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Research Article

Existence of Solutions for Elliptic Systems with Nonlocal Terms in One Dimension

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We study the solvability of a system of second-order differential equations with Dirichlet boundary conditions and non-local terms depending upon a parameter. The main tools used are a dual variational method and the topological degree.

1. Introduction

In the past decade there has been a lot of interest on boundary value problems for elliptic systems. For general systems of the form

$$\begin{gathered} -\Delta u = f(x, u, v, \nabla u, \nabla v), \quad x \in \Omega, \\ -\Delta v = g(x, u, v, \nabla u, \nabla v), \quad x \in \Omega, \\ u = v = 0, \quad \text{in } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, a survey was given by De Figueiredo in [1]. The specific case of one-dimensional systems, motivated by the problem of finding radial solutions to an elliptic system on an annulus of $\mathbb{R}^n$, has been considered by Dunninger and Wang [2] and by Lee [3], who have obtained conditions under which such a system may possess multiple positive solutions.

On the other hand, systems of two equations that include non-local terms have also been considered recently. These are of importance because they appear in the applied sciences, for example, as models for ignition of a compressible gas, or general physical phenomena where temperature has a central role in triggering a reaction. In fact their interest ranges from physics and engineering to population dynamics. See for instance [4]. The related parabolic problems are also of great interest in reaction-diffusion theory; see [5–7] where the approach to existence and blow-up for evolution systems with integral terms may be found.

In this paper we are interested in a simple one-dimensional model: the two-point boundary value problem for the system of second order differential equations with a linear integral term

$$\begin{gathered} -u''(t) + c\int_0^1 v(s)ds + g(v(t)) = 0, \quad \text{for a.e. } t \in [0,1], \\ -v''(t) + c\int_0^1 u(s)ds + h(u(t)) = 0, \quad \text{for a.e. } t \in [0,1], \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{gathered} \tag{1.2}$$

where $c \in \mathbb{R}$, $c \neq 0$ and $g, h : \mathbb{R} \to \mathbb{R}$. First we consider (1.2) as a perturbation of the nonlocal system and prove that if g and h grow linearly, then (1.2) has a solution provided $|c|$ is not too large. Afterwards, assuming that g and h are monotone, we will give estimates on the growth of these functions in terms of the parameter c to ensure solvability. This will be done on the basis of some spectral analysis for the linear part and a dual variational setting.

2. Preliminaries

Let us introduce some notation: we define $L^2(0,1)$ as the Hilbert space of the Lebesgue measurable functions f such that $\int_0^1 f^2(x)dx < \infty$ with the usual inner product

$$\langle f, g\rangle_{L^2} = \int_0^1 f(x)g(x)dx. \tag{2.1}$$

We also define

$$\begin{gathered} AC(I) = \{f : [0,1] \longrightarrow \mathbb{R} : f \text{ is absolutely continuous on } [0,1]\}, \\ H_0^2(0,1) := \{f \in C^1[0,1] : f' \in AC[0,1],\ f'' \in L^2(0,1),\ f(0) = 0 = f(1)\}, \end{gathered} \tag{2.2}$$

with the inner product

$$\langle f, g\rangle_{H_0^2} := \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2} + \langle f'', g''\rangle_{L^2}. \tag{2.3}$$

If $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ are both Hilbert spaces, we will consider the Hilbert product space $X \times Y$ with the inner product

$$\big\langle (x_1, y_1), (x_2, y_2) \big\rangle_{X\times Y} = \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y. \tag{2.4}$$

We first study the invertibility of the linear part of (1.2).

Lemma 2.1. *The linear operator* $L : H_0^2(0,1) \times H_0^2(0,1) \to L^2(0,1) \times L^2(0,1)$*, defined by*

$$L_c(u,v) = \left(-u'' + c\int_0^1 v(s)ds,\ -v'' + c\int_0^1 u(s)ds \right), \tag{2.5}$$

is invertible if and only if $c \neq \pm 12$.

Moreover, L_c *and* L_c^{-1} *are both continuous for* $c \neq \pm 12$.

