POSITIVE SOLUTIONS OF THREE-POINT BOUNDARY VALUE PROBLEMS FOR HIGHER-ORDER p-LAPLACIAN WITH INFINITELY MANY SINGULARITIES

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We study a three-point nonlinear boundary value problem with higher-order p-Laplacian. We show that there exist countable many positive solutions by using the fixed point index theorem for operators in a cone.

1. Introduction

In this paper, we study a higher-order quasilinear equation with p-Laplacian

$$(\phi_p(u^{(n-1)}))' + g(t)f(t,u,u',\dots,u^{(n-2)}) = 0, \quad 0 < t < 1,\ n \geq 3, \tag{1.1}$$

subject to the following three-point boundary conditions:

$$\begin{gathered} u^i(0) = 0, \quad 0 \leq i \leq n-3, \\ \alpha\phi_p(u^{(n-2)}(0)) - \beta\phi_p(u^{(n-1)}(\eta)) = 0, \\ \gamma\phi_p(u^{(n-2)}(1)) - \delta\phi_p(u^{(n-1)}(1)) = 0, \end{gathered} \tag{1.2}$$

where $\phi_p(s)$ is a p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \in (0,1)$ is a given constant, $\alpha > 0$, $\gamma > 0$, $\beta \geq 0$, $\delta \geq 0$, $g : [0,1] \to [0,\infty)$ has countable many singularities on $(0,1/2)$.

In recent years, because of the wide mathematical and physical backgrounds [7, 8], the existence of positive solutions for nonlinear boundary value problems with p-Laplacian received wide attention. Especially, when $p = 2$, the existence of positive solutions for nonlinear singular boundary value problems has been obtained (see [5, 6, 10]); when $p \neq 2$ and the nonlinearities are continuous, many results of the existence of positive solutions

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have been obtained [1–4, 9] by using comparison results, topological degree theorem, respectively. Recently, on the existence of positive solutions of multipoint boundary value problems for second-order ordinary differential equation, some authors have obtained the existence results (see [5–8, 10]). However, all of the above-mentioned references dealt with the case of the nonlinearity without singularities. For the singular case of multipoint boundary value problems, to our acknowledge, no one has studied the existence of positive solutions in this case.

Very recently, Kaufmann and Kosmatov [3] established the result of countably many positive solutions for the two-point boundary value problems with infinitely many singularities of the following form:

$$\begin{gathered} u''(t)+a(t)f(u(t))=0, \quad 0<t<1, \\ u'(0)=0, \qquad u(1)=0, \end{gathered} \tag{1.3}$$

where $a \in L^p[0,1]$, $p \geq 1$, and $a(t)$ can have countably singularities on $[0,1/2)$.

Lian and Ge in [4] investigated the following boundary value problem:

$$\begin{gathered} \left(\phi_p\left(u^{(n-1)}\right)\right)'+g(t)f\left(t,u,u',\dots,u^{(n-2)}\right)=0, \quad 0<t<1, \\ u^i(0)=0, \quad 0\leq i\leq n-3, \\ \alpha u^{(n-2)}(0)-\beta u^{(n-1)}(0)=0, \\ \gamma u^{(n-2)}(1)-\delta u^{(n-1)}(1)=0, \quad n\geq 3, \end{gathered} \tag{1.4}$$

where $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\alpha,\beta,\gamma,\delta\geq 0$, $\alpha\gamma+\alpha\delta+\gamma\beta>0$ and obtained that the problem has at least one positive solution by using the fixed point theorem of the compression and expansion of norm in the cone.

Motivated by the results mentioned above, in this paper, we extend the results obtained in [4] to the more general three-point boundary value problems (1.1)-(1.2) which are generalization of problems (1.4). We would stress that the results presented in this paper complement and improve those obtained in [3, 4], since we allow nonlinearity to have infinitely many singularities and the boundary value conditions are more general. We will show that the problems (1.1)-(1.2) have infinitely many solutions if g and f satisfy some suitable conditions.

