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Fixed Points of Log-Linear Discrete Dynamics

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In this paper we study the fixed points of the Log-linear discrete dynamics. We show that almost all Log-linear dynamics have at most two fixed points which is a generalization of Soni's result.

Keywords: Log-linear discrete dynamics, Fixed points

1 INTRODUCTION

The log-linear discrete dynamics

$$f_i(x_1,\ldots,x_n) = \frac{c_i x_1^{a_{i1}} \cdots x_n^{a_{in}}}{\sum_{j=1}^n c_j x_1^{a_{j1}} \cdots x_n^{a_{jn}}}, \quad i = 1,\ldots,n,$$

have been studied originally as a socio-spacial dynamic model by Dendrinos and Sonis [1]. Many interesting phenomena, for example strange attractors, pitch folk like bifurcations and invariant circles [1–5] have been found to be contained in them.

The log-linear dynamics maps depict a family of dynamics defined systematically by matrix $A = (a_{ij})$ and vector $\vec{c} = (c_1,\ldots,c_n)^{\mathrm{T}}$; like other such families of dynamics (for instance the Lotka–Volterra dynamics) they are a definitive object of mathematical studies. Therefore a thorough analysis of the log-linear dynamics is necessary because of the importance not only from an applicational view point but also from a pure mathematical view point.

In this paper we investigate the fixed points of the dynamics as our first step of a more extended mathematical study of the log-linear discrete dynamics. We define a real valued function on R, which plays a key role in counting the number of the fixed points found in the map, and we prove that almost all dynamics have at most two fixed points. This result is a generalization of Sonis's result [4].

2 DEFINITIONS AND NOTATIONS

We begin with some notations and definitions.

For an n-dimensional vector $\vec{x} = (x_1,\ldots,x_n)^{\mathrm{T}}$, let $(\vec{x})_i$ be the ith component of $\vec{x}$, i.e., $(\vec{x})_i = x_i$.

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Let

$$E = \mathrm{diag}(1,\ldots,1),$$
the n dimensional unit matrix,
$$\vec{u} = (1,\ldots,1)^{\mathrm{T}} \in R^n,$$

$$R^{n+} = \{\vec{x} \in R^n \mid x_i > 0 \text{ for } i = 1,\ldots,n\},$$
$$\mathring{\Delta}^{n-1} = \{\vec{x} \in R^n \mid \vec{x}\cdot\vec{u} = 1,\ x_i > 0 \text{ for } i = 1,\ldots,n\}.$$

For an $n\times n$ matrix $A=(a_{ij})$, $\vec{a}_i$ denotes the ith column vector of A, i.e.,

$$\vec{a}_i = (a_{1i},\ldots,a_{ni})^{\mathrm{T}}, \quad A = (\vec{a}_1,\ldots,\vec{a}_n).$$

Given an $n\times n$ matrix $A=(a_{ij})$ and n positive real numbers $c_1,\ldots,c_n$, we define a vector $\vec{\gamma}$ and functions g_i, $\vec{g}$, g, f_i and $\vec{f}$ defined on R^{n+} as follows:

$$\vec{\gamma} = (\log c_1,\ldots,\log c_n)^{\mathrm{T}},$$

$$g_i(\vec{x}) = c_i x_1^{a_{i1}}\cdots x_n^{a_{in}}, \quad i = 1,\ldots,n,$$

$$\vec{g}(\vec{x}) = (g_1(\vec{x}),\ldots,g_n(\vec{x}))^{\mathrm{T}},$$

$$g(\vec{x}) = \vec{g}\cdot\vec{u} = \sum_{i=1}^{n} g_i(\vec{x}),$$

$$f_i(\vec{x}) = \frac{g_i(\vec{x})}{g(\vec{x})},$$

$$\vec{f}(\vec{x}) = (f_1(\vec{x}),\ldots,f_n(\vec{x}))^{\mathrm{T}}.$$

Since $\vec{f}(\vec{x})\cdot\vec{u} = 1$, the map $\vec{f}$ gives dynamics on the $(n-1)$-simplex $\mathring{\Delta}^{n-1}$.