Proof. Let $(x,y) \in L^2(0,1) \times L^2(0,1)$. The equation $L_c(u,v) = (x,y)$ is equivalent to

$$\begin{gathered} -u''(t) = x(t) - c\int_0^1 v(s)ds, \quad \text{for a.e. } t \in [0,1], \\ -v''(t) = y(t) - c\int_0^1 u(s)ds, \quad \text{for a.e. } t \in [0,1], \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0. \end{gathered} \tag{2.6}$$

We denote $\int_0^1 v(s)ds = a$, $\int_0^1 u(s)ds = b$, $X(t) = \int_0^1 G(t,s)x(s)ds$, and $Y(t) = \int_0^1 G(t,s)y(s)ds$, where

$$G(t,s) = \begin{cases} t(1-s), & \text{if } 0 \le t < s \le 1, \\ s(1-t), & \text{if } 0 \le s \le t \le 1 \end{cases} \tag{2.7}$$

is the Green's function associated to $-u'' = h(t)$, $u(0) = 0 = u(1)$. Notice that $X, Y \in H_0^2(0,1)$ are the solutions of $-u'' = x(t)$, $u(0) = 0 = u(1)$ and $-u'' = y(t)$, $u(0) = 0 = u(1)$, respectively.

Now it is easy to see that (u,v) is a solution of (2.6) if and only if

$$u(t) = X(t) + \frac{ca}{2}t(t-1), \quad v(t) = Y(t) + \frac{cb}{2}t(t-1), \tag{2.8}$$

for some $a, b \in \mathbb{R}$ such that

$$\begin{aligned} a &= \int_0^1 Y(s)ds - \frac{cb}{12}, \\ b &= \int_0^1 X(s)ds - \frac{ca}{12}, \end{aligned} \tag{2.9}$$

Clearly this linear system is uniquely solvable for each pair of functions $X, Y \in H_0^2(0,1)$ if and only if $c \neq \pm 12$.

In order to prove the continuity of L_c it is easy to show that there exists $k > 0$ such that

$$\|L_c(u,v)\|_{L^2\times L^2} \leq k\|(u,v)\|_{H_0^2\times H_0^2} \quad \forall (u,v) \in H_0^2 \times H_0^2. \tag{2.10}$$

By the open mapping theorem we deduce that L_c^{-1}, $c \neq \pm 12$, is continuous too. □

In view of the previous lemma we will assume

(C0) $c \in \mathbb{R} \setminus \{-12, 0, 12\}$.

Lemma 2.2. *Assume* (C0).*Then the operator* $K_c = U \circ i \circ L_c^{-1} : L^2(0,1) \times L^2(0,1) \to L^2(0,1) \times L^2(0,1)$ *is compact and self-adjoint, where* $i : H_0^2(0,1) \times H_0^2(0,1) \hookrightarrow L^2(0,1) \times L^2(0,1)$ *is the inclusion and* $U(x,y) = (y,x)$.

Proof. Since the inclusion i is compact (see [8, Theorem VIII.7]) and L_c^{-1} and U are continuous we obtain the compactness of K_c. On the other hand an easy computation shows that

$$\langle K_c u, v\rangle_{L^2\times L^2} = \langle u, K_c v\rangle_{L^2\times L^2}, \tag{2.11}$$

so K_c is a self-adjoint operator. □

3. An Existence Result of Perturbative Type

Let us introduce the basic assumption

(H) $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions,

and set

$$l := \limsup_{|v|\to\infty} \left|\frac{g(v)}{v}\right|, \qquad m := \limsup_{|u|\to\infty} \left|\frac{h(u)}{u}\right|. \tag{3.1}$$

Theorem 3.1. *Assume* (H), $c \neq 0$, *and* $|c| + (l+m)/2 < \pi^2$.

Then problem (1.2) *has a solution.*

Proof. Consider the homotopy $I - \lambda(K_c \circ \mathcal{N})$ for all $\lambda \in [0,1]$, where $\mathcal{N}$ is the Nemitskii operator $\mathcal{N} : L^2(0,1) \times L^2(0,1) \to L^2(0,1) \times L^2(0,1)$ given by

$$\mathcal{N}(u,v) = \big(-g(u(\cdot)), -h(v(\cdot))\big). \tag{3.2}$$

It is easy to check that $[I-\lambda(K_c\circ\mathcal{N})](v,u)=(0,0)$ if and only if (u,v) is a solution of problem

$$\begin{gathered}
-u''(t)+c\int_0^1 v(s)ds+\lambda g(v(t))=0, \quad \text{for a.e. } t\in[0,1],\\
-v''(t)+c\int_0^1 u(s)ds+\lambda h(u(t))=0, \quad \text{for a.e. } t\in[0,1],\\
u(0)=u(1)=0, \quad v(0)=v(1)=0.
\end{gathered} \tag{3.3}$$