In the rest of the paper, we make the following assumptions:

(H_1) $f\in \mathbb{C}([0,1]\times[0,+\infty)^{n-1},[0,+\infty))$,

(H_2) $g\in L^1[0,1]$ is nonnegative and $g(t)$ does not vanish identically on any subinterval of $[0,1]$,

(H_3) there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_{i+1}<t_i$, $t_1<1/2$, $\lim_{i\to\infty}t_i=t^*\geq 0$, $\lim_{t\to t_i}g(t)=\infty$ $(i=1,2,\dots)$, and

$$0<\int_0^1 g(t)dt<\infty. \tag{1.5}$$

It is easy to check that condition (H_3) implies that

$$0 < \int_0^1 \phi_q\left(\int_0^s g(s_1)\,ds_1\right)ds < +\infty, \tag{1.6}$$

where $\phi_q = \phi_p^{-1}$ and $1/p + 1/q = 1$.

2. Preliminaries and lemmas

We denote

$$\begin{gathered} B = \{u \in \mathbb{C}^{(n-2)}[0,1] : u^i(0) = 0,\ 0 \le i \le n-3\}, \\ K = \{u \in B : u^{(n-2)}(t) \ge 0,\ u^{(n-2)}(t) \text{ is concave function}, t \in [0,1]\}, \end{gathered} \tag{2.1}$$

and the norm $\|u\| = \max_{t\in[0,1]} |u^{(n-2)}(t)|$. Set $K_r = \{u \in K : \|u\| \le r\}$, then it is obvious that K is a cone. Our main tool of this paper is the following fixed point theorem of cone expansion and compression of norm type.

Lemma 2.1 [1]. *Suppose E is a banach space, $K \subset E$ is a cone, let Ω_1, Ω_2 be two bounded open sets of E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be completely continuous. Suppose that one of the following two conditions holds:*
(i) *$\|Tx\| \le \|x\|$, for all $x \in K \cap \partial\Omega_1$, $\|Tx\| \ge \|x\|$, for all $x \in K \cap \partial\Omega_2$,*
(ii) *$\|Tx\| \ge \|x\|$, for all $x \in K \cap \partial\Omega_1$, $\|Tx\| \le \|x\|$, for all $x \in K \cap \partial\Omega_2$.*
Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now we define a mapping $T : K \to \mathbb{C}^{(n-1)}[0,1] \cap B$,

$$(Tu)(t) = \int_0^{t_1}\int_0^{s_1}\cdots\int_0^{s_{n-3}} w(s_{n-2})\,ds_{n-2}s_{n-3}\cdots ds_1, \tag{2.2}$$

where $w(t)$ is given by

$$w(t) = \begin{cases} \phi_q\left(\dfrac{\beta}{\alpha}\displaystyle\int_\eta^\delta g(s)f\left(s,u(s),u'(s),\dots,u^{(n-2)}(s)\right)ds\right) \\ \quad + \displaystyle\int_0^t \phi_q\left(\int_s^\delta g(r)f(r,u(r),u'(r),\dots,u^{(n-2)}(r))\,dr\right)ds, & 0 \le t \le \delta, \\ \phi_q\left(\dfrac{\delta}{\gamma}\displaystyle\int_\delta^1 g(s)f(s,u(s),u'(s),\dots,u^{(n-2)}(s))\,ds\right) \\ \quad + \displaystyle\int_t^1 \phi_q\left(\int_\delta^s g(r)f(r,u(r),u'(r),\dots,u^{(n-2)}(r))\,dr\right)ds, & \delta \le t \le 1, \end{cases} \tag{2.3}$$

where δ is a solution of the equation $y_0(x) = y_1(x)$, here

$$\begin{aligned}
y_0(x) &= \phi_q\left(\frac{\beta}{\alpha}\int_\eta^x g(s)f\left(s,u(s),u'(s),\dots,u^{(n-2)}(s)\right)ds\right) \\
&\quad + \int_0^x \phi_q\left(\int_s^x g(r)f\left(r,u(r),u'(r),\dots,u^{(n-2)}(r)\right)dr\right)ds, \\
y_1(x) &= \phi_q\left(\frac{\delta}{\gamma}\int_x^1 g(s)f\left(s,u(s),u'(s),\dots,u^{(n-2)}(s)\right)ds\right) \\
&\quad + \int_x^1 \phi_q\left(\int_x^s g(r)f\left(r,u(r),u'(r),\dots,u^{(n-2)}(r)\right)dr\right)ds.
\end{aligned} \tag{2.4}$$

