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Research Article

Existence of Triple Positive Solutions for Second-Order Discrete Boundary Value Problems

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By using a new fixed-point theorem introduced by Avery and Peterson (2001), we obtain sufficient conditions for the existence of at least three positive solutions for the equation $\Delta^2 x(k-1)+q(k)f(k,x(k),\Delta x(k))=0$, for $k\in\{1,2,\dots,n-1\}$, subject to the following two boundary conditions: $x(0)=x(n)=0$ or $x(0)=\Delta x(n-1)=0$, where $n\geq 3$.

1. Introduction

The second-order differential and difference boundary value problems arise in many branches of both applied and basic mathematics and have been extensively studied in the literature. We refer the reader to [1–4] for some recent results for second-order nonlinear two-point boundary value problems. The main tools used in the above works are fixed-point theorems.

Avery and Peterson [1] generalize the fixed-point theorem of Leggett-Williams by using theory of fixed-point index and Dugundji extension theorem. Recently, Bai et al. [5] have applied this theorem to prove the existence of three positive solutions for the second-order differential equation $x''(t)+q(t)f(t,x(t),x'(t))=0$, $0<t<1$.

In this paper, the aim of this work is to establish the existence of three positive solutions for the second-order difference equation

$$\Delta^2 x(k-1)+q(k)f\big(k,x(k),\Delta x(k)\big)=0,\quad \text{for } k\in\{1,2,\dots,n-1\},\tag{1.1}$$

subject to one of the following two pairs of boundary conditions:

$$x(0)=x(n)=0,\tag{1.2}$$

$$x(0)=\Delta x(n-1)=0,\tag{1.3}$$

where $\Delta x(k) = x(k+1) - x(k)$, for $k \in \{0,1,\dots,n-1\}$, and $\Delta^2 x(k) = x(k+2) - 2x(k+1) + x(k)$, for $k \in \{0,1,\dots,n-2\}$.

We are concerned with positive solutions to the above problem, that is, $x(k) \geq 0$, for $k \in \{0,1,\dots,n\}$, and assume that

(C1) $f : \{1,2,\dots,n-1\} \times [0,\infty) \times R \to [0,\infty)$ is continuous;

(C2) $q(k) \geq 0$ but $q(k)$ does not identically equal to zero, for $k \in \{1,2,\dots,n-1\}$.

We will depend on an application of a fixed-point theorem due to Avery and Peterson, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space to obtain our main results, and an example to illustrate the main results in this paper.

2. Background materials and definitions

In this section, we present some background materials that will be needed in our discussion.

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P \subset E$ is said to be a cone of E, if it satisfies the following conditions:

(i) $x \in P$, $\lambda \geq 0$, implies $\lambda x \in P$,

(ii) $x \in P$, $-x \in P$, implies $x = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \tag{2.1}$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, the map β is called a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y) \tag{2.2}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$\begin{gathered} P(\gamma,d) = \{x \in P \mid \gamma(x) < d\}, \\ P(\gamma,\alpha,b,d) = \{x \in P \mid b \leq \alpha(x),\ \gamma(x) \leq d\}, \\ P(\gamma,\theta,\alpha,b,c,d) = \{x \in P \mid b \leq \alpha(x),\ \theta(x) \leq c,\ \gamma(x) \leq d\}, \end{gathered} \tag{2.3}$$

and a closed set

$$R(\gamma,\psi,a,d)=\{x\in P \mid a\le\psi(x),\ \gamma(x)\le d\}. \tag{2.4}$$

The following fixed-point theorem of Avery and Peterson plays an important role in this paper.