We are going to prove that the possible solutions of $[I-\lambda(K_c\circ\mathcal{N})](u,v)=(0,0)$ are bounded independently of $\lambda\in[0,1]$. By our assumptions, there exist $l'>0$, $m'>0$ and k such that

$$|c|+\frac{l'+m'}{2}<\pi^2, \qquad \left|g(u)\right|\le l'|u|+k \quad \forall u\in\mathbb{R}, |h(v)|\le m'|v|+k \quad \forall v\in\mathbb{R}. \tag{3.4}$$

Multiplying the first equation of (3.3) by u, the second one by v, integrating between 0 and 1 and adding both equations we obtain

$$\begin{aligned}
\int_0^1\left(\left(u'(s)^2\right)+\left(v'(s)^2ds\right)\right) &\le 2|c|\int_0^1|u(s)|ds\int_0^1|v(s)|ds\\
&\quad+\lambda\int_0^1\left(\left|g(v(s))\right||u(s)|+|h(u(s))||v(s)|\right)ds\\
&\le |c|\left(\int_0^1 u^2(s)ds+\int_0^1 v^2(s)ds\right)+\frac{l'+m'}{2}\int_0^1\left(u^2(s)+v^2(s)\right)ds\\
&\quad+k\int_0^1(|u(s)|+|v(s)|)ds\\
&=\left(|c|+\frac{l'+m'}{2}\right)\left(\int_0^1\left(u^2(s)+v^2(s)\right)ds\right)\\
&\quad+k\left(\left(\int_0^1 u^2(s)ds\right)^{1/2}+\left(\int_0^1 v^2(s)ds\right)^{1/2}\right).
\end{aligned} \tag{3.5}$$

On the other hand, by the Poincaré inequality (see [9, Chapter 2])

$$\pi^2\int_0^1 u^2(s)+v^2(s)ds\le\int_0^1 u'(s)^2+v'(s)^2ds, \tag{3.6}$$

so we have

$$\pi^2 \int_0^1 \left(u^2(s) + v^2(s)\right) ds \le \left(|c| + \frac{l' + m'}{2}\right)\left(\int_0^1 \left(u^2(s) + v^2(s)\right) ds\right) + k\left(\left(\int_0^1 u^2(s) ds\right)^{1/2} + \left(\int_0^1 v^2(s) ds\right)^{1/2}\right) \tag{3.7}$$

and since $|c|+(l'+m')/2 < \pi^2$ we obtain that $(\int_0^1 u^2(s)ds)^{1/2} = \|u\|_{L^2}$ and $(\int_0^1 v^2(s)ds)^{1/2} = \|v\|_{L^2}$ are bounded.

Thus we may invoke the properties of the Leray-Schauder degree (see, e.g., [10]) to deduce the existence of a solution for (3.3) with $\lambda = 1$ which is our problem (1.2). □

Remark 3.2. Notice that when $g(0) = h(0) = 0$ the solution given by Theorem 3.1 may be the trivial one $(0, 0)$. However, under our assumptions if moreover $g(0) \neq 0$ or $h(0) \neq 0$ we obtain a proper solution.

4. Monotone Nonlinearities

In the following lemma we give some estimates for the minimum eigenvalue of K_c.

