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Research Article

A Dynamic Economic Model with Discrete Time and Consumer Sentiment

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The paper describes a dynamical economic model with discrete time and consumer sentiment in the deterministic and stochastic cases. We seek to demonstrate that consumer sentiment may create fluctuations in the economical activities. The model possesses a flip bifurcation and a Neimark-Sacker bifurcation, after which the stable state is replaced by a (quasi) periodic motion. We associate the difference stochastic equation to the model by randomizing the control parameter d and by adding one stochastic control. Numerical simulations are made for the deterministic and stochastic models, for different values of the control parameter d.

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1. Introduction

The economic empirical evidence [1–3] suggests that consumer sentiment influences household expenditure and thus confirms Keynes' assumption that consumer "attitudes" and "animalic spirit" may cause fluctuations in the economic activity. On the other hand, Dobrescu and Opris [4, 5] analyzed the bifurcation aspects in a discrete-delay Kaldor model of business cycle, which corresponds to a system of equations with discrete time and delay. Following these studies, we develop a dynamic economic model in which the agents' consumption expenditures depend on their sentiment. As particular cases, the model contains the Hick-Samulson [6], Puu [7], and Keynes [3] models as well as the model from [3]. The model possesses a flip and Neimark-Sacker bifurcation, if the autonomous consumption is variable.

The implications of the stochastic noise on the economic process are studied. The stochastic difference equation with noise terms is scaled appropriately to account for intrinsic as well as extrinsic fluctuations. Under the influence of noise the difference equation behaves qualitatively different compared to its deterministic counterpart.

The paper is organized as follows. In Section 2, we describe the dynamic model with discrete time using investment, consumption, sentiment, and saving functions. For different values of the model parameters we obtain well-known dynamic models (Hick-Samuelson, Keynes, Pu). In Section 3, we analyze the behavior of the dynamic system in the fixed point's neighborhood for the associated map. We establish asymptotic stability conditions for the flip and Neimark-Sacker bifurcations. In both the cases of flip and Neimark-Sacker bifurcations, the normal forms are described in Section 4. Using the QR method, the algorithm for determining the Lyapunov exponents is presented in Section 5. In Section 6, a stochastic model with multiplicative noise is associated to the deterministic model. These equations are obtained by randomizing one parameter of the deterministic equation or by adding one stochastic control. Finally, the numerical simulations are done for the deterministic and stochastic equations. The obtained simulations show major changes between the deterministic and stochastic cases. The analysis of the present model proves its complexity and allows the description of the different scenarios which depend on autonomous consumption.

2. The Mathematical Model with Discrete Time and Consumer Sentiment

Let $y(t)$, $t \in N$ be the income at time step t and let

(1) the investment function $I(t)$, $t \in \mathbb{N}$, be given by

$$I(t) = v\big(y(t-1) - y(t-2)\big) - w\big(y(t-1) - y(t-2)\big)^3, v > 0, w \geq 0; \tag{2.1}$$

(2) the consumption function $C(t)$, $t \in \mathbb{N}$, be given by

$$C(t) = a + y(t-1)(b + cS(t-1)), \quad a \geq 0, b > 0,\ c \geq 0, \tag{2.2}$$

where $S(t), t \in \mathbb{N}$, is the sentiment function given by

$$S(t) = \frac{1}{1 + \varepsilon \exp\big(y(t-1) - y(t)\big)}, \quad \varepsilon \in [0,1]; \tag{2.3}$$

(3) the saving function $E(t)$, $t \in \mathbb{N}$, be given by

$$E(t) = d\big(y(t-2) - y(t-1)\big) + mS(t-1), \quad d \geq 0, m \geq 0. \tag{2.4}$$

The mathematical model is described by the relation:

$$y(t) = I(t) + C(t) + E(t). \tag{2.5}$$

From (2.1), (2.2), (2.3), (2.4), and (2.5) the mathematical model with discrete time and consumer sentiment is given by

$$y(t) = a + dy(t-2) + (b-d)y(t-1) + v\big(y(t-1) - y(t-2)\big)$$
$$-w(y(t-1) - y(t-2))^3 + \frac{cy(t-1) + m}{1 + \varepsilon \exp\big(y(t-2) - y(t-1)\big)}, \quad t \in \mathbb{N}. \tag{2.6}$$