Obviously, $y_0(x)$ is a nondecreasing continuous function defined on $[0,1]$ with $y_0(0) = 0$ and $y_1(x)$ is a nonincreasing continuous function defined on $[0,1]$ with $y_1(1) = 0$. Moveover, if $\delta_1,\delta_2 \in [0,1]$ $(\delta_1 < \delta_2)$ are solutions of the equation $y_0(x) = y_1(x)$, then we have

$$g(t)f\left(t,u(t),u'(t),\dots,u^{(n-2)}(t)\right) \equiv 0. \tag{2.5}$$

As $t \in [\delta_1,\delta_2]$, we choose $\delta \in [\delta_1,\delta_2]$ and can have

$$(Tu)^{(n-1)}(t) = w'(t) = \begin{cases} \phi_q\left(\displaystyle\int_t^\delta g(s)f\left(s,u(s),u'(s),\dots,u^{(n-2)}(s)\right)ds\right) \geq 0, & 0 \leq t \leq \delta, \\ -\phi_q\left(\displaystyle\int_\delta^t g(s)f\left(s,u(s),u'(s),\dots,u^{(n-2)}(s)\right)ds\right) \leq 0, & \delta \leq t \leq 1. \end{cases} \tag{2.6}$$

Obviously, we can obtain the following results:

$$\alpha\phi_p(w(0)) - \beta\phi_p(w'(\eta)) = 0, \qquad \gamma\phi_p(w(1)) + \delta\phi_p(w'(1)) = 0. \tag{2.7}$$

Let

$$\begin{aligned}
f_0 &= \lim_{u_{n-1}\to 0}\ \max_{0\leq u_1\leq\cdots\leq u_{n-2}\leq(1/\theta)u_{n-1},\ t\in[0,1]} \frac{f(t,u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}}, \\
f_\infty &= \lim_{u_{n-1}\to\infty}\ \min_{0\leq u_1\leq\cdots\leq u_{n-2}\leq(1/\theta)u_{n-1},\ t\in[0,1]} \frac{f(t,u_1,u_2,\dots,u_{n-1})}{(u_{n-1})^{p-1}},
\end{aligned} \tag{2.8}$$

where $\theta \in (0,1/2)$ is a given constant. We can easily get the following lemmas.

Lemma 2.2. *Let $u \in K$ and $\theta \in (0,1/2)$. Then*

$$u^{(n-2)}(t) \geq \theta\|u\|, \quad t \in [\theta, 1-\theta]. \tag{2.9}$$

The proof of Lemma 2.2 is similar to the proof of lemma in [9], so we omit the details.

Lemma 2.3 [4]. *Let (H_1), (H_2), (H_3) hold. Then $T : K \to K$ is completely continuous.*

Lemma 2.4. *Suppose condition (H_3) holds. Then the function*

$$A(t) = \int_{t_1}^{t} \phi_q\left(\int_s^t g(s_1)ds_1\right)ds + \int_t^{1-t_1} \phi_q\left(\int_t^s g(s_1)ds_1\right)ds, \quad t \in [t_1, 1-t_1], \tag{2.10}$$

is positive continuous functions on $[t_1, 1-t_1]$, therefore, $A(t)$ has minimum on $[t_1, 1-t_1]$, and hence it is supposed that there exists $L > 0$ such that $A \geq L$, $t \in [t_1, 1-t_1]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[t_1, 1-t_1]$. Next, let

$$A_1(t) = \int_{t_1}^{t} \phi_q\left(\int_s^t g(s_1)ds_1\right)ds, \qquad A_2(t) = \int_t^{1-t_1} \phi_q\left(\int_t^s g(s_1)ds_1\right)ds. \tag{2.11}$$

