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Shape Parameterization of the Time-Dependent Geometry of the Heart for Steady Fluid Dynamical Analysis

C.J. EVANS*, M.I.G. BLOOR and M.J. WILSON

Department of Applied Mathematics University of Leeds LS2 9JT United Kingdom

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A parametric model of the complex time-dependent geometry of the ventricles of the human heart is constructed. The geometry model is created by means of a boundary value approach, solving an elliptic partial differential equation to generate a representation of the inner surface of the ventricles. The technique provides a closed-form description of the geometry with the advantage that the geometry can be readily changed without introducing holes or discontinuities in the surface. It also allows a straightforward link to analysis, facilitating the calculation of physical properties such as those relevant to fluid dynamics. As an application of this work, the geometry model is combined with commercial CFD software to analyse the blood flow in the heart. Steady-state calculations are performed at various time steps to follow the evolution of the fluid flow.

Keywords: geometry; shape; parameterization; ventricles; fluid; dynamics

1 INTRODUCTION

This paper deals with the geometric modelling of the left and right ventricles of the heart. Pedley [10] describes the left ventricle as roughly circular in cross-section, and shaped rather like a blunted arrow-head. When it contracts, there is an initial phase in which the long axis shortens slightly and the transverse cross section expands. Then the aortic valve opens and the long axis remains roughly the same size while the transverse axes shorten by around a third. The right ventricle is more complex in shape. The interventricular wall is functionally part of the left ventricle but the outer wall of the right ventricle is significantly thinner and has a much larger area, resulting in a crescent-like shaped cavity wrapped around the left ventricle. Both chambers have inlet and outlet valves which act in such a way that one closes before the other opens so that (ideally) no backflow occur. More information on cardiac physiology can be found in any standard text, such as Smith and Kampine [14].

The shape of the ventricles is crucial to their functionality so a realistic parametric geometric model is extremely valuable in any related investigation. There are two main avenues of approach when considering the geometric modelling of biological systems. Firstly, scan data can be directly used in order to make possible accurate representations of the object in question. With regard to the ventricles of the heart

* Email: chrise@amsta.leeds.ac.uk

this approach has been used by Taylor and Yamaguchi [16], Park et al. [9] and Haber et al. [5] amongst others. Often a simple geometric primitive is deformed to somehow match the scan surface data as closely as possible. While these techniques can provide good approximations to the actual geometry in an individual case, a general investigation into the effects of modifications to the shape can be more difficult to perform, due to the way the individual data is directly involved in the surface generation.

The second avenue is to create generic representations of the object to be modelled (informed by available information, including scan data). The work of Peskin and McQueen [11] is some of the most advanced to date. Their model encompasses both ventricles and atria and also the major arteries connected to the heart. They build up the heart surface by specifying the position of muscle fibres in the heart walls which are connected to the fluid flow using their Immersed Boundary Method. This technique considers the elastic fibres to occupy zero volume and move at the local fluid velocity. A set of differential equations which couple the fibre mechanics and fluid dynamics are then solved. The calculations required to do this are extremely computationally intensive due to the complexity of the model. Yoganathan et al. [18] have also adapted Peskin's method to study a thin-walled left ventricle during early systole. The computational time required to perform these calculations, however, makes it difficult to conduct general investigations into different aspects of the motion so other work has used greatly simplified generic ventricle geometry to look at the effects of disease upon the fluid flow in the heart. For example Schoephoerster [4], [13] uses a spherical left ventricle to examine the effects of abnormal wall motion on the flow dynamics.

With this in mind, the main aim of the present study was to create a generic parametric model of the ventricles of the heart which lay in between the above extremes. Parametric, in this sense, means that the geometry is defined by a set of 'shape' or 'design' parameters and can be altered by varying these numbers in a controlled way. The particular parameterization is of crucial importance because the 'parameter space' must include enough suitable variation of the surfaces to allow a general investigation to proceed, whilst not being too large to make this practical. In particular, in the case of the ventricles of the heart, we wished the parameterization to easily allow

1. the contraction of the ventricles during the cardiac cycle to be realistically modelled, only requiring the variation of a small number of the design parameters, and
2. enough flexibility in the ventricle shape to make possible investigation into the effects of both differences in individual hearts and also abnormal shape and/or motion such as that caused by various types of heart disease.

By allowing parameters to be functions of time, we create a time-dependent model of the beating ventricles of the heart.