We call this dynamics the log-linear discrete dynamics.

For a vector $\vec{d} = (d_1,\ldots,d_n) \in R^n$, let

$$A[\vec{d}] = (\vec{a}_1 + d_1\vec{u},\ldots,\vec{a}_n + d_n\vec{u}).$$

If we modify a matrix A to a matrix $A[\vec{d}]$, then the function $g_i(\vec{x})$ becomes

$$c_i x_1^{a_{i1}+d_1}\cdots x_n^{a_{in}+d_n} = g_i(\vec{x})x_1^{d_1}\cdots x_n^{d_n}$$

and the function $g(\vec{x})$ becomes

$$g(\vec{x})x_1^{d_1}\cdots x_n^{d_n}.$$

This implies that the dynamics $\vec{f}$ do not change under the modification A to $A[\vec{d}]$.

Therefore as the canonical form of a matrix A, we can consider, for example [1],

$$\begin{bmatrix} 0, & \ldots, & 0 \\ *, & \ldots, & * \\ \vdots & & \vdots \\ *, & \ldots, & * \end{bmatrix}.$$

However we will not restrict a matrix A in the canonical form, to keep a free hand for perturbations in the set of $n\times n$ matrices $M(n)$.

Let $V=\{A\in M(n)\mid \det(A-E)=0\}$ and $\widetilde{M}(n)=M(n)-V=\{A\in M(n)\mid \det(A-E)\neq 0\}$. Then since $\det(A-E)$ is a polynomial function of a_{ij}'s, V is a (n^2-1)-dimensional surface in n^2-dimensional space $M(n)$. Hence V is a thin set in $M(n)$ and almost all matrices belong to $\widetilde{M}(n)$. Moreover even if A is in V, one can modify A to $A[\vec{d}]$ in $\widetilde{M}(n)$ except for the few and rare cases discussed later.

Suppose that $A\in\widetilde{M}(n)$. We define functions of a positive variable t as follows:

$$\varphi_i(t) = \frac{t^{(B\vec{u})_i}}{e^{(B\vec{\gamma})_i}}, \quad i = 1,\ldots,n,$$

$$\vec{\varphi}(t) = (\varphi_1(t),\ldots,\varphi_n(t)),$$

and

$$\Phi(t) = \vec{\varphi}(t)\cdot\vec{u},$$

where $B=(A-E)^{-1}$. Note that $\Phi(t)$ is not a constant function and that $\vec{\varphi}(t)\in\mathring{\Delta}^{n-1}$ if and only if $\Phi(t)=1$.

3 FIXED POINTS

Suppose that $\vec{x}$ is a fixed point of $\vec{f}$, that is $\vec{f}(\vec{x}) = \vec{x}$. We can find this fixed point of $\vec{f}$ by solving the nonlinear equation system

$$\begin{aligned} g_1(\vec{x}) &= x_1 g(\vec{x}), \\ &\vdots \\ g_n(\vec{x}) &= x_n g(\vec{x}), \\ \vec{x}\cdot\vec{u} &= 1. \end{aligned}$$

However we note that it is difficult to solve this nonlinear equation system even numerically.

The following theorem shows that we can find all fixed points of $\vec{f}$ by solving a single nonlinear equation,

$$\Phi(t) = 1, \quad t > 0 \qquad (*)$$

whose numerical solutions can be easily obtained.

THEOREM 1 *Let* $A \in \widetilde{M}(n)$. *Suppose that the equation* $(*)$ *has* m *distinct solutions* $t_1, \ldots, t_m$. *Then* $\vec{f}$ *has just* m *fixed points* $\vec{\varphi}(t_1), \ldots, \vec{\varphi}(t_m)$.