Then, from condition (H_3), we have that the function $A_1(t)$ is strictly monotone increasing on $[t_1, 1-t_1]$ and $A_1(t_1) = 0$, and that the function $A_2(t)$ is strictly monotone decreasing on $[t_1, 1-t_1]$ and $A_2(1-t_1) = 0$, which implies $L = \min_{t\in[t_1,1-t_1]} A(t) > 0$. The proof is complete. □

Lemma 2.5. *Suppose that conditions (H_1), (H_2) hold. Then the solution $u(t)$ of problem (1.1), (1.2) satisfies*

$$u(t) \leq u'(t) \leq \cdots \leq u^{(n-3)}(t), \quad t \in [0,1], \tag{2.12}$$

and for $\theta \in (0,1/2)$ in Lemma 2.2,

$$u^{(n-3)}(t) \leq \frac{1}{\theta}u^{(n-2)}(t), \quad t \in [\theta, 1-\theta]. \tag{2.13}$$

Proof. If $u(t)$ is the solution of problem (1.1), (1.2), then $u^{(n-2)}(t)$ is a concave function, and $u^i(t) \geq 0$, $i = 0,1,\dots,n-2$, $t \in [0,1]$. Thus we have

$$u^i(t) = \int_0^t u^{(i+1)}(s)ds \leq tu^{(i+1)}(t) \leq u^{(i+1)}(t), \quad i = 0,1,\dots,n-4, \tag{2.14}$$

that is, $u(t) \leq u'(t) \leq \cdots \leq u^{(n-3)}(t), t \in [0,1]$. Next, by Lemma 2.2, for $t \in [\theta, 1-\theta]$, we have $u^{(n-2)}(t) \geq \theta\|u^{(n-2)}\|$. Then by $u^{(n-3)}(t) = \int_0^t u^{(n-2)}(s)ds \leq \|u^{(n-2)}\|$, we have

$$u^{(n-3)}(t) \leq \frac{1}{\theta}u^{(n-2)}(t), \quad t \in [\theta, 1-\theta]. \tag{2.15}$$

The proof is complete. □

3. The main result

In this section, we present our main results, and also provide an example of family of functions $a(t)$ that satifies condition (H_3). For convenience, we set

$$\theta^* = \frac{2}{L}, \qquad \theta_* = \frac{1}{(1+\phi_q(\beta/\alpha))\phi_q\left(\int_0^1 g(r)dr\right)}. \tag{3.1}$$

Theorem 3.1. *Suppose conditions (H_1), (H_2), (H_3) hold. Let $\{\theta_k\}_{k=1}^{\infty}$ be such that $\theta_k \in (t_{k+1}, t_k)$ $(k = 1,2,\ldots)$, let $\{r_k\}_{k=1}^{\infty}$ and $\{R_k\}_{k=1}^{\infty}$ be such that*

$$R_{k+1} < \theta_k r_k < r_k < m r_k < R_k, \quad k = 1,2,\ldots, \tag{3.2}$$

and for each natural number k, assume that f satisfy

(A_1) $f(t,u_1,u_2,\ldots,u_{n-1}) \geq (mr_k)^{p-1}$, *for $\theta_k r_k \leq u_{n-1} \leq r_k$, where $m \in (\theta^*,\infty)$,*

(A_2) $f(t,u_1,u_2,\ldots,u_{n-1}) \leq (MR_k)^{p-1}$, *for $0 \leq u_{n-1} \leq R_k$, where $M \in (0,\theta_*)$.*

Then, the boundary value problem (1.1), (1.2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$ such that

$$r_k \leq \|u_k\| \leq R_k, \quad k = 1,2,\ldots. \tag{3.3}$$

Proof. From (2.6), we have $(Tu)^{(n-2)}(\delta) = \max_{t\in[0,1]}(Tu)^{(n-2)}(t) = \|Tu\|$. Because $t_0 < t_{k+1} < \theta_k < t_k < 1/2$, $k = 1,2,\ldots$, for any $k \in \mathbb{N}$, $u \in K$, by Lemma 2.2, we have

$$u^{(n-2)}(t) \geq \theta_k\|u\|, \quad t \in [\theta_k, 1-\theta_k]. \tag{3.4}$$

