then W_a is a Banach space under the norm $|v(\cdot)|_\infty = \max_{t\in[0,T]}|v(t)|$. For each $v(\cdot) \in W_a$, consider the following equation:

$$u'(t) = Av(t) + f(t), \quad t \in R,$$
$$u(t) = -u(t+T), \quad t \in R. \tag{E 2.3}$$

It is easy to see that $u(t) = -(1/2)\int_0^T [Av(s) + f(s)]ds + \int_0^t [Av(s) + f(s)]ds$ is the unique solution of (E 2.3).

We define a mapping $K : W_a \to W_a$ as follows:

$$\text{for any } v(\cdot) \in W_a, \quad Kv(\cdot) = u(\cdot), \quad u(\cdot) \text{ is the solution of (E 2.3)}. \tag{2.1}$$

First we prove that K is a continuous compact mapping. Now assume $v_n(\cdot) \in W_a$, $n = 1, 2, \ldots$, and $v_n(\cdot) \to v(\cdot) \in W_a$. Then $|Av_n(\cdot) - Av(\cdot)|_\infty \to 0$ as $n \to \infty$. This immediately implies that $\int_0^T |(Kv_n(t))' - (Kv(t))'|^2 dt \to 0$ as $n \to \infty$.

We have $Kv_n(t) - Kv(t) = (1/2)\{\int_0^t [(Kv_n(s))' - (Kv(s))']ds - \int_t^T [(Kv_n(s))' - (Kv(s))']ds]\}$, and so $Kv_n(\cdot) \to Kv(\cdot)$ in W_a.

Now since $(Kv(t))' = Av(t) + f(t)$, $t \in R$, it is easy to see that

$$\left(\int_0^T \left|(Kv(t))'\right|^2 dt\right)^{1/2} \le \sqrt{T}|Av(\cdot)|_\infty + \left(\int_0^T \left|f(t)\right|^2 dt\right)^{1/2}. \tag{2.2}$$

Thus K maps a bounded subset of W_a to a bounded equicontinuous subset in W_a, therefore K is completely continuous.

Next take $r_0 > (1-(T/2)\Sigma_{1\le i<j\le n}|a_{ij}-a_{ji}|)^{-1}(\sqrt{T}/2)(\int_0^T|f(t)|^2dt)^{1/2}$. We show that $Kv(\cdot)\neq\lambda v(\cdot)$ for all $\lambda\ge 1$, and $|v(\cdot)|_\infty = r_0$. If this is not true, there exist $\lambda_0\ge 1$, $w(\cdot)\in W_a$ with $|w(\cdot)|_\infty = r_0$ such that $Kw(\cdot)=\lambda_0 w(\cdot)$, that is, $w(t)=-w(t+T)$, $t\in R$ and

$$\lambda_0 w'(t) = Aw(t) + f(t), \quad t\in R. \tag{2.3}$$

Multiply (2.3) by $w'(t)$ (i.e., take inner product) and integrate over $[0,T]$, and notice that $\int_0^T w_i(t)w_j'(t)dt = -\int_0^T w_i'(t)w_j(t)dt$ to get

$$\lambda_0\int_0^T |w'(t)|^2dt \le \Sigma_{1\le i<j\le n}\left|a_{ij}-a_{ji}\right|\int_0^T\left|w_i(t)w_j'(t)\right|dt + \left(\int_0^T\left|f(t)\right|^2dt\right)^{1/2}\left(\int_0^T\left|w'(t)\right|^2dt\right)^{1/2}, \tag{2.4}$$

where $w(t)=(w_i(t))$, $i=1,2,\dots,n$. Notice that $w(t)=(1/2)[\int_0^t w'(s)ds - \int_t^T w'(s)]ds$, so we have

$$|w(\cdot)|_\infty \le \frac{\sqrt{T}}{2}\left(\int_0^T\left|w'(t)\right|^2dt\right)^{1/2}. \tag{2.5}$$