Theorem 2.4 [1]. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x)\le\lambda\psi(x)$, for $0\le\lambda\le 1$, such that for some positive numbers M and d,*

$$\alpha(x)\le\psi(x),\qquad \|x\|\le M\gamma(x), \tag{2.5}$$

for all $x\in\overline{P(\gamma,d)}$. Suppose $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a, b, and c with $a<b$ such that

(S1) $\{x\in P(\gamma,\theta,\alpha,b,c,d)\mid\alpha(x)>b\}\neq\emptyset$ *and* $\alpha(Tx)>b$, *for* $x\in P(\gamma,\theta,\alpha,b,c,d)$;

(S2) $\alpha(Tx)>b$, *for* $x\in P(\gamma,\alpha,b,d)$ *with* $\theta(Tx)>c$;

(S3) $0\notin R(\gamma,\psi,a,d)$ *and* $\psi(Tx)<a$, *for* $x\in R(\gamma,\psi,a,d)$ *with* $\psi(x)=a$.

Then T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\gamma,d)}$, such that

$$\begin{gathered}\gamma(x_i)\le d,\quad \text{for } i=1,2,3;\ b<\alpha(x_1);\\ a<\psi(x_2),\quad \text{with } \alpha(x_2)<b;\ \psi(x_3)<a.\end{gathered} \tag{2.6}$$

3. Main results

In this section, we will impose suitable growth conditions on f, which enable us to apply Theorem 2.4 with respect to obtaining triple positive solutions of BVP (1.1)-(1.2) and (1.1)–(1.3).

Now, we deal with the first problem. Let $X=\{x:\{0,1,\dots,n\}\to R\}$ be endowed with the ordering $x\le y$ if $x(k)\le y(k)$, for all $k\in\{0,1,\dots,n\}$ with the norm

$$\|x\|=\max\left\{\max_{k\in\{0,1,\dots,n\}}|x(k)|,\ \max_{k\in\{0,1,\dots,n-1\}}|\Delta x(k)|\right\}. \tag{3.1}$$

Then, we define the cone P in E by

$$P=\{x\in X: x(k)\ge 0,\ k\in\{0,1,\dots,n\};\ x(0)=x(n)=0,\ \Delta^2x(k)\le 0,\ k\in\{0,1,\dots,n-2\}\}. \tag{3.2}$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ, γ, and the nonnegative continuous functional ψ be defined on the cone P

by

$$\gamma(x) = \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|, \qquad \psi(x) = \theta(x) = \max_{k\in\{0,1,\dots,n\}} |x(k)|,$$
$$\alpha(x) = \min_{k\in\{[n/4]+1,\dots,n-[n/4]-1\}} |x(k)|. \tag{3.3}$$

In order to prove our main results,we need the following lemma.

Lemma 3.1. *If $x \in P$, then*

$$\max_{k\in\{0,1,\dots,n\}} |x(k)| \le \frac{n}{2} \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|, \quad \textit{that is, } \theta(x) \le \frac{n}{2}\gamma(x),$$
$$\max_{k\in\{0,1,\dots,n\}} |x(k)| \ge \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|, \quad \textit{that is, } \theta(x) \ge \gamma(x). \tag{3.4}$$

Proof. Suppose the maximum of x occurs at $k_0 \in \{1,2,\dots,n-1\}$, by the definition of the cone P, we know $\Delta x(k+1) \le \Delta x(k)$, for $k \in \{0,1,\dots,n-2\}$, then $\Delta x(k) \ge 0$, for $k \in \{0,1,\dots,k_0-1\}$, and $\Delta x(k) \le 0$, for $k \in \{k_0,k_0+1,\dots,n-1\}$. Then,

$$x(k_0) = x(k_0) - x(0) = \Delta x(0) + \Delta x(1) + \cdots + \Delta x(k_0-1)$$
$$\le k_0 \Delta x(0) \le k_0 \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|,$$
$$x(k_0) = |x(n) - x(k_0)| = |\Delta x(k_0) + \cdots + \Delta x(n-1)|$$
$$\le (n-k_0)|\Delta x(n-1)| \le (n-k_0) \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|. \tag{3.5}$$