We define two open subset sequences $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of B,

$$\Omega_{1,k} = \{u \in K : \|u\| < r_k\}, \quad k = 1,2,\ldots, \qquad \Omega_{2,k} = \{u \in K : \|u\| < R_k\}, \quad k = 1,2,\ldots. \tag{3.5}$$

For a fixed k and $u \in \partial\Omega_{1,k}$, by (3.4), we have

$$r_k = \|u\| \geq u^{(n-2)}(t) \geq \theta_k\|u\| = \theta_k r_k, \quad t \in [\theta_k, 1-\theta_k]. \tag{3.6}$$

For $t \in [t_1, 1-t_1] \subseteq [\theta_k, 1-\theta_k]$, we will discuss it from three cases.

(i) If $\delta \in [t_1, 1-t_1]$, then for $u \in \partial\Omega_{1,k}$, by ($\mathrm{A}_1$) and Lemma 2.3, we have

$$\begin{aligned}
2\|Tu\| &= 2(Tu)^{(n-2)}(\delta) \\
&\geq \int_0^\delta \phi_q\left(\int_s^\delta g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\quad + \int_\delta^1 \phi_q\left(\int_\delta^s g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\geq \int_{t_1}^\delta \phi_q\left(\int_s^\delta g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\quad + \int_\delta^{1-t_1} \phi_q\left(\int_\delta^s g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\geq m r_k A(\delta) \geq m r_k L(t_1) > 2 r_k = 2\|u\|.
\end{aligned} \tag{3.7}$$

(ii) If $\delta \in (1-t_1, 1]$, thus for $u \in \partial\Omega_{1,k}$, by ($\mathrm{A}_1$) and Lemma 2.3, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta) \\
&\geq \phi_q\left(\frac{\beta}{\alpha} \int_\eta^\delta g(r) f\left(t, u(r), u'(r), \ldots u^{(n-2)}(r)\right) dr\right) \\
&\quad + \int_0^\delta \phi_q\left(\int_s^\delta g(r) f\left(t, u(r), u'(r), \ldots u^{(n-2)}(r)\right) dr\right) ds \\
&\geq \int_{t_1}^{1-t_1} \phi_q\left(\int_s^{1-t_1} g(r) f\left(t, u(r), u'(r), \ldots u^{(n-2)}(r)\right) dr\right) ds \\
&\geq m r_k A(1-t_1) \geq m r_k L > 2 r_k > r_k = \|u\|.
\end{aligned} \tag{3.8}$$

(iii) If $\delta \in (0, t_1)$, then for $u \in \partial\Omega_{1,k}$, by ($\mathrm{A}_1$) and Lemma 2.3, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(\delta) \\
&\geq \phi_q\left(\frac{\delta}{\gamma} \int_\delta^1 g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) \\
&\quad + \int_\delta^1 \phi_q\left(\int_s^\delta g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\geq \phi_q\left(\frac{\delta}{\gamma} \int_{t_1}^{1-t_1} g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) \\
&\quad + \int_{t_1}^{1-t_1} \phi_q\left(\int_{t_1}^s g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\geq \int_{t_1}^{1-t_1} \phi_q\left(\int_{t_1}^s g(r) f\left(t, u(r), u'(r), \ldots, u^{(n-2)}(r)\right) dr\right) ds \\
&\geq m r_k A(t_1) \geq m r_k L > 2 r_k > r_k = \|u\|.
\end{aligned} \tag{3.9}$$

Therefore, no matter under which condition, we all have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{1,k}. \tag{3.10}$$