The technique which is used to generate the geometry in this work has been applied in the past to many other design problems, including the efficient parameterization of aircraft [1], propeller blades [3] and ship hulls [7]. Surfaces are produced by specifying boundary curves which reflect the key features of the shape to be modelled, and then forming smooth surface patches between them. A consequence of this approach is that only a small number of design parameters are required because the shape is entirely determined by the information specified around the boundary curves. Moreover, even using a small parameter space, a wide range of geometries can be generated. The current work differs from previous design work in that parameters can be functions of time as stated previously. Details of this method and the particular parameterization used can be found in section 2. This method improves upon many conventional CAD techniques in that

- the parameterization is efficient. Conventional Computer Aided Design (CAD) methods can require hundreds if not thousands of control points (parameters) whereas this technique uses a few tens (around twenty).
- the surface can be readily modified. Surfaces comprising several PDE patches maintain their connectivity and continuity as the shape is

changed. Again, this differs from many current CAD methods.

- it provides a straightforward link to analysis, facilitating the calculation of physical properties of the surface, such as velocity, curvature or mechanical stress.

As an initial application of our geometric model, we have employed computational fluid dynamics (CFD) software to investigate the blood flow in the ventricles during the cardiac cycle.

In this paper, we shall describe the parameterization method, demonstrate how it is applied to the problem of the ventricles and show the resulting geometry. We will also give an example of the type of analysis that can be performed using the model, in this case a fluid dynamical investigation. The following section describes our methodology for producing the geometry model, while in sections 4 and 5 we give the fluid flow results and some discussion.

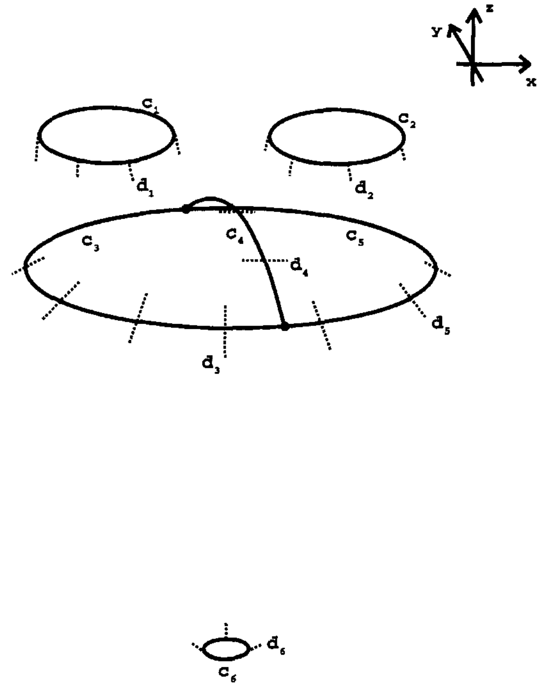

FIGURE 1 Boundary conditions used for left ventricle geometry. d_0, d_1 etc. represent the derivative conditions associated with each curve

2 GENERAL SHAPE PARAMETERIZATION

As mentioned previously, the technique for shape parameterization that is used defines surfaces in terms of boundary curves and derivative information about the surface on these curves. This information forms the boundary conditions used in solving a particular elliptic partial differential equation (PDE). The solution to this PDE represents a smooth surface passing through the specified boundary curves and combinations of these surfaces define the shape being modelled.

The PDE used in most of the work to date is a version of the biharmonic equation:

$$\left(\frac{\partial^2}{\partial u^2} + a^2 \frac{\partial^2}{\partial v^2}\right)^2 \boldsymbol{X}(u,v) = 0 \qquad (1)$$

where $\boldsymbol{X}$: $(u, v) \rightarrow (x, y, z)$ with (u, v) contained in some finite subset of R^2. a is called the smoothing parameter and controls the relative scaling between the u and v directions.

Suitable boundary conditions would then be of the form:

$$\boldsymbol{X}(0,v) = \boldsymbol{f}_0(v), \boldsymbol{X}(1,v) = \boldsymbol{f}_1(v) \qquad (2)$$
$$\boldsymbol{X}_u(0,v) = \boldsymbol{s}_0(v), \boldsymbol{X}_u(1,v) = \boldsymbol{s}_1(v)$$

Here, f_0 and f_1 define the shapes of the edge curves parametrised in terms of the variable v. s_0 and s_1 are the corresponding derivatives, again parametrised by v. The geometric parameters of the model appear within the functions f_0, f_1, s_0, s_1. Note that more complex surfaces can easily be created by simply having separate PDE surfaces with a common boundary curve. Continuity is assured by setting consistent (in this case identical) derivative conditions on the mutual curve. As mentioned previously, this means that if the curve is altered, both adjacent patches are automatically updated, maintaining a smoothly connected surface without holes.