Since

$$k_0 \le \frac{n}{2} \quad \text{or} \quad n - k_0 \le \frac{n}{2}, \tag{3.6}$$

so, we have

$$\max_{k\in\{0,1,\dots,n\}} |x(k)| \le \frac{n}{2} \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)|. \tag{3.7}$$

The proof is complete. □

By Lemma 3.1 and the definitions, the functionals defined above satisfy

$$\frac{1}{4}\theta(x) \le \alpha(x) \le \theta(x) = \psi(x), \qquad \|x\| = \max\{\theta(x), \gamma(x)\} = \theta(x) \le \frac{n}{2}\gamma(x), \tag{3.8}$$

for all $x \in \overline{P(\gamma,d)} \subset P$. Therefore, condition (2.5) is satisfied.

Now, we show that $(1/4)\theta(x) \le \alpha(x)$. Here, we also suppose $\theta(x) = x(k_0)$, and by the definitions of α and the cone P, we can distinguish two cases.

(i) $\alpha(x) = x([n/4]+1)$, then we certainly have $k_0 \ge [n/4]+1$, and

$$\begin{aligned} x(k_0) &= \Delta x(0) + \cdots + \Delta x\left(\left[\frac{k_0}{4}\right]\right) + \Delta x\left(\left[\frac{k_0}{4}\right]+1\right) + \cdots + \Delta x\left(\left[\frac{k_0}{2}\right]\right) \\ &\quad + \Delta x\left(\left[\frac{k_0}{2}\right]+1\right) + \cdots + \Delta x\left(\left[\frac{3k_0}{4}\right]\right) + \Delta x\left(\left[\frac{3k_0}{4}\right]+1\right) + \cdots + \Delta x(k_0-1) \\ &\le 4x\left(\left[\frac{k_0}{4}\right]+1\right) \le 4x\left(\left[\frac{n}{4}\right]+1\right), \end{aligned} \tag{3.9}$$

that is, $(1/4)x(k_0) \le x([n/4]+1)$.

(ii) $\alpha(x) = x(n-[n/4]-1)$, then $k_0 \le n-[n/4]-1$, and

$$\begin{aligned} x(k_0) &= -\Bigg(\Delta x(n-1) + \cdots + \Delta x\left(n - \left[\frac{n-k_0}{4} - 1\right]\right) + \Delta x\left(n - \left[\frac{n-k_0}{4}\right] - 2\right) \\ &\quad + \cdots + \Delta x\left(n - \left[\frac{n-k_0}{2} - 1\right]\right) + \Delta x\left(n - \left[\frac{n-k_0}{2}\right] - 2\right) \\ &\quad + \cdots + \Delta x\left(n - \left[\frac{3(n-k_0)}{4} - 1\right]\right) + \Delta x\left(n - \left[\frac{3(n-k_0)}{4}\right] - 2\right) + \cdots + \Delta x(k_0)\Bigg) \\ &\le 4x\left(n - \left[\frac{n-k_0}{4}\right]\right) \le 4x\left(n - \left[\frac{n}{4}\right] - 1\right), \end{aligned} \tag{3.10}$$

that is, $(1/4)x(k_0) \le x(n-[n/4]-1)$. So, we have $(1/4)\theta(x) \le \alpha(x)$.

$G(k,i)$ is Green's function for boundary value problem

$$\begin{gathered} -\Delta^2 x(k-1) = 0, \quad \text{for } k \in \{1,2,\ldots,n-1\}, \\ x(0) = x(n) = 0. \end{gathered} \tag{3.11}$$

Then $G : \{0,1,\ldots,n\} \times \{1,2,\ldots,n-1\} \to R$ is given by

$$G(k,i) = \begin{cases} (n-k)\dfrac{i}{n}, & \text{for } 0 \le i \le k \le n, \\ (n-i)\dfrac{k}{n}, & \text{for } 0 \le k \le i \le n. \end{cases} \tag{3.12}$$