Lemma 4.1. *Assume* (C0). *If one denotes by $\mu(c)$ the minimum of the eigenvalues of K_c, one has $\mu(c) = -\lambda_0^2$, where λ_0 is the maximum value between $1/2\pi$ and the greater positive solution of the equation*

$$\left(-1 - c\lambda^2 + 2c\lambda^3 \tan\left(\frac{1}{2\lambda}\right)\right)\left(-1 - c\lambda^2 + 2c\lambda^3 \tanh\left(\frac{1}{2\lambda}\right)\right) = 0. \tag{4.1}$$

More precisely, if one denotes by

$$c_0 = -\frac{\pi^3}{\pi - 2\tanh(\pi/2)} \approx -23.718, \qquad c_1 = -\frac{4\pi^3}{\pi - 2\tanh\pi} \approx -57.811, \tag{4.2}$$

one obtains that

(i) $|\mu(c)| = 1/4\pi^2$ *if* $c \in (-\infty, c_1] \cup (12, \infty)$,

(ii) $1/4\pi^2 < |\mu(c)| < 1/\pi^2$ *if* $c \in (c_1, c_0) \cup (-12, 0)$,

(iii) $|\mu(c)| = 1/\pi^2$ *if* $c = c_0$,

(iv) $1/\pi^2 < |\mu(c)|$ *if* $c \in (c_0, -12) \cup (0, 12)$.

Proof. By Lemma 2.2 the operator K_c is compact, so its set of eigenvalues is bounded and nonempty (see [8, Theorem VI.8]). Moreover we have that $\mu = -\lambda^2$ is a negative eigenvalue of K_c if and only if there exists a pair $(x, y) \in H_0^2 \times H_0^2$, $(x, y) \neq (0, 0)$, such that

$$\begin{aligned} -\lambda^2 x''(t) + y(t) &= -c\lambda^2 \int_0^1 y(s)ds, \\ -\lambda^2 y''(t) + x(t) &= -c\lambda^2 \int_0^1 x(s)ds, \\ x(0) = x(1) = 0, &\quad y(0) = y(1) = 0. \end{aligned} \tag{D}$$

Differentiating twice on the first equation of (D) and replacing on the second one, we arrive at the following equality:

$$-\lambda^4 x^{(4)}(t) + x(t) = -c\lambda^2 \int_0^1 x(t)dt. \tag{4.3}$$

In consequence

$$x(t) = a_1 \cos\left(\frac{t}{\lambda}\right) + b_1 \sin\left(\frac{t}{\lambda}\right) + c_1 e^{t/\lambda} + d_1 e^{-t/\lambda} + e_1. \tag{4.4}$$

Analogously, differentiating twice on the second equation of (D) and replacing on the first one, we arrive at

$$y(t) = a_2 \cos\left(\frac{t}{\lambda}\right) + b_2 \sin\left(\frac{t}{\lambda}\right) + c_2 e^{t/\lambda} + d_2 e^{-t/\lambda} + e_2. \tag{4.5}$$

Now, by means of the expression

$$y''(t) = \lambda^2 x^{(4)}(t), \tag{4.6}$$

we deduce that

$$a_2 = -a_1, \quad b_2 = -b_1, \quad c_2 = c_1, \quad d_2 = d_1, \tag{4.7}$$

and thus

$$y(t) = -a_1 \cos\left(\frac{t}{\lambda}\right) - b_1 \sin\left(\frac{t}{\lambda}\right) + c_1 e^{t/\lambda} + d_1 e^{-t/\lambda} + e_2. \tag{4.8}$$

So, we have that in the expression of the solutions of the two equations on system (D) six real parameters are involved. Now, to fix the value of such parameters, we use the four boundary value conditions imposed on problem(D) together with the fact that

$$e_1 = -c\lambda^2 \int_0^1 x(t)dt, \quad e_2 = -c\lambda^2 \int_0^1 y(t)dt. \tag{4.9}$$

Therefore, we arrive at the following six-dimensional homogeneous linear system:

$$\begin{aligned}
a_1 + c_1 + d_1 + e_1 &= 0,\\
a_1 \cos\left(\frac{1}{\lambda}\right) + b_1 \sin\left(\frac{1}{\lambda}\right) + c_1 e^{1/\lambda} + d_1 e^{-1/\lambda} + e_1 &= 0,\\
-a_1 + c_1 + d_1 + e_2 &= 0,\\
-a_1 \cos\left(\frac{1}{\lambda}\right) - b_1 \sin\left(\frac{1}{\lambda}\right) + c_1 e^{1/\lambda} + d_1 e^{-1/\lambda} + e_2 &= 0,\\
a_1\lambda \sin\left(\frac{1}{\lambda}\right) + b_1\left(1-\cos\left(\frac{1}{\lambda}\right)\right)\lambda + c_1\left(e^{1/\lambda}-1\right)\lambda + d_1\left(1-e^{-1/\lambda}\right)\lambda + e_1\left(1+\frac{1}{c\lambda^2}\right) &= 0,\\
a_1\lambda \sin\left(\frac{1}{\lambda}\right) + b_1\left(\cos\left(\frac{1}{\lambda}\right)-1\right)\lambda + c_1\left(e^{1/\lambda}-1\right)\lambda + d_1\left(1-e^{-1/\lambda}\right)\lambda + e_2\left(1+\frac{1}{c\lambda^2}\right) &= 0.
\end{aligned} \tag{4.10}$$