The solution of this PDE is a straightforward process resulting in an (in general) infinite Fourier series which computationally is of limited value. A method

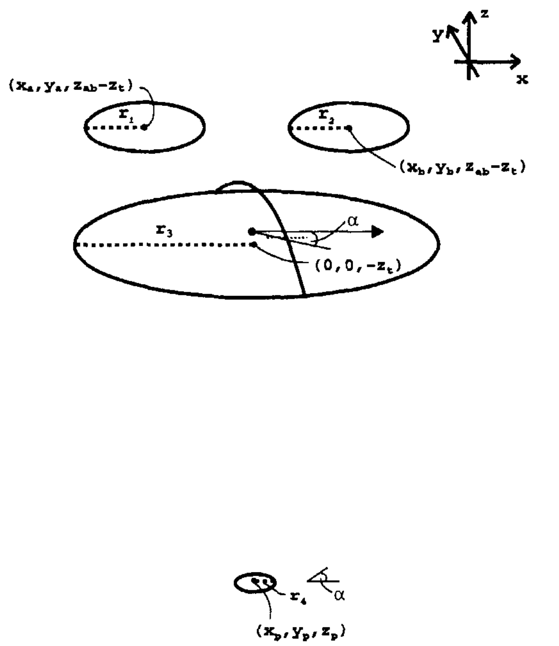

FIGURE 2 Schematic diagram of the curves and parameters used for the left ventricle

has therefore been developed in previous work to form a surface which exactly meets the boundary conditions of the problem and closely approximates the exact solution. Essentially the Fourier series is terminated after a suitable number of terms and then a 'remainder' function is added whose purpose is to ensure that the surface satisfies the boundary conditions. The new surface will is 'close' to the true solution of the PDE and is still represented in closed form. Details can be found in the appendix but the crucial point is that the new surface exactly satisfies the boundary conditions (even though it will not in general be an exact solution of the PDE), since to not do so would cause holes and/or discontinuities between surface patches.

The generation of the geometry using this technique is a very quick process, taking a fraction of a second (quick enough for real-time animation) for the ventricles. It must be performed at each time step of interest, so first the time step to be considered is chosen, then the time-dependent parameters are calculated at this stage, then the PDE method is applied to generate the surface of the ventricles.