The parameters from (2.6) have the following economic interpretations. The parameter a represents the autonomous expenditures. The parameter d is the control, $d \in [0,1]$, and it characterizes a part of the difference between the incomes obtained at two time steps $t-2$ and $t-1$, which is used for consumption or saving in the time step t. The parameter c, $c \in [0,1]$, is the trend towards consumption. The parameter m, $m \in [0,1]$, is the trend towards the saving. The parameter b, $b \in (0,1)$, represents the consumer's reaction against the increase or decrease of his income. When the income (strongly) decreases, the consumer becomes pessimistic and consumes $0 < b < 1$ of his income. When the income (strongly) increases, the consumer becomes optimistic and consumes $b < b + c < 1$ of his income. Note that Souleles [8] finds, in fact, that higher consumer confidence is correlated with less saving and increases in relation to expected future resources. The parameters v and w, $v > 0$, $w \geq 0$ describe the investment function. If $w = 0$, the investment function is linear. The parameter ε, $\varepsilon \in [0,1]$, describes a family of the sentiment functions.

For different values of the model parameters, we obtain the following classical models:

(1) for $a = 0, b = 1 - s, d = 0, m = 0, \varepsilon = 0, s \in (0,1)$ from (2.6) we obtain the Hick-Samuelson model [6]:

$$y(t) = (1 + v - s)y(t-1) - vy(t-2), \quad v > 0, s \in (0,1); \tag{2.7}$$

(2) for $v = w = 0, \varepsilon = 0, m = 0$, (2.6) gives us the Keynes model [6]:

$$y(t) = d\big(y(t-2) - y(t-1)\big) + a + by(t-1); \tag{2.8}$$

(3) for $v = w = 0, \varepsilon = 1$, (2.6) leads to the model from [4]:

$$y(t) = d\big(y(t-2) - y(t-1)\big) + a + by(t-1) + \frac{cy(t-1) + m}{1 + \exp\big(y(t-2) - y(t-1)\big)}, \quad t \in \mathbb{N}; \tag{2.9}$$

(4) for $a = 0, b = s - 1, d = 0, m = 0, w = 1 + v, \varepsilon = 0$, from (2.6) we get the Puu model [7]:

$$y(t) = v\big(y(t-1) - y(t-2)\big) - (1 + v)(y(t-1) - y(t-2))^3 - (1 - s)y(t-1). \tag{2.10}$$

3. The Dynamic Behavior of the Model (2.6)

Using the method from Kusnetsov [4, 6, 9], we will analyze the system (2.6), considering the parameter a as bifurcation parameter. The associated map of (2.6) is $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a + (b - d + v)y + (d - v)z - w(y - z)^3 + \dfrac{(cy + m)}{1 + \varepsilon \exp(z - y)} \\ y \end{pmatrix}. \tag{3.1}$$

Using the methods from [5, 6, 9], the map (3.1) has the following properties.

Proposition 3.1. (i) *If* $(1+\varepsilon)(1-b) - c > 0$, *then, for the map* (3.1), *the fixed point with the positive components is* $E_0(y_0, z_0)$, *where*

$$y_0 = p_1 a + p_2, \qquad z_0 = y_0, \tag{3.2}$$

$$p_1 = \frac{(1+\varepsilon)}{(1+\varepsilon)(1-b) - c}, \qquad p_2 = \frac{m}{(1+\varepsilon)(1-b) - c}. \tag{3.3}$$

(ii) *The Jacobi matrix of the map* F *in* E_0 *is given by*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \end{pmatrix}, \tag{3.4}$$

where: $a_{11} = p_3 a - p_4 + p_5$, $a_{12} = -p_3 a + p_4$,

$$p_3 = \frac{\varepsilon c}{(1+\varepsilon)^2} p_1, \qquad p_4 = d - v - \frac{\varepsilon m}{(1+\varepsilon)^2} - \frac{\varepsilon c}{(1+\varepsilon)^2} p_2, \qquad p_5 = b + \frac{c}{(1+\varepsilon)}. \tag{3.5}$$