In consequence, the values of $\lambda > 0$ for which there exist nontrivial solutions of system (D) coincide with the zeroes of the determinant of the matrix

$$m = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0\\
\cos\left(\frac{1}{\lambda}\right) & \sin\left(\frac{1}{\lambda}\right) & e^{1/\lambda} & e^{-1/\lambda} & 1 & 0\\
-1 & 0 & 1 & 1 & 0 & 1\\
-\cos\left(\frac{1}{\lambda}\right) & -\sin\left(\frac{1}{\lambda}\right) & e^{1/\lambda} & e^{-1/\lambda} & 0 & 1\\
c\lambda^3 \sin\left(\frac{1}{\lambda}\right) & c\lambda^3\left(1-\cos\left(\frac{1}{\lambda}\right)\right) & c(-1+e^{1/\lambda})\lambda^3 & c\lambda^3(1-e^{-1/\lambda}) & c\lambda^2+1 & 0\\
-c\lambda^3 \sin\left(\frac{1}{\lambda}\right) & c\lambda^3\left(\cos\left(\frac{1}{\lambda}\right)-1\right) & c(-1+e^{1/\lambda})\lambda^3 & c\lambda^3(1-e^{-1/\lambda}) & 0 & c\lambda^2+1
\end{pmatrix}, \tag{4.11}$$

that is

$$\operatorname{Det}(m) = 4e^{-1/\lambda}\left(-1+e^{1/\lambda}\right)d(c,\lambda) = 0, \tag{4.12}$$

where

$$d(c,\lambda) = \left(-c(2\lambda+1)\lambda^2 + e^{1/\lambda}\left(c\lambda^2(2\lambda-1)-1\right)-1\right) \\ \times \left(2c\left(\cos\left(\frac{1}{\lambda}\right)-1\right)\lambda^3 + \left(c\lambda^2+1\right)\sin\left(\frac{1}{\lambda}\right)\right). \tag{4.13}$$

We notice that for all $n \in \mathbb{N}$ we have

$$d\left(c, \frac{1}{2n\pi}\right) = 0, \tag{4.14}$$

and for all $\lambda \neq 1/n\pi$, with n odd,

$$d(c,\lambda) = \left(-1-e^{1/\lambda}\right)\sin\left(\frac{1}{\lambda}\right)\left(2c\lambda^3\tan\left(\frac{1}{2\lambda}\right)-c\lambda^2-1\right)\left(2c\lambda^3\tanh\left(\frac{1}{2\lambda}\right)-c\lambda^2-1\right). \tag{4.15}$$

Hence, $\lambda_1 = 1/2\,\pi$ is the greatest zero among the sequence $1/2n\pi$. On the other hand, since $c \neq 0$, $\lambda = 1/\pi$ is solution of (4.15) if and only if $c = c_0$ and the remaining solutions $\lambda > 1/2\pi$ are the zeroes of the last two factors on (4.15). A careful study shows that function

$$p(c,\lambda) = -1 - c\lambda^2 + 2c\lambda^3\tanh\left(\frac{1}{2\lambda}\right) \tag{4.16}$$

is such that $cp(c,\lambda)$ is strictly decreasing on $(0,+\infty)$. Moreover

$$\lim_{\lambda\to\infty} p(c,\lambda) = -\frac{1}{12}(c+12), \qquad p\left(c,\frac{1}{\pi}\right) = -\frac{\pi c - 2c\tanh(\pi/2)+\pi^3}{\pi^3}. \tag{4.17}$$

In consequence, there is a (unique) solution greater than $1/\pi$ of the equation $p(c,\lambda) = 0$ if and only if $c \in (c_0, -12)$. Moreover the greatest zero of function $p(c,\cdot)$ belongs to the interval $(1/2\,\pi, 1/\pi)$ if and only if $c \in (c_1, c_0)$.