Let

$$\begin{gathered} M = \max\left\{\sum_{i=1}^{n-1} \frac{n-i}{n} q(i), \sum_{i=1}^{n-1} \frac{i}{n} q(i)\right\}, \\ \delta = \min\left\{\sum_{i=1}^{n-1} G\left(\left[\frac{n}{4}\right]+1, i\right) q(i), \sum_{i=1}^{n-1} G\left(n - \left[\frac{n}{4}\right] - 1, i\right) q(i)\right\}, \\ N = \max_{k\in\{0,1,\ldots,n\}} \sum_{i=1}^{n-1} G(k,i) q(i). \end{gathered} \tag{3.13}$$

To present our main result, we assume there exist constants $0 < a < b \leq d/4$ such that
(A1) $f(k,u,v) \leq d/M$, for $(k,u,v) \in \{1,2,\dots,n-1\} \times [0,(n/2)d] \times [-d,d]$;
(A2) $f(k,u,v) > b/\delta$, for $(k,u,v) \in \{[n/4]+1,\dots,n-[n/4]-1\} \times [b,4b] \times [-d,d]$;
(A3) $f(k,u,v) < a/N$, for $(k,u,v) \in \{1,2,\dots,n-1\} \times [0,a] \times [-d,d]$.

Theorem 3.2. *With the assumptions (A1)–(A3), the boundary value problem (1.1)-(1.2) has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\begin{gathered}\max_{k\in\{0,1,\dots,n-1\}} |\Delta x_i(k)| \leq d, \quad \text{for } i = 1,2,3; \\ b < \min_{k\in\{[n/4]+1,\dots,n-[n/4]-1\}} x_1(k); \quad a < \max_{k\in\{0,1,\dots,n\}} x_2(k), \\ \text{with} \min_{k\in\{[n/4]+1,\dots,n-[n/4]-1\}} x_2(k) < b; \quad \max_{k\in\{0,1,\dots,n\}} x_3(k) < a.\end{gathered} \tag{3.14}$$

Proof. x is a solution of problem (1.1)-(1.2) if and only if

$$x(k) = Tx(k) = \sum_{i=1}^{n-1} G(k,i)q(i)f(i,x(i),\Delta x(i)). \tag{3.15}$$

Using the continuity of f and the definition of T, it is easy to see that $T: P \to P$ is continuous. Next, we prove T is completely continuous.

Suppose that the sequence $\{x_i\} \subseteq P$ is bounded, then there exists $M > 0$, such that $\|x_i\| \leq M$, for any $i = 1,2,\dots$. By the continuity of f with Green's function, G is bounded, we know that there exists $M' > 0$, such that $|Tx_i(k)| \leq M'$, for $k \in \{0,1,\dots,n\}$ and $i = 1,2,\dots$. In view of the bounded sequence $\{Tx_i(0)\}$, there exists $\{x_{i0}\} \subseteq \{x_i\}$, such that $\lim_{i\to\infty} Tx_{i0}(0) = a_0$. For the bounded sequence $\{Tx_{i0}(1)\}$, there exists $\{x_{i1}\} \subseteq \{x_{i0}\}$, such that $\lim_{i\to\infty} Tx_{i1}(1) = a_1$. By repetition in this way, we have that there exists $\{x_{ij}\} \subseteq \{x_{ij-1}\}$, for $j = 2,3,\dots,n$, such that $\lim_{i\to\infty} Tx_{ij}(j) = a_j$. Let $y = \{a_0,a_1,\dots,a_n\}$, by the definition of the norm on X, there exists $\{x_{in}\} \subseteq \{x_i\}$, such that $\lim_{i\to\infty} Tx_{in} = y$.

Hence, $T: P \to P$ is completely continuous.