(iii) *The characteristic equation of matrix* A *is given by*

$$\lambda^2 - a_{11}\lambda - a_{12} = 0. \tag{3.6}$$

(iv) *If the model parameters* $d, v, \varepsilon, b, c, m$ *satisfy the following inequality:*

$$(1 + d - v)(1+\varepsilon)((1+\varepsilon)(1-b) - c) - m\varepsilon(1-b) > 0, \tag{3.7}$$

then, for (3.6), *the roots have the modulus less than 1, if and only if* $a \in (a_1, a_2)$, *where*

$$a_1 = \frac{2p_4 - p_5 - 1}{2p_3}, \qquad a_2 = \frac{1 + p_4}{p_3}. \tag{3.8}$$

(v) *If the model parameters* d, v, ε, b, c, m *satisfy the inequality* (3.7) *and* $a = a_1$, *then, one of equation* (3.6)*'s roots is* -1, *while the other one has the modulus less than 1.*

(vi) *If the model parameters* $d, v, \varepsilon, b, c, m$ *satisfy the inequality* (3.7) *and* $a = a_2$, *then,* (3.6) *has the roots* $\mu_1(a) = \mu(a), \mu_2(a) = \overline{\mu}(a)$, *where* $|\mu(a)| = 1$.

Using [7] and Proposition 3.1, with respect to parameter a, the asymptotic stability conditions of the fixed point, the conditions for the existence of the flip and Neimark-Sacker bifurcations are presented in the following.

Proposition 3.2. (i) *If* $(1+\varepsilon)(1-b) - c > 0$, *the inequality* (3.7) *holds and* $2p_4 - p_5 - 1 > 0$, *then for* $a \in (a_1, a_2)$ *the fixed point* E_0 *is asymptotically stable. If* $(1+\varepsilon)(1-b) - c > 0$, *the inequality* (3.7) *holds, and* $2p_4 - p_5 - 1 < 0$, *then for* $a \in (0, a_2)$ *the fixed point* E_0 *is asymptotically stable.*

(ii) *If* $(1+\varepsilon)(1-b) - c > 0$, *the inequality* (3.7) *holds, and* $2p_4 - p_5 - 1 > 0$, *then* $a = a_1$ *is a flip bifurcation and* $a = a_2$ *is a Neimark-Sacker bifurcation.*

(iii) *If* $(1+\varepsilon)(1-b) - c > 0$, *the inequality* (3.7) *holds and* $2p_4 - p_5 - 1 < 0$, *then* $a = a_2$ *is a Neimark-Sacker bifurcation.*

4. The Normal Form for Flip and Neimark-Sacker Bifurcations

In this section, we describe the normal form in the neighborhood of the fixed point E_0, for the cases $a = a_1$ and $a = a_2$.

We consider the transformation:

$$u_1 = y - y_0, \qquad u_2 = z - z_0, \tag{4.1}$$

where y_0, z_0 are given by (3.2). With respect to (4.1), the map (3.1) is $G : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$G(u_1, u_2) = \begin{pmatrix} g_1(u_1, u_2) \\ g_2(u_1, u_2) \end{pmatrix}, \tag{4.2}$$

$$\begin{aligned} g_1(u_1, u_2) &= -\frac{r}{1+\varepsilon} + (b - d + v)u_1 + (d - v)u_2 - w(u_1 - u_2)^3 + \frac{cu_1 + r}{1 + \varepsilon \exp(u_2 - u_1)}, \\ g_2(u_1, u_2) &= u_1, \qquad r = cy_0 + m. \end{aligned} \tag{4.3}$$

The map (4.2) has $O(0,0)$ as fixed point.