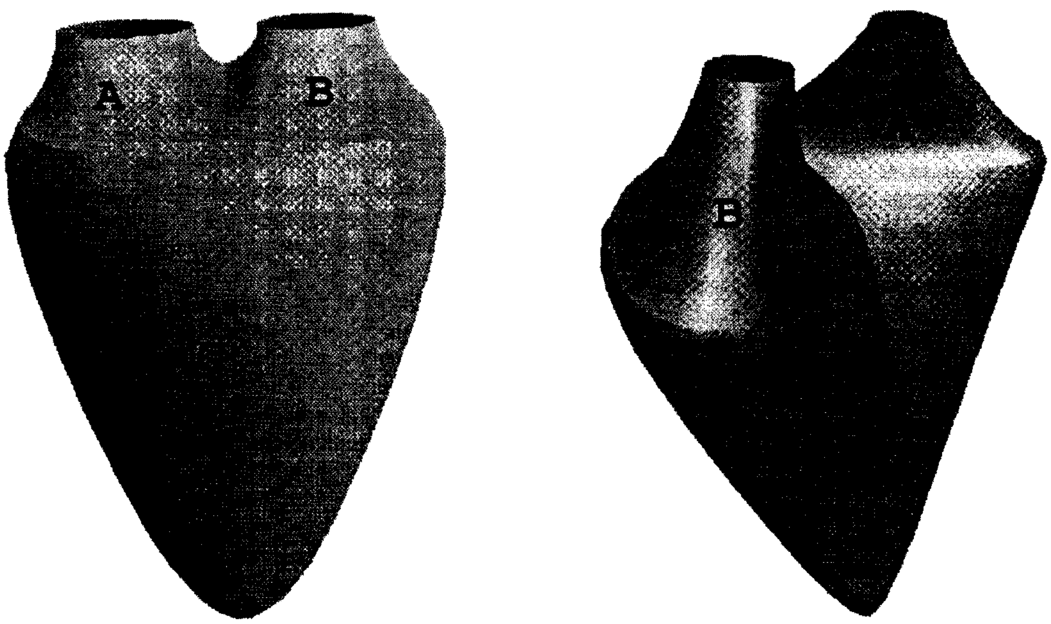

FIGURE 3 Resulting geometry for the (a) left and (b) right ventricles

3 MODELLING VENTRICULAR SHAPE

To demonstrate the above, boundary conditions for the modelling of the left ventricle are pictured in a schematic in figure 1. This case involves three seperate PDE patches. The dotted lines illustrate the derivative boundary conditions which define the surface tangent at every point on the curves. This situation would translate to the following mathematical formulation (using the notation of equation 2).

$$\begin{array}{llll} \text{Patch A} & f_0 = \boldsymbol{c}_1 & f_1 = \boldsymbol{c}_3 \cup \boldsymbol{c}_4 & \\ & s_0 = \boldsymbol{d}_1 & s_1 = \boldsymbol{d}_3 \cup \boldsymbol{d}_4 & \\ \text{Patch B} & f_0 = \boldsymbol{c}_2 & f_1 = \boldsymbol{c}_4 \cup \boldsymbol{c}_5 & \\ & s_0 = \boldsymbol{d}_2 & s_1 = \boldsymbol{d}_4 \cup \boldsymbol{d}_5 & \\ \text{Patch C} & f_0 = \boldsymbol{c}_3 \cup \boldsymbol{c}_5 & f_1 = \boldsymbol{c}_6 & \\ & s_0 = \boldsymbol{d}_3 \cup \boldsymbol{d}_5 & s_1 = \boldsymbol{d}_6 & (3) \end{array}$$

The precise definitions of these curves are based on information about the actual shape to be modelled. Since the left ventricle is roughly circular in cross-section, the majority of curves are circles or parts of circles. Figure 2 illustrates the design parameters that form part of the definition of the curves and the form of the curve definitions are shown below.

$$\boldsymbol{c}_1(v) = (x_a + r_1 \cos(v + \alpha(t)), \\ y_a + r_1 \sin(v + \alpha(t)), z_{ab} - z(t))$$

$$\boldsymbol{c}_2(v) = (x_b + r_2 \cos(v + \alpha(t)), \\ y_b + r_2 \sin(v + \alpha(t)), z_{ab} - z(t))$$

$$\boldsymbol{c}_3(v) = (r_3(t) \cos(v + \alpha(t)), \\ r_3(t) \sin(v + \alpha(t)), -z(t)) \\ (-\pi/2 \leq v \leq \pi/2)$$

$$\boldsymbol{c}_5(v) = (r_3(t) \cos(v + \alpha(t)), \\ r_3(t) \sin(v + \alpha(t)), -z(t)) \\ (\pi/2 \leq v \leq 3\pi/2)$$

$$\boldsymbol{c}_6(v) = (x_p + r_4(t) \cos(v - \alpha(t)), \\ y_p + r_4(t) \sin(v - \alpha(t)), z_p - z(t))$$

where t is the time and

$$z(t) = z_{ch} * \sin^2(t\pi/t_m)$$

$$r_3(t) = r_3(1 - r_{ch} \sin^2(t\pi/t_m))$$

$$\alpha(t) = \alpha_{ch} * \sin^2(t\pi/t_m)$$

t_m is the time period of the cardiac cycle (taken to be about 0.8sec).

The conditions used for the right ventricle were similar but the largest curve is a crescent type shape rather than a circle as for the left ventricle.