Proof Suppose that $\vec{f}(\vec{x}) = \vec{x}$. Then $g_i(\vec{x}) = x_i t$, $i = 1, \ldots, n$, where $t = g(\vec{x})$ i.e.,

$$c_i x_1^{a_{i1}} \cdots x_n^{a_{in}} = x_i t, \quad i = 1, \ldots, n.$$

Taking logarithms on both sides, we have

$$\gamma_i + \sum_j a_{ij} \log x_j = \log x_i + \log t, \quad i = 1, \ldots, n,$$

$$(A - E)(\log x_1, \ldots, \log x_n)^{\mathrm{T}} = -\vec{\gamma} + (\log t)\vec{u}.$$

Since $A - E$ has the inverse matrix B, one has

$$(\log x_1, \ldots, \log x_n)^{\mathrm{T}} = -B\vec{\gamma} + \log t \cdot B\vec{u},$$

$$\log x_i = -(B\vec{\gamma})_i + \log t \cdot (B\vec{u})_i, \quad i = 1, \ldots, n.$$

Therefore one obtains

$$x_i = \frac{t^{(B\vec{u})_i}}{e^{(B\vec{\gamma})_i}} = \varphi_i(t), \quad i = 1, \ldots, n.$$

Since $\vec{x}\cdot\vec{u} = 1$, t is a solution of $\Phi(t) = 1$.

Conversely we show that if $\hat{t}$ is a solution of the equation $(*)$, then $\vec{\varphi}(\hat{t})$ is a fixed point of $\vec{f}$.

First we notice that $AB = B + E$ since $E = (A - E)B = AB - B$. Then

$$\begin{aligned} g_i(\varphi(t)) &= c_i(\varphi(t))^{a_{i1}} \cdots (\varphi_n(t))^{a_{in}} \\ &= c_i\left(\frac{t^{(B\vec{u})_1}}{e^{(B\vec{\gamma})_1}}\right)^{a_{i1}} \cdots \left(\frac{t^{(B\vec{u})_n}}{e^{(B\vec{\gamma})_n}}\right)^{a_{in}} \\ &= c_i \frac{t^{a_{i1}(B\vec{u})_1 + \cdots + a_{in}(B\vec{u})_n}}{e^{a_{i1}(B\vec{\gamma})_1 + \cdots + a_{in}(B\vec{\gamma})_n}} \\ &= c_i \frac{t^{(AB\vec{u})_i}}{e^{(AB\vec{\gamma})_i}} = c_i \frac{t^{(B\vec{u})_i + 1}}{e^{(B\vec{\gamma})_i + \gamma_i}} \\ &= t\varphi_i(t), \quad i = 1, \ldots, n, \end{aligned}$$

and

$$\begin{aligned} g(\vec{\varphi}(t)) &= \vec{g}(\vec{\varphi}(t)) \cdot \vec{u} \\ &= (t\vec{\varphi}(t)) \cdot \vec{u} = t\Phi(t). \end{aligned}$$

Hence

$$f_i(\vec{\varphi}(t)) = \frac{g_i(\vec{\varphi}(t))}{g(\vec{\varphi}(t))} = \frac{\varphi_i(t)}{\Phi(t)}, \quad i = 1, \ldots, n.$$

Therefore if $\hat{t}$ is a solution of the equation, then

$$f_i(\vec{\varphi}(\hat{t})) = \frac{\varphi_i(\hat{t})}{\Phi(\hat{t})} = \varphi_i(\hat{t}), \quad i = 1, \ldots, n,$$

so that $\vec{\varphi}(\hat{t})$ is a fixed point of $\vec{f}$.

Finally if $\hat{t}$ and $\tilde{t}$ are distinct solutions of the equation, then $\varphi_1(\hat{t}) \neq \varphi_1(\tilde{t})$ since $\varphi_1(t)$ is a monotone function. Hence $\vec{\varphi}(t_1), \ldots, \vec{\varphi}(t_m)$ are distinct.