From (2.4), (2.5), we have

$$\lambda_0\left(\int_0^T\left|w'(t)\right|^2dt\right)^{1/2} \le \sqrt{T}\Sigma_{1\le i<j\le n}\left|a_{ij}-a_{ji}\right||w(\cdot)|_\infty + \left(\int_0^T\left|f(t)\right|^2dt\right)^{1/2}. \tag{2.6}$$

This with (2.5) gives

$$\lambda_0|w(\cdot)|_\infty \le \frac{T}{2}\Sigma_{1\le i<j\le n}\left|a_{ij}-a_{ji}\right||w(\cdot)|_\infty + \frac{\sqrt{T}}{2}\left(\int_0^T\left|f(t)\right|^2dt\right)^{1/2}. \tag{2.7}$$

As a result

$$|w(\cdot)|_\infty \le \left(1-\frac{T}{2}\Sigma_{1\le i<j\le n}\left|a_{ij}-a_{ji}\right|\right)^{-1}\frac{\sqrt{T}}{2}\left(\int_0^T\left|f(t)\right|^2dt\right)^{1/2}, \tag{2.8}$$

which contradicts $|w(\cdot)|_\infty = r_0$.

Thus the Leray-Schauder degree $\deg(I-K, B(0,r_0),0)=1$, where $B(0,r_0)$ is the open ball centered at 0 with radius r_0 in C_a. Consequently, K has a fixed point in $B(0,r_0)$, that is, (E 2.2) has a solution. For the uniqueness, if $u(\cdot)$, $v(\cdot)$ are two solutions of (E 2.2), set $w(t) = u(t)-v(t)$, then $w'(t) = Aw(t)$, and $w(t) = -w(t+T)$, for $t\in R$. Following the obvious strategy above (see the clear adjustment of (2.8)) gives $|w(\cdot)|_\infty = 0$. Thus the solution of (E 2.2) is unique. □

From Theorem 2.1 we have immediately the following result.

Corollary 2.2. *Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix, $f : R \to R^n$ is continuous and $f(t+T) = -f(t)$ for $t \in R$. Then*

$$u'(t) = Au(t) + f(t), \quad t \in R, \qquad u(t) = -u(t+T), \quad t \in R, \tag{E 2.4}$$

has a unique solution.

Using a proof similar to Theorem 2.1, we have the following result.

Theorem 2.3. *Let $A = (a_{ij})$ be an $n \times n$ matrix, $G : R^n \to R^n$ is an even continuously differentiable function, and $f(t,u) : R \times R^n \to R^n$ is continuous and $f(t+T,u) = -f(t,u)$ for $(t,u) \in R \times R^n$. Suppose the following conditions are satisfied:*

(1) $|f(t,x)| \le M|x| + g(t)$, *for a.e.* $(t,x) \in R \times R^n$, *where* $M > 0$ *is a constant, and* $g(\cdot) \in L^2(0,T)$;

(2) $(T/2)[\Sigma_{1\le i<j\le n}|a_{ij} - a_{ji}| + M] < 1$.

Then

$$u'(t) = Au(t) + \partial Gu(t) + f(t,u(t)), \quad t \in R, \qquad u(t) = -u(t+T), \quad t \in R \tag{E 2.5}$$

has a solution.

Remark 2.4. Equation (E 2.5) was studied by Haraux [18] and Chen et al. [14] in the case $A = 0$, and also by Chen [12] with different assumptions on f and A.