We now show that all the conditions of Theorem 2.4 are satisfied. If $x \in \overline{P(\gamma,d)}$, then

$$\gamma(x) = \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)| \leq d. \tag{3.16}$$

With Lemma 3.1,

$$\max_{k\in\{0,1,\dots,n\}} |x(k)| \leq \frac{n}{2}d, \tag{3.17}$$

then assumption (A1) implies $f(k,u,v) \leq d/M$, and

$$\begin{aligned}
\gamma(Tx) &= \max_{k\in\{0,1,\ldots,n-1\}} |\Delta Tx(k)| = \max\Big\{|\Delta Tx(0)|, |\Delta Tx(n-1)|\Big\} \\
&= \max\Big\{Tx(1), Tx(n-1)\Big\} \\
&= \max\left\{\sum_{i=1}^{n-1} G(1,i)q(i)f\big(i,x(i),\Delta x(i)\big), \sum_{i=1}^{n-1} G(n-1,i)q(i)f\big(i,x(i),\Delta x(i)\big)\right\} \\
&\leq \frac{d}{M}\max\left\{\sum_{i=1}^{n-1} G(1,i)q(i), \sum_{i=1}^{n-1} G(n-1,i)q(i)\right\} \\
&= \frac{d}{M}\max\left\{\sum_{i=1}^{n-1} \frac{n-i}{n}q(i), \sum_{i=1}^{n-1} \frac{i}{n}q(i)\right\} = \frac{d}{M}\cdot M = d.
\end{aligned} \tag{3.18}$$

Thus, $T:\overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

To check condition (S1) of Theorem 2.4, let $x(k) = 4b$, for $k \in \{1,2,\ldots,n-1\}$ and $x(0) = x(n) = 0$. It is easy to see that $x \in P(\gamma,\theta,\alpha,b,4b,d)$ and $\alpha(x) = 4b > b$, so $\{x \in P(\gamma,\theta,\alpha,b,4b,d) \mid \alpha(x) > b\} \neq \emptyset$. If $x \in P(\gamma,\theta,\alpha,b,4b,d)$, then $b \leq x(k) \leq 4b$, $|\Delta x(k)| \leq d$, for $k \in \{[n/4]+,\ldots,n-[n/4]-1\}$. From assumption (A2), we have $f(k,x(k),\Delta x(k)) > b/\delta$ for $k \in \{[n/4]+1,\ldots,n-[n/4]-1\}$, then

$$\begin{aligned}
\alpha(Tx) &= \min\left\{Tx\left(\left[\frac{n}{4}\right]+1\right), Tx\left(n-\left[\frac{n}{4}\right]-1\right)\right\} \\
&= \min\left\{\sum_{i=1}^{n-1} G\left(\left[\frac{n}{4}\right]+1,i\right)q(i)f\big(i,x(i),\Delta x(i)\big),\right. \\
&\qquad \left.\sum_{i=1}^{n-1} G\left(n-\left[\frac{n}{4}\right]-1,i\right)q(i)f\big(i,x(i),\Delta x(i)\big)\right\} \\
&> \frac{b}{\delta}\min\left\{\sum_{i=1}^{n-1} G\left(\left[\frac{n}{4}\right]+1,i\right)q(i), \sum_{i=1}^{n-1} G\left(n-\left[\frac{n}{4}\right]-1,i\right)q(i)\right\} = \frac{b}{\delta}\cdot\delta = b.
\end{aligned} \tag{3.19}$$

Therefore, the condition (S1) of Theorem 2.4 is satisfied.

Secondly, with (3.8), we have

$$\alpha(Tx) \geq \frac{1}{4}\theta(Tx) > \frac{1}{4}\cdot 4b = b, \tag{3.20}$$

for all $x \in P(\gamma,\alpha,b,d)$ with $\theta(Tx) > 4b$. Thus, condition (S2) of Theorem 2.4 is satisfied.

Finally, we show that (S3) of Theorem 2.4 also holds. As $\psi(0) = 0 < a$, we know $0 \notin R(\gamma,\psi,a,d)$. Suppose that $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$. Then $0 \leq x(k) \leq a$, $-d \leq \Delta x(k) \leq d$, for $k \in \{1,2,\ldots,n-1\}$, from the assumption (A3), we have $f(k,x(k),\Delta x(k)) < a/N$.