These equations are the means by which the ventricular contraction is implemented using a simple sinusoidal type variation in time (although more complex time dependence could easily be used). Information about the changes in ventricular shape during a heartbeat can be found in various physiology texts, eg Smith and Kampine [14] and the three components of the motion that have been included above are:

- 'axial' contraction. The axis in question stretches from the plane of the valves to the apex of the ventricles and is controlled by the function $z(t)$. z_{ch}- controls the amount of contraction and is typically taken to be 0.7 cm.
- 'radial' contraction. Controlled by $r_3(t)$. r_{ch} controls the amount of contraction and is typically taken to be 1/3.
- 'twisting' or 'wringing' motion. This is caused in actuality by the contraction (expansion) of the muscle fibres which form the heart wall during systole (diastole). The fibres stretch round the ventricle in a helical-like curve so as they shorten the heart 'twists'. While the grid lines forming the model surface do not correspond to the fibres, the effect is mimicked by rotating boundary curves and this is controlled by $\alpha(t)$. α_{ch} represents the amount of twisting included in the motion and a value of $\pi/4$ has been used.

With these three modes of contraction included, realistic changes in volume and dimensions can be achieved.

Figure 3 shows pictures of the final ventricle surfaces. The lines on the pictures are symptoms of the computer graphics rather than discontinuities of the surface. Movies of the ventricles beating are available on the internet.

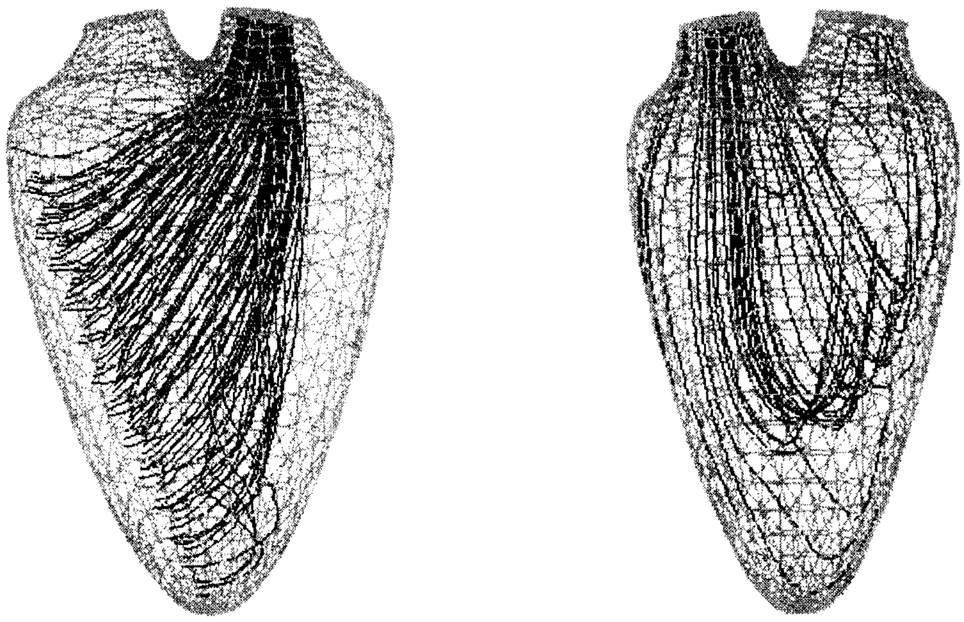

FIGURE 4 Particle paths in the left ventricle during (a) systole and (b) diastole

4 STEADY-STATE FLUID DYNAMICAL ANALYSIS

As an initial application of the parameterized geometry, a computational fluid dynamics (CFD) analysis was carried out. A commercial software package called Fluent was used for this purpose. Fluent uses finite volume techniques to solve conservation equations for mass and momentum by means of a first order upwind scheme. Prior to this process, an internal tetrahedral volume mesh must be generated. The software modules PreBFC and Tgrid were used to automatically perform this process. Steady state calculations were performed at time steps throughout the cardiac cycle, treating blood as a Newtonian, incompressible fluid with density $\rho = 1050\ kgm^{-3}$ and viscosity $\nu = 0.004\ kgm^{-1}s^{-1}$. The flow was assumed to be laminar. These calculations were carried out on modest Silicon Graphics workstations on which the added computation required to solve the unsteady problem would not have been practical.

4.1 Flow Boundary Conditions

In the Fluent solver, a boundary can be specified as (amongst other things):

- a velocity inlet. Here the velocity of the fluid in each cell on the boundary is specified.
- a wall. Usual no-slip conditions are enforced
- an outflow. This condition is used when velocities and pressures on the boundary are not known prior to the solution of the problem. Fluent extrapolates the required information from the interior.

The simplest case was in systole where one valve was taken to be an outflow, the other a wall and all other regions were treated as velocity inlets. As we specify how the heart wall moves throughout the cycle, the instantaneous velocities of the grid points on the surface can easily be calculated.