In Section 5 we give Example 5 in which the coefficients c_1, c_2, c_3 are all equal to 1. Then the equation has no solution. In general:

PROPOSITION 1 *Suppose that* $A \in \widetilde{M}(n)$ *and* $c_1 = \cdots = c_n = 1$. *Then the equation* $(*)$ *has:*

1. *one solution if* $(B\vec{u})_1 > 0, \ldots, (B\vec{u})_n > 0$,
2. *one solution if* $(B\vec{u})_1 < 0, \ldots, (B\vec{u})_n < 0$,
3. *no solution if* $(B\vec{u})_i \geq 0, (B\vec{u})_j \leq 0$ *for some* $1 \leq i, j \leq n$.

Proof In case (1) (resp. (2)), $\Phi(t)$ is an increasing (resp. decreasing) function and

$$\lim_{t\to+0}\Phi(t)=0\ (\textit{resp.}\ \infty),\ \lim_{t\to\infty}\Phi(t)=\infty\ (\textit{resp.}\ 0).$$

Therefore the equation has unique solution. In case (3)

$$\Phi(t)>\varphi_i(t)=\frac{t^{(B\vec{u})_i}}{e^{(B\vec{\gamma})_i}}=t^{(B\vec{u})_i}\geq 1 \quad \text{for any } t\geq 1$$

and

$$\Phi(t)>\varphi_j(t)=\frac{t^{(B\vec{u})_j}}{e^{(B\vec{\gamma})_j}}=t^{(B\vec{u})_j}\geq 1 \quad \text{for any } t<1$$

since $\vec{\gamma}=\vec{0}$. Therefore the equation has no solution.

4 THE NUMBER OF FIXED POINTS

In this section we prove that *almost all* log-linear dynamics have at most two fixed points.

We first prove:

LEMMA 1 *Suppose that*

$$h(t)=a_1t^{\alpha_1}+a_2t^{\alpha_2}+\cdots+a_nt^{\alpha_n},$$

where $\alpha_1>\alpha_2>\cdots>\alpha_n$ *and* $\alpha_n=0$.

(1) *If* $a_1,\ldots,a_n>0$, *then* $h(t)>0$ *for all* $t>0$.

(2) *If* $a_1,\ldots,a_k>0$, $a_{k+1},\ldots,a_n<0$ *for some* $k\,(1\leq k<n)$, *then*

$$\begin{aligned} &h(t)<0, && 0<t<t_0,\\ &h(t_0)=0, && t=t_0,\\ &h(t)>0, && t_0<t \end{aligned}$$

for some $t_0>0$.

Proof Note that

$$\lim_{t\to\infty}h(t)=\infty.$$

Lemma 1 is true when $n=2$. We may therefore proceed by induction, assuming Lemma 1 true for n. Let

$$h(t)=a_1t^{\alpha_1}+a_2t^{\alpha_2}+\cdots+a_nt^{\alpha_n}+a_{n+1}t^{\alpha_{n+1}}$$
$$(\alpha_1>\alpha_2>\cdots>\alpha_n>\alpha_{n+1}=0).$$

Then

$$h'(t)=a_1\alpha_1t^{\alpha_1-1}+a_2\alpha_2t^{\alpha_2-1}+\cdots+a_n\alpha_nt^{\alpha_n-1}$$

since $\alpha_{n+1}=0$. We write $h'(t)$ in the form

$$h'(t)=t^{\alpha_n-1}k(t),$$

where

$$\begin{aligned} &k(t)=b_1t^{\beta_1}+\cdots+b_nt^{\beta_n},\\ &b_1=a_1\alpha_1,\ldots,b_n=a_n\alpha_n,\\ &\beta_1=\alpha_1-\alpha_n,\ldots,\beta_n=\alpha_n-\alpha_n=0. \end{aligned}$$

Note that $\beta_1>\beta_2>\cdots>\beta_n=0$. If $a_1,\ldots,a_{n+1}>0$, then $b_1,\ldots,b_n>0$, so $k(t)>0$, $t>0$ by the assumption. Since $h(0)=a_{n+1}>0$ and $h'(t)>0$ for all $t>0$, $h(t)>0$ for all $t>0$, so that (1) holds.