On the other hand, when $u \in \partial\Omega_{2,k}$, we have $u(t) \leq \|u\| = R_k$, and by (A_2), we know

$$\begin{aligned} \|Tu\| &= (Tu)^{(n-2)}(\delta) \\ &= \phi_q\left(\frac{\beta}{\alpha}\int_\eta^\delta g(r)f(t,u(r),u'(r),\dots,u^{(n-2)}(r))\,dr\right) \\ &\quad + \int_0^\delta \phi_q\left(\int_s^\delta g(r)f(t,u(r),u'(r),\dots,u^{(n-2)}(r))\,dr\right)ds \\ &\leq \left[1+\phi_q\left(\frac{\beta}{\alpha}\right)\right]MR_k\phi_q\left(\int_0^1 g(r)\,dr\right) = R_k = \|u\|. \end{aligned} \tag{3.11}$$

Thus

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{2,k}. \tag{3.12}$$

For $0 \in \Omega_{1,k} \subset \overline{\Omega}_{1,k} \subset \Omega_{2,k}$, by (3.10), (3.12), and Lemma 2.1, operator T has a fixed point $u_k \in (\overline{\Omega}_{2,k} \setminus \Omega_{1,k})$, and $r_k \leq \|u_k\| \leq R_k$. By the randomness of k, we know that Theorem 3.1 holds. This completes the proof of Theorem 3.1. □

THEOREM 3.2. *Suppose conditions (H_1), (H_2), (H_3) hold. Let $\{\theta_k\}_{k=1}^\infty$ be such that $\theta_k \in (t_{k+1},t_k)$ $(k=1,2,\dots)$. Let $\{r_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ be such that*

$$R_{k+1} < \theta_k r_k < r_k < mr_k < R_k, \quad k=1,2,\dots. \tag{3.13}$$

For each natural number k, assume that f satisfies
(A_3) $f_\infty = \lambda \in ((2\theta^*/\theta_k)^{p-1},\infty)$,
(A_4) $f_0 = \varphi \in [0,(\theta_*/4)^{p-1})$.
Then, the boundary value problem (1.1), (1.2) has infinitely many solutions $\{u_k\}_{k=1}^\infty$ such that

$$r_k \leq \|u_k\| \leq R_k, \quad k=1,2,\dots. \tag{3.14}$$

Proof. First, by $f_0 = \varphi \in [0,(\theta_*/4)^{p-1})$, for $\epsilon = (\theta_*/4)^{p-1} - \varphi$, there exists an adequately small positive number ρ such that, as $0 \leq u_{n-1} \leq \rho$, $u_{n-1} \neq 0$, we have

$$f(t,u_1,u_2,\dots,u_{n-1}) \leq (\varphi+\epsilon)(u_{n-1})^{p-1} \leq \left(\frac{\theta_*}{4}\right)^{p-1}\rho^{p-1} = \left(\frac{\theta_*}{4}\rho\right)^{p-1}. \tag{3.15}$$

Then let $R_k = \rho$, $M = \theta_*/4 \in (0,\theta_*)$, thus by (3.15),

$$f(t,u_1,u_2,\dots,u_{n-1}) \leq (MR_k)^{p-1}, \quad 0 \leq u_{n-1} \leq R_k. \tag{3.16}$$

So condition (A_2) holds.

Next, by condition $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$, for $\epsilon = \lambda - (2\theta^*/\theta)^{p-1}$, there exists an adequately big positive number $r_k \neq R_k$ such that, as $u_{n-1} \geq \theta r_k$, we have

$$f(t,u_1,u_2,\dots,u_{n-1}) \geq (\lambda - \epsilon)(u_{n-1})^{p-1} \geq \left(\frac{2\theta^*}{\theta_k}\right)^{p-1}(\theta_k r_k)^{p-1} = (2\theta^* r_k)^{p-1}. \quad (3.17)$$

Let $m = 2\theta^* > \theta^*$, then by (3.17), condition (A_1) holds. Therefore, by Theorem 3.1, we know that the result of Theorem 3.2 holds. The proof of Theorem 3.2 is complete. □

THEOREM 3.3. *Suppose conditions (H_1), (H_2), (H_3) hold. Let $\{\theta_k\}_{k=1}^\infty$ be such that $\theta_k \in (t_{k+1}, t_k)$ $(k = 1,2,\dots)$. Let $\{r_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ be such that*

$$R_{k+1} < \theta_k r_k < r_k < m r_k < R_k, \quad k = 1,2,\dots. \quad (3.18)$$

For each natural number k, assume that f satisfies

(A_5) $f_0 = \varphi \in ((2\theta^*/\theta_k)^{p-1}, \infty)$,

(A_6) $f_\infty = \lambda \in [0, (\theta_*/4)^{p-1})$.