3. Antiperiodic Boundary Value Problem for Impulsive ODE

In this section, we prove an existence result for the equation

$$\begin{gathered} u'(t) = Gu(t) + f(t,u(t)), \quad \text{a.e. } t \in J = [0,T],\ t \ne t_k, \\ u(0) = -u(T), \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,p, \end{gathered} \tag{E 3.1}$$

where $G : R^n \to R^n$ is a Lipschitz function. We first introduce some notations. Let $J = [0,T]$, and $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$. $PC(J) = \{u : J \to R^n, u_{(t_k,t_{k+1}]} \in C((t_k,t_{k+1}],R^n), k = 0,1,\ldots,p,\ u(t_k^-)$ exist for $k = 1,2,\ldots,p$, and $u(0^+) = u(0)\}$, and $PW^{1,2}(J) = \{u \in PC(J) : u_{(t_k,t_{k+1})} \in W^{1,2}((t_k,t_{k+1}),R^n), k = 1,\ldots,p\}$. It is clear that $PC(J)$

and $PW^{1,2}(J)$ are Banach spaces with the respective norm $\|u\|_{PC(J)} = \sup\{|u(t)|, t \in J\}$, and $\|u\|_{PW^{1,2}(J)} = \sum_{k=0}^{p} \|u_k\|_{W^{1,2}(t_k,t_{k+1})}$, where $u_k : (t_k, t_{k+1}] \to R$ is defined by $u_k(t) = u(t)$ for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, p$.

We say a function u is a solution of (E 3.1) if $u \in PW^{1,2}(J)$ and u satisfies (E 3.1).

We first prove the following result.

Lemma 3.1. *Let $I_i : R^n \to R^n$ be continuous functions for $i = 1, 2, \ldots, p$, and $\Sigma_{k=1}^{p} |I_k(x_k)| \le \alpha\{\max_{1\le k\le p}|x_k|\} + \delta$ for all $x_k \in R^n$, $k = 1, 2, \ldots, p$, where $\alpha, \delta > 0$ are constants, and $\alpha < 2$. Suppose $u \in PW^{1,2}(J)$ with $u(0) = -u(T)$, and $\Delta u(t_i) = I_i(u(t_i))$, for $i = 1, 2, \ldots, p$. Then*

$$\|u\|_{PC(J)} \le \left(1 - \frac{1}{2}\alpha\right)^{-1} \left[\frac{1}{2}\delta + \frac{\sqrt{T}}{2}\left(\int_0^T |u'(s)|^2 ds\right)^{1/2}\right]. \tag{3.1}$$

Proof. By assumption, we have $u(t) = u(0) + \int_0^t u'(s)ds$ for $t \in [0, t_1)$, and

$$u(t) = u(0) + \Sigma_{i=1}^{k} I_i(u(t_i)) + \int_0^t u'(s)ds \tag{3.2}$$

for $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots, p$. Since $u(0) = -u(T)$, it follows that $u(t) = -(1/2)[\Sigma_{i=1}^{p} I_i(u(t_i)) + \int_0^T u'(s)ds] + \int_0^t u'(s)ds$ for $t \in [0, t_1)$, and

$$u(t) = -\frac{1}{2}\left[\Sigma_{i=1}^{p} I_i(u(t_i)) + \int_0^T u'(s)ds\right] + \Sigma_{i=1}^{k} I_i(u(t_i)) + \int_0^t u'(s)ds \tag{3.3}$$

for $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots, p$. Hence we have

$$\|u\|_{PC(J)} \le \frac{1}{2}\left[\alpha\|u\|_{PC(J)} + \delta\right] + \frac{\sqrt{T}}{2}\left(\int_0^T |u'(s)|^2 ds\right)^{1/2}. \tag{3.4}$$

Thus

$$\|u\|_{PC(J)} \le \left(1 - \frac{1}{2}\alpha\right)^{-1} \left[\frac{1}{2}\delta + \frac{\sqrt{T}}{2}\left(\int_0^T |u'(s)|^2 ds\right)^{1/2}\right]. \tag{3.5}$$

□

Theorem 3.2. *Let $G : R^n \to R^n$ be a function satisfying $G0 = 0$, and $f : [0, T] \to R^n$ such that $f(\cdot) \in L^2([0, T])$, and let $I_k : R^n \to R^n$ be continuous functions for $k = 1, 2, \ldots, p$. Suppose the following conditions are satisfied:*