Then,

$$\begin{aligned}\psi(Tx) &= \max_{k\in\{0,1,\dots,n\}} |Tx(k)| = \max_{k\in\{0,1,\dots,n\}} \sum_{i=1}^{n-1} G(k,i)q(i)f(i,x(i),\Delta x(i)) \\ &< \frac{a}{N} \max_{k\in\{0,1,\dots,n\}} \sum_{i=1}^{n-1} G(k,i)q(i) = \frac{a}{N}\cdot N = a.\end{aligned} \tag{3.21}$$

So, condition (S3) of Theorem 2.4 is satisfied.

Applying Theorem 2.4, we know the boundary value problem (1.1)-(1.2) has at least three positive solutions x_1, x_2, and x_3 satisfying (3.14). The proof is complete. □

Remark 3.3. To apply Theorem 2.4, we only need $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, therefore, condition (C1) can be substituted with a weaker condition:

(C1)' $f : \{1,2,\dots,n-1\}\times[0,(n/2)d]\times[-d,d]\to[0,\infty)$ is continuous.

Now we deal with problem (1.1)–(1.3). The method is just similar to what we have done above. So, the proof is omitted. Define the cone $P_1 \subset X$ by

$$\begin{aligned}P_1 = \Big\{&x\in X : x(k)\ge 0,\ k\in\{0,1,\dots,n\};\ x(0)=\Delta x(n-1)=0,\ \Delta^2 x(k)\le 0, \\ &k\in\{0,1,\dots,n-2\}\Big\}.\end{aligned} \tag{3.22}$$

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functional θ_1, γ, and the nonnegative continuous functional ψ be defined on the cone P_1 by

$$\begin{aligned}\gamma_1(x) &= \max_{k\in\{0,1,\dots,n-1\}} |\Delta x(k)| = \Delta x(0) = x(1), \\ \psi_1(x) = \theta_1(x) &= \max_{k\in\{0,1,\dots,n\}} |x(k)| = x(n-1) = x(n), \\ \alpha_1(x) &= \min_{k\in\{[n/2],\dots,n\}} |x(k)| = x\left(\left[\frac{n}{2}\right]\right), \quad \text{for } x\in P_1.\end{aligned} \tag{3.23}$$

Lemma 3.4. *If $x\in P_1$, then $\theta_1(x)\le(n-1)\gamma_1(x)$.*

With Lemma 3.4 and the definitions, the functionals defined above satisfy

$$\frac{1}{2}\theta_1(x)\le\alpha_1(x)\le\theta_1(x)=\psi_1(x), \qquad \|x\|=\max\{\theta_1(x),\gamma_1(x)\}=\theta_1(x)\le(n-1)\gamma_1(x), \tag{3.24}$$

for all $x\in\overline{P_1(\gamma_1,d)}\subset P_1$. Therefore, condition (2.5) is satisfied.

$G_1(k,i)$ is Green's function for boundary value problem

$$\begin{aligned}-\Delta^2 x(k-1) &= 0, \quad \text{for } k\in\{1,2,\dots,n-1\}, \\ x(0) &= \Delta x(n-1) = 0.\end{aligned} \tag{3.25}$$

Then, $G_1 : \{0,1,\dots,n\} \times \{1,2,\dots,n-1\} \to R$ is given by

$$G_1(k,i) = \begin{cases} i, & \text{for } 0 \le i \le k \le n, \\ k, & \text{for } 0 \le k \le i \le n. \end{cases} \tag{3.26}$$

Let

$$\begin{aligned} M_1 &= \sum_{i=1}^{n-1} q(i), \\ \delta_1 &= \sum_{i=1}^{[n/2]} iq(i) + \left[\frac{n}{2}\right] \sum_{i=[n/2]+1}^{n-1} q(i), \\ N_1 &= \sum_{i=1}^{n-1} iq(i). \end{aligned} \tag{3.27}$$