Then, the boundary value problem (1.1), (1.2) has infinitely many solutions $\{u_k\}_{k=1}^\infty$ such that

$$r_k \leq \|u_k\| \leq R_k, \quad k = 1,2,\dots. \quad (3.19)$$

Proof. First, by condition $f_0 = \varphi \in ((2\theta^*/\theta_k)^{p-1}, \infty)$, for $\epsilon = \varphi - (2\theta^*/\theta_k)^{p-1}$, there exists an adequately small positive number r_k such that, as $0 \leq u_{n-1} \leq r_k$, $u_{n-1} \neq 0$, we have

$$f(t,u_1,u_2,\dots,u_{n-1}) \geq (\varphi - \epsilon)(u_{n-1})^{p-1} = \left(\frac{2\theta^*}{\theta_k}\right)^{p-1}(u_{n-1})^{p-1}, \quad (3.20)$$

thus when $\theta_k r_k \leq u_{n-1} \leq r_k$, we have

$$f(t,u_1,u_2,\dots,u_{n-1}) \geq \left(\frac{2\theta^*}{\theta_k}\right)^{p-1}(\theta_k r_k)^{p-1} = (2\theta^* r_k)^{p-1}. \quad (3.21)$$

Let $m = 2\theta^* > \theta^*$, so by (3.21), condition (A_1) holds.

Next, by condition $f_\infty = \lambda \in [0, (\theta_*/4)^{p-1})$, for $\epsilon = (\theta_*/4)^{p-1} - \lambda$, there exists an adequately small positive number $\rho \neq r_k$ such that, as $u_{n-1} \geq \rho$, we have

$$f(t,u_1,u_2,\dots,u_{n-1}) \leq (\lambda + \epsilon)(u_{n-1})^{p-1} \leq \left(\frac{\theta_*}{4}\right)^{p-1}(u_{n-1})^{p-1}. \quad (3.22)$$

If f is unboundary, by the continuation of f on $[0,1]\times[0,\infty)^{n-1}$, there exist constant $R_k(\neq r_k)\geq\rho$, and a point $(t_0,u_{01},u_{02},\dots,u_{0(n-1)})\in[0,1]\times[0,\infty)^{n-1}$ such that $\rho\leq u_{0(n-1)}\leq R_k$ and $f(t,u_1,u_2,\dots,u_{n-1})\leq f(t_0,u_{01},u_{02},\dots,u_{0(n-1)})$, $0\leq u_{n-1}\leq R_k$. Thus, by $\rho\leq u_{0(n-1)}\leq R_k$, we know

$$\begin{aligned} f(t,u_1,u_2,\dots,u_{n-1}) &\leq f(t_0,u_{01},u_{02},\dots,u_{0(n-1)}) \\ &\leq \left(\frac{\theta_*}{4}\right)^{p-1}\left(u_{0(n-1)}\right)^{p-1}\leq\left(\frac{\theta_*}{4}R_k\right)^{p-1}. \end{aligned} \tag{3.23}$$

Let $M=\theta_*/4\in(0,\theta_*)$, we have $f(t,u_1,u_2,\dots,u_{n-1})\leq(MR_k)^{p-1}$, $0\leq u_{n-1}\leq R_k$. If f is bounded, we suppose $f(t,u_1,u_2,\dots,u_{n-1})\leq\overline{M}^{p-1}$, $u_{n-1}\in[0,\infty)$, and there exists an adequately big positive number $R_k>4/\theta_*\overline{M}$. Then letting $M=\theta_*/4\in(0,\theta_*)$, we have

$$f(t,u_1,u_2,\dots,u_{n-1})\leq\overline{M}^{p-1}\leq\left(\frac{\theta_*}{4}R_k\right)^{p-1}=(MR_k)^{p-1},\quad 0\leq u_{n-1}\leq R_k. \tag{3.24}$$