(1) *$|Gu - Gv| \le L|u - v|$ for all $u, v \in R^n$, and $L > 0$ is a constant;*

(2) *$\Sigma_{k=1}^{p} |I_k(x_k)| \le \gamma\{\max_{1\le k\le p}|x_k|\} + \delta$ for all $x_k \in R^n$, $k = 1, 2, \ldots, p$, where $\gamma, \delta > 0$ are constants;*

(3) *$\gamma + TL < 2$.*

Then the problem

$$
\begin{aligned}
u'(t) &= Gu(t) + f(t), \quad \text{a.e. } t \in J = [0,T],\ t \neq t_k, \\
u(0) &= -u(T), \\
\Delta u(t_k) &= I_k(u(t_k)), \quad k = 1,2,\dots,p
\end{aligned}
\tag{E 3.2}
$$

has a solution.

Proof. For each $v \in PC(J)$, consider the problem

$$
\begin{aligned}
u'(t) &= Gv(t) + f(t) \quad \text{a.e. } t \in J = [0,T],\ t \neq t_k, \\
u(0) &= -u(T), \\
\Delta u(t_k) &= I_k(v(t_k)), \quad k = 1,2,\dots,p.
\end{aligned}
\tag{E 3.3}
$$

One can easily show that the solution u of (E 3.3) is given by the following:

$$
\begin{aligned}
u(t) &= -\frac{1}{2}\left[\Sigma_{i=1}^{p} I_i(v(t_i)) + \int_0^T \big(Gv(s) + f(s)\big)ds\right] \\
&\quad + \int_0^t \big(G(v(s)) + f(s)\big)ds, \quad \text{for } t \in [0,t_1), \\
u(t) &= -\frac{1}{2}\left[\Sigma_{i=1}^{p} I_i(v(t_i)) + \int_0^T \big(Gv(s) + f(s)\big)ds\right] + \Sigma_{i=1}^{k} I_i(v(t_i)) \\
&\quad + \int_0^t \big(Gv(s) + f(s)\big)ds,
\end{aligned}
\tag{3.6}
$$

for $t \in [t_k, t_{k+1})$, $k = 1,\dots,p$.

Obviously, the solution of (E 3.3) is unique. Now we define $K : PC(J) \to PW^{1,2}(J) \subset PC(J)$ by $u = Kv$. We prove that K is continuous. Let $v_n \in PC(J)$ and $v_n \to v$ in $PC(J)$. It is easy to see that

$$
\int_0^T \left|(Kv_n(t) - Kv(t))'\right|^2 dt = \int_0^T |Gv_n(t) - Gv(t)|^2 dt \leq L^2 \int_0^T |v_n(t) - v(t)|^2 dt. \tag{3.7}
$$

Therefore $\left(\int_0^T |(Kv_n(t) - Kv(t))'|^2 dt\right)^{1/2} \leq \sqrt{T}L\|v_n - v\|_{PC(J)} \to 0$ as $n \to \infty$.

Note that $\Delta(Kv_n - Kv)(t_k) = I_k(v_n(t_k)) - I_k(v(t_k))$, and we have

$$
\begin{aligned}
Kv_n(t) - Kv(t) &= -\frac{1}{2}\left[\Sigma_{i=1}^{p}(I_i(v_n(t_i)) - I_i(v(t_i))) + \int_0^T (Kv_n - Kv)'(s)ds\right] \\
&\quad + \int_0^t (Kv_n - Kv)'(s)ds, \quad \text{for } t \in [0, t_1), \\
Kv_n(t) - Kv(t) &= -\frac{1}{2}\left[\Sigma_{i=1}^{p}(I_i(v_n(t_i)) - I_i(v(t_i))) + \int_0^T (Kv_n - Kv)'(s)ds\right] \\
&\quad + \Sigma_{i=1}^{k}(I_i(v_n(t_i)) - I_i(v(t_i))) + \int_0^t (Kv_n - Kv)'(s)ds
\end{aligned}
\tag{3.8}
$$

for $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots, p$. From the continuity of I_i, $i = 1, 2, \ldots, p$, and $\int_0^T |(Kv_n(t) - Kv(t))'|^2 dt \to 0$ as $n \to \infty$, we deduce that K is continuous.