Suppose there exist constants $0 < a < b \le (1/2)d$ such that

(A4) $f(k,u,v) \le d/M_1$, for $(k,u,v) \in \{1,2,\dots,n-1\} \times [0,nd] \times [0,d]$;

(A5) $f(k,u,v) > b/\delta_1$, for $(k,u,v) \in \{[n/2],\dots,n-1\} \times [b,2b] \times [0,d]$;

(A6) $f(k,u,v) < a/N_1$, for $(k,u,v) \in \{1,2,\dots,n-1\} \times [0,a] \times [0,d]$.

Theorem 3.5. *Under assumptions (A4)–(A6), the boundary-value problem (1.1)–(1.3) has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\max_{k\in\{0,1,\dots,n-1\}} \Delta x_i(k) \le d, \quad \text{for } i = 1,2,3;$$

$$b < \min_{k\in\{[n/2],\dots,n\}} x_1(k); \quad a < \max_{k\in\{0,1,\dots,n\}} x_2(k), \quad \text{with} \min_{k\in\{[n/2],\dots,n\}} x_2(k) < b; \quad \max_{k\in\{0,1,\dots,n\}} x_3(k) < a. \tag{3.28}$$

Example 3.6. Consider the following boundary value problem:

$$\Delta^2 x(k-1) + f(k,x(k),\Delta x(k)) = 0, \quad \text{for } k \in \{1,2,3,4\}\ x(0) = \Delta x(4) = 0, \tag{3.29}$$

with $a = 12$, $b = 15$, $d = 60$, where

$$f(k,u,v) = \begin{cases} \ln k + \left(\dfrac{u}{12}\right)^2 + \left(\dfrac{v}{60}\right)^2, & \text{for } u \le 12, \\ \ln k + \left(\dfrac{u}{5}\right)^2 + \left(\dfrac{v}{60}\right)^3, & \text{for } 12 < u \le 60, \\ \ln k + \left(\dfrac{u}{30}\right)^2 + \left(\dfrac{v}{60}\right)^2, & \text{for } u > 60, \end{cases} \tag{3.30}$$

and note $M = 2$, $\delta = 3$, $N = 3$. Then, $f(k,u,v)$ satisfies

(i) $f(k,u,v) < a/N = 4$, for $(k,u,v) \in \{1,2,3,4\} \times [0,12] \times [-60,60]$;

(ii) $f(k,u,v) > b/\delta = 5$, for $(k,u,v) \in \{2,3\} \times [15,60] \times [-60,60]$;

(iii) $f(k,u,v) \le d/M = 30$, for $(k,u,v) \in \{1,2,3,4\} \times [0,150] \times [-60,60]$.

It is clear that all the assumptions of Theorem 3.5 are satisfied. Therefore, by Theorem 3.5, we know that problem (3.29) has at least three positive solutions x_1, x_2,

x_3 such that

$$\max_{k\in\{0,1,\dots,4\}} \Delta x_i(k) \le 60, \quad \text{for } i = 1,2,3;$$
$$15 < \min_{k\in\{2,3\}} x_1(k); \quad 12 < \max_{k\in\{0,1,\dots,5\}} x_2(k), \quad \text{with } \min_{k\in\{2,3\}} x_2(k) < 15; \ \max_{k\in\{0,1,\dots,5\}} x_3(k) < 12. \tag{3.31}$$

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References

[1] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 313–322, 2001.
[2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, The Netherlands, 1999.
[3] R. I. Avery and J. Henderson, "Three symmetric positive solutions for a second-order boundary value problem," *Applied Mathematics Letters*, vol. 13, no. 3, pp. 1–7, 2000.
[4] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.
[5] Z. Bai, Y. Wang, and W. Ge, "Triple positive solutions for a class of two-point boundary-value problems," *Electronic Journal of Differential Equations*, vol. 2004, no. 6, pp. 1–8, 2004.

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