If $a_1,\ldots,a_n>0$ and $a_{n+1}<0$, then $b_1,\ldots,b_n>0$, so $h(0)=a_{n+1}<0$ and $h'(t)>0$ for all $t>0$. Since

$$\lim_{t\to\infty}h(t)=\infty,$$

there exists $t_0>0$ such that:

$$\begin{aligned} &h(t)<0, && 0<t<t_0,\\ &h(t_0)=0, && t=t_0,\\ &h(t)>0, && t_0<t. \end{aligned}$$

If $a_1,\ldots,a_k>0$ and $a_{k+1},\ldots,a_{n+1}<0$ for some $k\ (1\leq k<n)$, then $b_1,\ldots,b_k>0$ and $b_{k+1},\ldots,b_n<0$. Hence there exists $t_0>0$ such that:

$$\begin{aligned} &h'(t)<0, && 0<t<t_0,\\ &h'(t_0)=0, && t=t_0,\\ &h'(t)>0, && t_0<t. \end{aligned}$$

Moreover since $h(0)=a_{n+1}<0$, $h(t)<0$ for $0<t\leq t_0$.

Since $h'(t)>0$ for all $t>t_0$ and $\lim_{t\to\infty} h(t)=\infty$, there exits $t'_0>t_0>0$ such that:

$$\begin{aligned} h(t)&<0, & 0<t<t'_0,\\ h(t_0)&=0, & t=t'_0,\\ h(t)&>0, & t'_0<t. \end{aligned}$$

Therefore (2) holds.

THEOREM 2 *Almost all log-linear dynamics have at most two fixed points.*

Proof It suffices to show that the equation $(*)$

$$\Phi(t)=1, \quad t>0 \qquad (*)$$

has at most two solutions.

Without the loss of generality, we may write

$$\Phi(t)=a_1t^{\alpha_1}+\cdots+a_lt^{\alpha_l}+\text{const}.$$

where $a_1,\ldots,a_l>0$ and $\alpha_1>\alpha_2>\cdots\alpha_l$.

If $\alpha_1,\ldots,\alpha_l>0$ (*resp.* <0), then by the same arguments as the proof of Proposition 1, equation $(*)$ has a unique solution.

Suppose the $\alpha_1,\ldots,\alpha_k>0$ and $\alpha_{k+1},\ldots,\alpha_l<0$ for some k $(1\leq k<l)$. Then

$$\begin{aligned}\Phi'(t)&=a_1\alpha_1t^{\alpha_1-1}+\cdots+a_l\alpha_lt^{\alpha_l-1}\\ &=\alpha_1a_1t^{\alpha_l-1}(b_1t^{\beta_1}+\cdots+b_lt^{\beta_l}),\end{aligned}$$

where

$$\beta_1=\alpha_1-\alpha_l,\ldots,\beta_l=\alpha_l-\alpha_l,$$

$$b_1=1, \quad b_2=\frac{\alpha_2a_2}{\alpha_1a_1},\ldots,b_l=\frac{\alpha_la_l}{\alpha_1a_1}.$$

Note that $\beta_1>\cdots>\beta_l=0$, $b_1,\ldots,b_k>0$ and $b_{k+1},\ldots,b_l<0$. By Lemma 1, there exist $t_0>0$ such that:

$$\begin{aligned}\Phi'(t)&<0, & 0<t<t_0,\\ \Phi'(t_0)&=0, & t=t_0,\\ \Phi'(t)&>0, & t_0<t.\end{aligned}$$

Therefore $\Phi(t)$ is monotonically decreasing for $t<t_0$ and $\Phi(t)$ is monotonically increasing for $t>t_0$.

Since

$$\lim_{t\to+0}\Phi(t)=\infty, \qquad \lim_{t\to\infty}\Phi(t)=\infty,$$

the number of solutions is 0, 1 or 2 depending on the value of $\Phi(t_0)$. Hence the number of the fixed points is at most two.