For each $v \in PC(J)$, notice that $0 = G0$, so we have

$$
\left(\int_0^T |Kv|^2 dt\right)^{1/2} \le \sqrt{T} L \|v\|_{PC(J)} + \left(\int_0^T |f(s)|^2 ds\right)^{1/2}. \tag{3.9}
$$

From (3.9) and Lemma 3.1, we know that K maps bounded subsets of $PC(J)$ to relatively compact subsets of $PC(J)$.

Finally, for $\forall \lambda \in (0, 1]$, we prove that the set of solutions of $u = \lambda K u$ is bounded. If $u = \lambda K u$ for some $\lambda \in (0, 1)$, then

$$
\begin{gathered}
u'(t) = \lambda G u(t) + \lambda f(t) \quad \text{a.e. } t \in J = [0, T],\ t \neq t_k, \\
u(0) = -u(T), \\
\Delta u(t_k) = \lambda I_k(u(t_k)), \quad k = 1, 2, \ldots, p.
\end{gathered}
\tag{3.10}
$$

Therefore we have

$$
u(t) = -\frac{1}{2}\lambda\left[\Sigma_{i=1}^{p} I_i(u_i(t_i)) + \int_0^T (Gu(s) + f(s))ds\right] + \lambda \int_0^t (G(u(s)) + f(s))ds \tag{3.11}
$$

for $t \in [0, t_1)$, and

$$
\begin{aligned}
u(t) = &-\frac{1}{2}\lambda\left[\Sigma_{i=1}^{p} I_i(u_i(t_i)) + \int_0^T (Gu(s) + f(s))ds\right] + \lambda\, \Sigma_{i=1}^{k} I_i(u_i(t_i)) \\
&+ \lambda \int_0^t (G(u(s)) + f(s))ds
\end{aligned}
\tag{3.12}
$$

for $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, p$. This implies that

$$\|u\|_{PC(J)} \leq \frac{1}{2}\left[\gamma\|u\|_{PC(J)} + \delta + \int_0^T (|Gu(s)| + |f(s)|)\,ds\right]. \tag{3.13}$$

Since $0 = G0$, and $|Gu| \leq L|u|$, so we have

$$\|u\|_{PC(J)} \leq \frac{1}{2}\left[1 - \frac{1}{2}(\gamma + TL)\right]^{-1}\left(\delta + \int_0^T |f(s)|\,ds\right). \tag{3.14}$$

The Leray-Schauder principle guarantees a fixed point of K, which is easily seen to be a solution of (E 3.2). □

By using a similar method to Theorem 3.2, one can deduce the following result.

Theorem 3.3. *Let $G : R^n \to R^n$ be a function satisfying $G0 = 0$, and $f(t,x) : [0,T] \times R^n \to R^n$ a Caratheodory function, that is, f is measurable in t for each $x \in R^n$, and f is continuous in x for each $t \in [0,T]$, such that $|f(t,x)| \leq g(t)$ for $(t,x) \in [0,T] \times R^n$, where $g(\cdot) \in L^2([0,T])$, and let $I_k : R^n \to R^n$ be continuous functions for $k = 1,2,\ldots,p$. Suppose the following conditions are satisfied:*

(1) $|Gu - Gv| \leq L|u - v|$ *for all* $u, v \in R^n$, *and* $L > 0$ *is a constant;*

(2) $\Sigma_{k=1}^{p} |I_k(x_k)| \leq \gamma\{\max_{1 \leq k \leq p} |x_k|\} + \delta$ *for all* $x_k \in R^n$, $k = 1,2,\ldots,p$, *where* $\gamma, \delta > 0$ *are constants;*

(3) $\gamma + TL < 2$.