For the diastolic case we again wished to specify the velocities of each point of the grid as boundary conditions for the flow. Therefore the majority of the surface was once again treated as a velocity inlet with

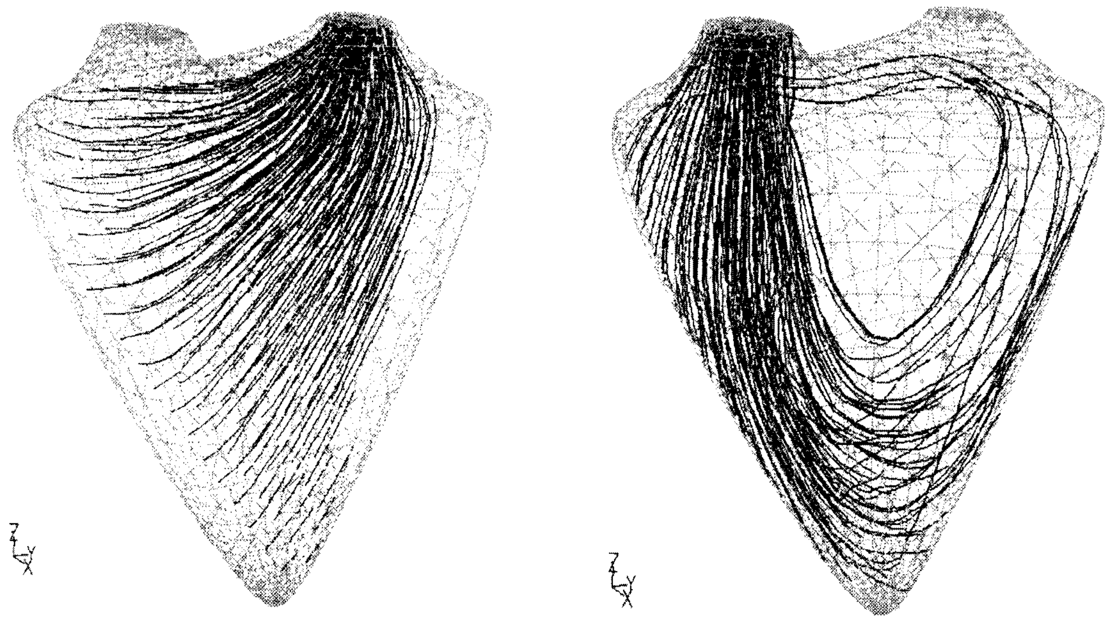

FIGURE 5 Particle paths in the right ventricle during (a) systole and (b) diastole

'negative' velocity since the ventricle is expanding during this part of the cycle. The inflow valve was treated as an outflow (but fluid still enters through it) in order that velocities did not have to be previously specified, and the aortic valve, since it shuts for the majority of diastole, was said to be a wall.

4.2 Results of Flow Calculations

Pictures of particle path lines for the left and right ventricles in systole and diastole are shown in figures 4 and 5. Figure 6 shows contour plots of velocity magnitude in the left ventricle. Similar pictures can be produced for any instant during the cardiac cycle.

5 DISCUSSION

A generic model of the geometry of the ventricles of the heart has been obtained using the PDE method, and time-dependence has been built in to the parameterization allowing the model to realistically beat. The geometry has a closed form representation providing a straightforward link to methods for physical analysis. To demonstrate this the model has been used in conjunction with the Fluent computational fluid dynamics software to perform steady state flow calculations at stages throughout the cardiac cycle.

There are many possible applications of this work. Firstly, as has already been mentioned, it should be possible using more advanced CFD software to perform unsteady, turbulent calculations on the same basis as described above to better approximate the conditions in the ventricles.

Secondly, the modelled geometry could be extended to include the actual geometry of the valves of the ventricles, again using the PDE method.