So, condition (A_2) holds. Therefore, by Theorem 3.1, we know that the result of Theorem 3.3 holds. The proof of Theorem 3.3 is complete. □

Remark 3.4. We can check that there exists a function $g(t)$ satisfying condition (A_2). In fact, let

$$\Delta=\sqrt{2}\left(\frac{\pi^2}{3}-\frac{9}{4}\right),\qquad t_0=\frac{5}{16},\qquad t_n=t_0-\sum_{i=1}^{n-1}\frac{1}{(i+2)^4},\quad n=1,2,\dots. \tag{3.25}$$

Consider function $g(t):[0,1]\to(0,+\infty)$, given by $g(t)=\sum_{n=1}^{\infty}g_n(t)$, $t\in[0,1]$, where

$$g_n(t)=\begin{cases}\dfrac{1}{n(n+1)\left(t_{n+1}+t_n\right)}, & 0\leq t<\dfrac{t_{n+1}+t_n}{2},\\[2ex] \dfrac{1}{\Delta\left(t_n-t\right)^{1/2}}, & \dfrac{t_{n+1}+t_n}{2}\leq t<t_n,\\[2ex] \dfrac{1}{\Delta\left(t-t_n\right)^{1/2}}, & t_n\leq t\leq\dfrac{t_{n+1}+t_n}{2},\\[2ex] \dfrac{2}{n(n+1)\left(2-t_n-t_{n-1}\right)}, & \dfrac{t_{n-1}+t_n}{2}<t\leq 1.\end{cases} \tag{3.26}$$

It is easy to know $t_1=1/4<1/2$, $t_n-t_{n+1}=1/(n+2)^4$ $(n=1,2,\dots)$, and

$$t^*=\lim_{n\to\infty}t_n=\frac{5}{16}-\sum_{i=1}^{\infty}\frac{1}{(i+2)^4}=\frac{22}{16}-\frac{\pi^4}{90}>\frac{1}{5}, \tag{3.27}$$

where $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$. From $\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_0^1 g_n(t)dt &= \sum_{n=1}^{\infty}\frac{2}{n(n+1)} + \frac{1}{\Delta}\sum_{n=1}^{\infty}\left[\int_{(t_{n+1}+t_n)/2}^{t_n}\frac{1}{(t_n-t)^{1/2}}dt + \int_{t_n}^{(t_n-t_{n-1})/2}\frac{1}{(t-t_n)^{1/2}}dt\right] \\
&= 2 + \frac{\sqrt{2}}{\Delta}\sum_{n=1}^{\infty}\left[\left(t_n-t_{n+1}\right)^{1/2} + \left(t_{n-1}-t_n\right)^{1/2}\right] \\
&= 2 + \frac{\sqrt{2}}{\Delta}\sum_{n=1}^{\infty}\left[\frac{1}{(n+2)^2} + \frac{1}{(n+1)^2}\right] \\
&= 2 + \frac{\sqrt{2}}{\Delta}\sum_{n=1}^{\infty}\left[\left(\frac{\pi^2}{6}-\frac{5}{4}\right) + \left(\frac{\pi^2}{6}-1\right)\right] \\
&= 2 + \frac{\sqrt{2}}{\Delta}\left[\frac{\pi^2}{3}-\frac{9}{4}\right] = 3.
\end{aligned} \tag{3.28}$$

Therefore,

$$\int_0^1 g(t)dt = \int_0^1\sum_{n=1}^{\infty}g_n(t)dt = \sum_{n=1}^{\infty}\int_0^1 g_n(t)dt < \infty. \tag{3.29}$$

Then we know that $g(t)$ satisfies condition (H_2).

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