On the other hand, function

$$q(c,\lambda) = -1 - c\lambda^2 + 2c\lambda^3\tan\left(\frac{1}{2\lambda}\right) \tag{4.18}$$

satisfies that $cq(c,\lambda)$ is strictly decreasing on its domain $\{\lambda > 0 : \lambda \neq 1/n\pi \text{ with } n \text{ odd}\}$, and

$$\lim_{\lambda\to 1/\pi^-} cq(c,\lambda) = -\infty, \qquad \lim_{\lambda\to 1/\pi^+} cq(c,\lambda) = +\infty, \qquad \lim_{\lambda\to\infty} q(c,\lambda) = \frac{1}{12}(c-12). \tag{4.19}$$

So, there is a (unique) solution greater than $1/\pi$ of the equation $q(c,\lambda) = 0$ if and only if $c \in (0,12)$. Moreover, since

$$q\left(c, \frac{1}{2\pi}\right) = -\frac{c}{4\pi^2} - 1, \tag{4.20}$$

it has its greatest zero between $1/2\pi$ and $1/\pi$ if and only if $-4\pi^2 < c < 0$. □

Let $\mathscr{H}$ denote the class of strictly increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. We introduce the following assumption:

(H') $g \in \mathscr{H}$ and $h \in \mathscr{H}$.

Let us define the functional $J_c : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ given by

$$J_c(x,y) := \frac{1}{2}\langle K_c(x(s),y(s)),(x(s),y(s))\rangle_{L^2\times L^2} + \int_0^1 [G^*(x(s)) + H^*(y(s))]ds, \tag{4.21}$$

where $G^*(t) := \int_0^t g^{-1}(r)dr$ and $H^*(t) := \int_0^t h^{-1}(r)dr$ for all $t \in \mathbb{R}$.

Notice that G^* and H^* are the Fenchel transform of g and h (see [11]).

Theorem 4.2. *Assume* (H').*Let c satisfy* (C0) *and in addition*

$$\limsup_{|x|\to\infty}\left|\frac{g(x)}{x}\right| < \frac{1}{|\mu(c)|}, \qquad \limsup_{|x|\to\infty}\left|\frac{h(x)}{x}\right| < \frac{1}{|\mu(c)|}. \tag{4.22}$$

Then J_c attains a minimum at some point (x_0,y_0).

Moreover, $(-u_0,-v_0)$ is a solution of (1.2)*, where we put $(v_0,u_0) = K_c(x_0,y_0)$.*

Proof.

Claim 1 (J_c attains a minimum at some point (x_0,y_0)). The space $L^2(0,1)\times L^2(0,1)$ is reflexive, and by our assumptions J_c is weakly sequentially lower semicontinuous. In fact, J_c is the sum of a convex continuous functional (corresponding to the two last summands in the integrand) with a weakly sequentially continuous functional (because of the compactness of K_c). So, in order to prove that J_c has a minimum, it is enough to show that J_c is coercive. By (4.22) we take $\alpha > |\mu(c)|/2 > 0$ such that

$$\limsup_{|x|\to\infty}\left|\frac{g(x)}{x}\right| < \frac{1}{2\alpha}, \quad \limsup_{|y|\to\infty}\left|\frac{h(y)}{y}\right| < \frac{1}{2\alpha}. \tag{4.23}$$

So, there exists $k > 0$ such that

$$\begin{aligned} |g(x)| &\le \frac{|x|}{2\alpha} + k, \quad \forall x \in \mathbb{R}, \\ |h(y)| &\le \frac{|y|}{2\alpha} + k, \quad \forall y \in \mathbb{R}. \end{aligned} \tag{4.24}$$

Thus, for every $\epsilon > 0$, there exists $a > 0$ such that we have

$$G^*(x) \geq (\alpha - \epsilon)x^2 - a, \quad \forall x \in \mathbb{R},$$
$$H^*(y) \geq (\alpha - \epsilon)y^2 - a, \quad \forall y \in \mathbb{R}. \tag{4.25}$$