Finally, one major advantage of the parameterization is that changes in the geometry and motion of the ventricles can be easily introduced into the model. In particular it should be possible to adapt the flow calculation in order to incorporate the effect(s) of types of heart disease. One possibility is to examine *ischemia*, a condition in which a portion of the muscle

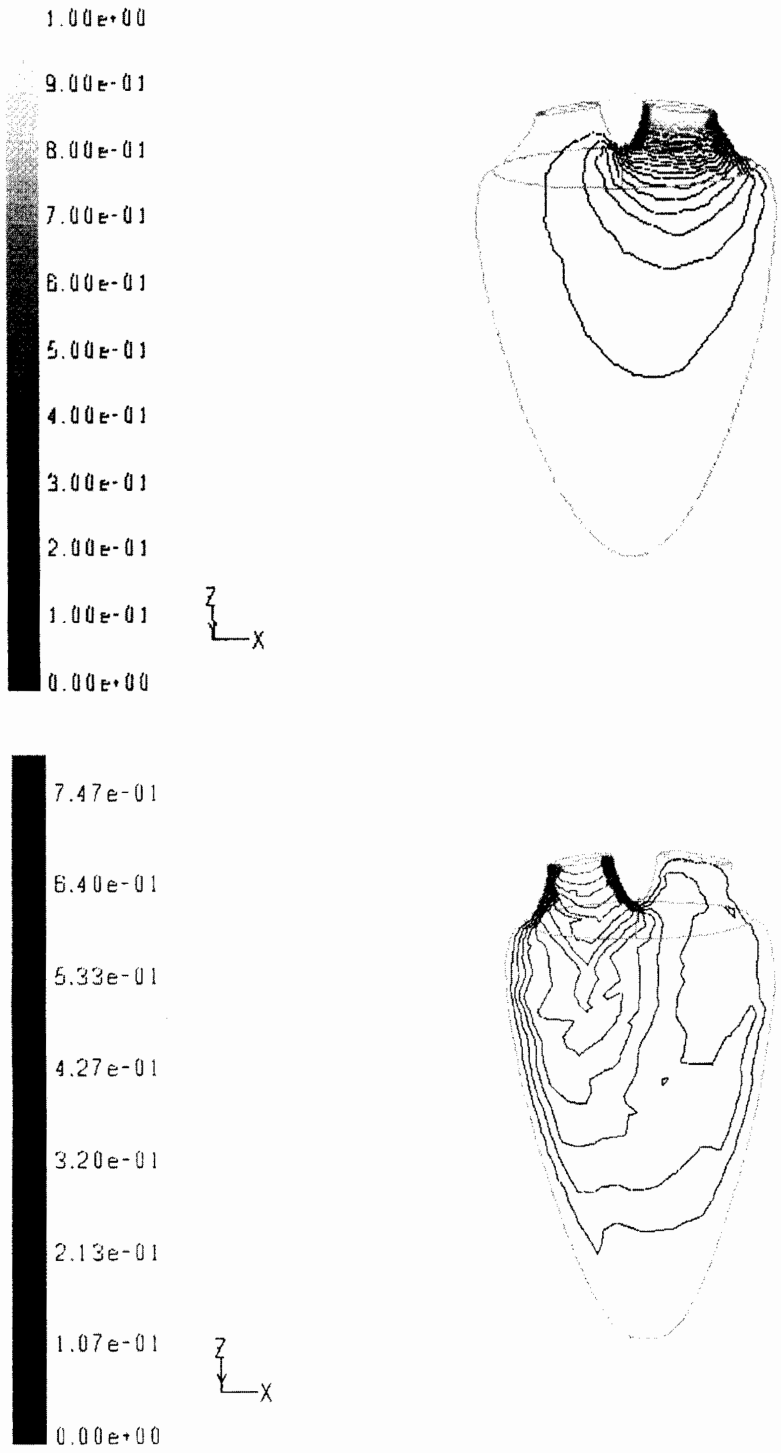

FIGURE 6 Contour plots of velocity magnitude (in m/s) in the left ventricle during (a) systole and (b) diastole

surrounding the heart dies, affecting the way in which the ventricles contract.

6 ACKNOWLEDGEMENTS

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APPENDIX – APPROXIMATING THE PDE SURFACE

We demonstrate here the method used to create a closed form approximation to the PDE surface that exactly meets our boundary conditions. We are interested in solving the PDE:

$$\left(\frac{\partial^2}{\partial u^2} + a^2\frac{\partial^2}{\partial v^2}\right)^2 \boldsymbol{X}(u,v) = 0 \qquad (4)$$

subject to boundary conditions typically of the form:

$$\boldsymbol{X}(0,v) = \boldsymbol{f}_0(v), \boldsymbol{X}(1,v) = \boldsymbol{f}_1(v)$$
$$\boldsymbol{X}_u(0,v) = \boldsymbol{s}_0(v), \boldsymbol{X}_u(1,v) = \boldsymbol{s}_1(v)$$