Remark We suppose in Theorems 1 and 2 that $A-E$ is invertible. As the coefficients of A are taken randomly, the probability that $A-E$ is noninvertible is zero. However, when the coefficients are restricted to integers, or when one changes an entry of A continuously, one often has to consider a matrix A with $\det(A-E)=0$. So we will study the case $A-E$ when it is noninvertible.

Suppose that $\det(A-E)=0$. In this case one may try to modify A to $A[\vec{d}]$ so that $\det(A[\vec{d}]-E)\neq 0$.

Let $C=(c_{ij})=A-E$. Since

$$\begin{aligned}\det(A[\vec{d}]-E)=\det(C[]\vec{d}])&=\det(C)\\ &+d_1\det(\vec{u},\vec{c}_2,\ldots,\vec{c}_n)\\ &\vdots\\ &+d_n\det(\vec{c}_1,\ldots,\vec{c}_{n-1},\vec{u}),\end{aligned}$$

one can choose $\vec{d}$ so that $\det(A[\vec{d}]-E)\neq 0$ except for the case where

$$\det(\vec{u},\vec{c}_2,\ldots,\vec{c}_n)=\cdots=\det(\vec{c}_1,\ldots,\vec{c}_{n-1},\vec{u})=0.$$

EXAMPLE Let

$$A=\begin{pmatrix}0&0&0\\0&2&1\\1&2&3\end{pmatrix}.$$

Then

$$\det(A-E)=\det\begin{pmatrix}-1&0&0\\0&1&1\\1&2&2\end{pmatrix}=0$$

and

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \det\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 0.$$

So one cannot modify A to $A[\vec{d}]$ with $\det(A[\vec{d}] - E) \neq 0$. For this example, one can get fixed points by simple calculations.

Suppose that $c_1 = 1$. Then $\vec{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ is a fixed point if and only if

$$x_1 + x_2 + x_3 = 1, \qquad x_1, x_2, x_3 > 0,$$

$$c_2 x_1 x_2 x_3 = c_3 (x_1 x_2 x_3)^2 = 1.$$

This system of equations has no solution except for the case where

$$c_3 = c_2^2, \quad c_2 > 27,$$

in which case the fixed points make a closed curve in the 2-simplex.

5 EXAMPLE

In this section we give some numerical examples illustrating the forms of the function $\Phi(t)$.

Example 1:

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1,$$

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -3 & -1 & 3 \end{pmatrix}.$$

Then

$$\Phi(t) = t + t^2 + t^3$$

is monotonically increasing and the equation has one solution.

Example 2:

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1,$$

$$A = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}.$$

Then

$$\Phi(t) = \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}$$

is monotonically decreasing and the equation has one solution.

Example 3:

$$c_1 = 1, \quad c_2 = 7, \quad c_3 = 50,$$

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 2.5 \\ 2 & 0.5 & 0 \end{pmatrix}.$$

Then

$$\Phi(t) = 0.209128 t^{0.2} + 0.123576 t^{0.488889} + \frac{0.768706}{t^{0.355556}}$$

has one minimum (< 1) and the equation has two solutions.

Example 4:

$$c_1 = 1, \quad c_2 = 7, \quad c_3 = 50,$$

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & -2.57419151135 \\ 2 & 0.5 & 0 \end{pmatrix}.$$

Then

$$\Phi(t) = 0.209128 t^{0.2} + 0.124635 t^{0.500423} + \frac{0.771993}{t^{0.349789}}$$

has one minimum ($= 1$) and the equation has one solution.

Example 5:

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1,$$

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & -2.5 \\ 2 & 0.5 & 0 \end{pmatrix}.$$

Then

$$\Phi(t) = t^{0.2} + t^{0.488889} + \frac{1}{t^{0.355556}}$$

has one minimum (> 1) and the equation has no solution.

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