On the other hand (see [8, Proposition VI.9]),

$$\begin{aligned}\langle K_c(x(s), y(s)), (x(s), y(s))\rangle_{L^2\times L^2} &\geq \mu(c)\langle (x(s), y(s)), (x(s), y(s))\rangle_{L^2\times L^2} \\ &= \mu(c)\int_0^1 \left(x^2(s) + y^2(s)\right)ds.\end{aligned} \tag{4.26}$$

Taking ϵ such that $2(\alpha - \epsilon) > |\mu(c)|$, we have

$$J_c(x,y) \geq \frac{\mu(c)}{2}\int_0^1 \left(x^2(s) + y^2(s)\right)ds + (\alpha - \epsilon)\int_0^1 \left(x^2(s) + y^2(s)\right)ds - 2a, \tag{4.27}$$

and therefore J_c is coercive.

Claim 2. If we denote $(v_0, u_0) = K_c(x_0, y_0)$ then $(-u_0, -v_0)$ is a solution of (1.2).

Since (x_0, y_0) is a critical point of J_c then for all $(\overline{x}, \overline{y}) \in L^2(0,1) \times L^2(0,1)$ we have

$$\begin{aligned}J'_c(x_0, y_0)(\overline{x}, \overline{y}) &= \langle K_c(x_0(s), y_0(s)), (\overline{x}(s), \overline{y}(s))\rangle_{L^2\times L^2} \\ &\quad + \int_0^1 \left[g^{-1}(x_0(s))\overline{x}(s) + h^{-1}(y_0(s))\overline{y}(s)\right]ds \\ &= 0,\end{aligned} \tag{4.28}$$

which implies that $v_0(s) + g^{-1}(x_0(s)) = 0$ and $u_0(s) + h^{-1}(y_0(s)) = 0$ for a.e. $s \in [0,1]$, where we put $(v_0, u_0) = K_c(x_0, y_0)$. Then $(-u_0, -v_0)$ is a solution of (1.2). □

Remark 4.3. Under the more restrictive assumption

$$\sup_{v,w\in\mathbb{R}} \frac{g(v) - g(w)}{v - w} < \frac{1}{|\mu(c)|}, \qquad \sup_{v,w\in\mathbb{R}} \frac{h(v) - h(w)}{v - w} < \frac{1}{|\mu(c)|}, \tag{4.29}$$

it follows that J'_c is a strictly monotone operator (see [11]). Hence, when (4.29) holds, J_c has a unique critical point. The argument of Claim 2 in previous theorem shows that there is a one-to-one correspondence between critical points of J_c and the solutions to (1.2). In consequence, the solution of problem (1.2) is unique.

Remark 4.4. Suppose that under the conditions of the theorem, $g(0) = h(0) = 0$. If moreover

$$\liminf_{z\to 0}\frac{g(z)}{z} > \frac{1}{|\mu(c)|}, \qquad \liminf_{z\to 0}\frac{h(z)}{z} > \frac{1}{|\mu(c)|}, \tag{4.30}$$

we claim that the solution given by the theorem is not the trivial one $(0,0)$. In fact let $(\overline{x},\overline{y})$ be a normalized eigenvector associated to $\mu(c)$. The properties of eigenvectors imply that x and y are in fact continuous functions. Since (4.30) implies $2G^*(z) \leq kz^2$ and $2H^*(z) \leq kz^2$ for some $k < -\mu(c)$ and $|z|$ small, an easy computation implies that

$$J_c\left(t(\overline{x},\overline{y})\right) < 0 \tag{4.31}$$

for t sufficiently small. Hence the minimum of J_c is not attained at $(0,0)$.

Remark 4.5. If $g(0) > 0$ and $h(0) \geq 0$ or $g(0) \geq 0$ and $h(0) > 0$, we have that $\gamma = (0,0)$ is a lower solution. Moreover if $0 < c < 12$ and

$$\limsup_{x\to\infty}\frac{g(x)}{x} < \frac{2}{3}(12-c), \qquad \limsup_{x\to\infty}\frac{h(x)}{x} < \frac{2}{3}(12-c), \tag{4.32}$$

then we can take an upper solution of the form $\beta = a(t(1-t), t(1-t))$ with $a > 0$ and then apply the monotone method.

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