Then the equation

$$\begin{gathered} u'(t) = Gu(t) + f(t,u(t)), \quad a.e.\ t \in J = [0,T],\ t \neq t_k, \\ u(0) = -u(T), \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,p \end{gathered} \tag{E 3.4}$$

has a solution.

4. Examples

In this section, we give examples to show the application of our results to differential and impulsive differential equations.

Example 4.1. Consider the antiperiodic problem

$$
\begin{gathered}
u_1'(t) = \lambda_1 u_1(t) + 5u_2(t) + \sin \pi t, \quad t \in R, \\
u_2'(t) = \frac{7}{2} u_1(t) + \lambda_2 u_2(t) + \cos \pi t, \quad t \in R, \\
u_1(t) = -u_1(t+1), \quad u_2(t) = -u_2(t+1), \quad t \in R.
\end{gathered} \tag{E 4.1}
$$

Set

$$
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} \sin \pi t \\ \cos \pi t \end{pmatrix}, \qquad A = \begin{pmatrix} \lambda_1 & 5 \\ \frac{7}{2} & \lambda_2 \end{pmatrix}. \tag{4.1}
$$

Now (E 4.1) is equivalent to

$$
\begin{gathered}
u'(t) = Au(t) + f(t), \quad t \in R, \\
u(t) = -u(t+1), \quad t \in R.
\end{gathered} \tag{E 4.2}
$$

Also $f(t) = -f(t+1)$, for $t \in R$ and $(1/2)|a_{12} - a_{21}| = 3/4$. By Theorem 2.1, (E 4.2) has a unique solution, so (E 4.1) has a unique solution.

Example 4.2. Consider the antiperiodic boundary value problem

$$
\begin{gathered}
u_1'(t) = \frac{1}{2 + u_1^2(t) + u_2^2(t)} [3u_1(t) - 2u_2(t)] + \sin \pi t, \quad t \in (0,1),\ t \neq \frac{1}{4}, \\
u_2'(t) = \frac{1}{2 + u_1^2(t) + u_2^2(t)} [2u_1(t) + 3u_2(t)] - \cos \pi t, \quad t \in (0,1),\ t \neq \frac{1}{4}, \\
\Delta u_1\left(\frac{1}{4}\right) = \frac{1}{5(1 + |u_2(1/4)|)}, \qquad \Delta u_2\left(\frac{1}{4}\right) = \frac{1}{8(1 + |u_1(1/4)|)}, \\
u_1(0) = -u_1(1), \qquad u_2(0) = -u_2(1).
\end{gathered} \tag{E 4.3}
$$

Set

$$
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} \sin \pi t \\ -\cos \pi t \end{pmatrix}, \qquad Gu = \begin{pmatrix} \dfrac{3u_1 - 2u_2}{2 + u_1^2 + u_2^2} \\ \dfrac{2u_1 + 3u_2}{2 + u_1^2 + u_2^2} \end{pmatrix}, \qquad I(u) \begin{pmatrix} \dfrac{1}{5(1 + |u_2|)} \\ \dfrac{1}{8(1 + |u_1|)} \end{pmatrix}. \tag{4.2}
$$

It is easy to check that $|Gu - Gv| \leq (\sqrt{13}/2)|u - v|$ for $u, v \in R^2$, $|I(u)| < 2/5$ for $u \in R^2$, and $\sqrt{13}/2 < 2$. Now (E 4.3) is equivalent to the equation

$$u'(t) = Gu(t) + f(t), \quad t \in (0,1),\ t \neq \frac{1}{4},$$
$$\Delta u\left(\frac{1}{4}\right) = I\left(u\left(\frac{1}{4}\right)\right), \qquad u(0) = -u(1). \tag{E 4.4}$$

By Theorem 3.2, we know that (E 4.4) has a solution, so (E 4.3) has a solution.

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