The general solution to this problem is of the form:

$$\boldsymbol{X}(u,v) = \boldsymbol{A}_0(u) + \sum_{n=1}^{\infty}[\boldsymbol{A}_n(u)\cos(nv) + \boldsymbol{B}_n(u)\sin(nv)]$$

where

$$\boldsymbol{A}_0(u) = \boldsymbol{a}_{00} + \boldsymbol{a}_{01}u + \boldsymbol{a}_{02}u^2 + \boldsymbol{a}_{03}u^3$$
$$\boldsymbol{A}_n(u) = \boldsymbol{a}_{n1}e^{anu} + \boldsymbol{a}_{n2}ue^{anu} + \boldsymbol{a}_{n3}e^{-anu} + \boldsymbol{a}_{n4}ue^{-anu}$$
$$\boldsymbol{B}_n(u) = \boldsymbol{b}_{n1}e^{anu} + \boldsymbol{b}_{n2}ue^{anu} + \boldsymbol{b}_{n3}e^{-anu} + \boldsymbol{b}_{n4}ue^{-anu}$$

Here, $\boldsymbol{a}_{nm}$ and $\boldsymbol{b}_{nm}$ are all constant vectors.

The solution to Equation (4) subject to (5) will sometimes be a finite series, but in general it will be necessary to Fourier analyse the functions f_0, f_1, s_0 and s_1 and the resulting series will be infinite. In order to numerically generate a surface, a means of providing an approximate (but nevertheless analytic) solution is necessary.

We now demonstrate a method (again devised by Bloor and Wilson) for doing this.

We call the function obtained by terminating the Fourier series after N terms $\boldsymbol{F}(u, v)$ and define a 'remainder' term, $\boldsymbol{R}(u, v)$, designed to make the solution satisfy the boundary conditions exactly. So we now have

$$\boldsymbol{X}(u,v) = \boldsymbol{F}(u,v) + \boldsymbol{R}(u,v) \quad \text{where} \qquad (7)$$
$$\boldsymbol{F}(u,v) = \boldsymbol{A}_0(u) + \sum_{n=1}^{N}[\boldsymbol{A}_n(u)\cos(nv) + \boldsymbol{B}_n(u)\sin(nv)]$$

In practise, taking N to be 5 is often sufficient.

We now need to define $\boldsymbol{R}$. There is leeway in this choice, but in most of the work to date the following form has been used.

$$\boldsymbol{R}(u,v) = \boldsymbol{r}_1(v)e^{\omega u} + \boldsymbol{r}_2(v)ue^{\omega u} + \boldsymbol{r}_3(v)e^{-\omega u} + \boldsymbol{r}_4(v)ue^{-\omega u} \qquad (8)$$

ω is an arbitrary constant. It acts as a kind of smoothing parameter local to the boundaries. This helps to give the designer/modeller even more control over the surface, and can be chosen to give the most suitable shape. $\boldsymbol{R}$ can now be completely determined by the insistence that X satisfy the boundary conditions exactly. This means that we need to define $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4$ such that

$$\boldsymbol{X}(0,v) = \boldsymbol{F}(0,v) + \boldsymbol{R}(0,v) = \boldsymbol{f}_0(v)$$
$$\boldsymbol{X}(1,v) = \boldsymbol{F}(1,v) + \boldsymbol{R}(1,v) = \boldsymbol{f}_1(v)$$
$$\boldsymbol{X}_u(0,v) = \boldsymbol{F}_u(0,v) + \boldsymbol{R}_u(0,v) = \boldsymbol{s}_0(v)$$
$$\boldsymbol{X}_u(1,v) = \boldsymbol{F}_u(1,v) + \boldsymbol{R}_u(1,v) = \boldsymbol{s}_1(v)$$

We have four equations for the four unknown functions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4$, which means that $\boldsymbol{R}$ is determined and X now satisfies the boundary conditions exactly.

The question does arise of how close this solution is to the exact solution of the original problem. In fact it can be shown that the higher the Fourier mode, the more quickly the coefficient functions decay away from the boundaries of the surface. Thus $\boldsymbol{R}$, which represents the high frequency Fourier terms is small except close to the boundaries for sufficiently large n.