We consider

$$\begin{gathered} a_{11} = \frac{\partial g_1}{\partial u_1}(0,0), \qquad a_{12} = \frac{\partial g_1}{\partial u_2}(0,0), \qquad l_{20} = \frac{\partial^2 g_1}{\partial u_1^2}(0,0), \qquad l_{11} = \frac{\partial^2 g_1}{\partial u_1 \partial u_2}(0,0), \\ l_{02} = \frac{\partial^2 g_1}{\partial u_2^2}(0,0), \qquad l_{30} = \frac{\partial^3 g_1}{\partial u_1^3}(0,0), \qquad l_{21} = \frac{\partial^3 g_1}{\partial u_1^2 \partial u_2}(0,0), \qquad l_{12} = \frac{\partial^3 g_1}{\partial u_1 \partial u_2^2}(0,0), \\ l_{03} = \frac{\partial^3 g_1}{\partial u_1^3}(0,0). \end{gathered} \tag{4.4}$$

We develop the function $G(u)$, $u = (u_1, u_2)^T$ in the Taylor series until the third order and obtain

$$G(u) = Au + \frac{1}{2}B(u,u) + \frac{1}{2}D(u,u,u) + O\left(|u|^4\right), \tag{4.5}$$

where A is given by (3.4) and

$$B(u,u) = \left(B^1(u,u), 0\right)^T, \qquad D(u,u,u) = \left(D^1(u,u,u), 0\right)^T, \tag{4.6}$$

$$\begin{gathered} B^1(u,u) = u^T \begin{pmatrix} l_{20} & l_{11} \\ l_{11} & l_{02} \end{pmatrix} u, \\ D^1(u,u,u) = u^T \left(u_1 \begin{pmatrix} l_{30} & l_{21} \\ l_{21} & l_{12} \end{pmatrix} + u_2 \begin{pmatrix} l_{21} & l_{12} \\ l_{12} & l_{03} \end{pmatrix} \right) u. \end{gathered} \tag{4.7}$$

For $a = a_1$, given by (3.8) with the condition (v) from Proposition 3.1, we have the following.

Proposition 4.1. (i) *The eigenvector $q \in \mathbb{R}^2$, given by $Aq = -q$, has the components:*

$$q_1 = 1, \qquad q_2 = -1. \tag{4.8}$$

(ii) *The eigenvector $h \in \mathbb{R}^2$, given by $h^T A = -h^T$, has the components:*

$$h_1 = \frac{1}{1 + a_{12}}, \qquad h_2 = -\frac{a_{12}}{1 + a_{12}}. \tag{4.9}$$

The relation $\langle q, h \rangle = 1$ holds.

(iii) *The normal form of the map* (3.1) *on the center manifold in* $O(0,0)$ *is given by*

$$\eta \longrightarrow -\eta + \frac{1}{6}\nu\eta^3 + O\left(\eta^4\right), \tag{4.10}$$

where $\nu = (1/(1 + a_{12}))(l_{30} - 3l_{21} + 3l_{12} - l_{03}) + (3/(1 - a_{11} - a_{12}))(l_{20} - 2l_{11} + l_{02})^2$.

The proof results from straight calculus using the formula (2.6):

$$\nu = \langle r, D(q,q,q) \rangle + 3B\left(q, (I - A)^{-1} B(q,q)\right), \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.11}$$

For $a = a_2$, given by (3.8) with the condition (vi) from Proposition 3.1, one has the following.

Proposition 4.2. (i) *The eigenvector $q \in \mathbb{C}^2$, given by $Aq = \mu_1 q$, where μ_1 is the eigenvalue of the matrix A, has the components:*

$$q_1 = 1, \qquad q_2 = \mu_2 = \overline{\mu}_1. \tag{4.12}$$

(ii) *The eigenvector $h \in \mathbb{C}^2$, given by $h^T A = \mu_2 h^T$, where $\mu_2 = \overline{\mu}_1$, has the components:*

$$h_1 = \frac{1}{1 + \mu_1^2 a_{12}}, \qquad h_2 = \frac{\mu_1 a_{12}}{1 + \mu_1^2 a_{12}}. \tag{4.13}$$

The relation $\langle q, \overline{h} \rangle = 1$ holds.

Using (4.6) and (4.8) one has

$$\begin{aligned} B^1(q,q) &= l_{20} + 2l_{11}\mu_2 + l_{02}\mu_2^2, \\ B^1(q,\overline{q}) &= l_{20} + l_{11}(\mu_1 + \mu_2) + l_{02}\mu_1\mu_2, \\ B^1(\overline{q},\overline{q}) &= l_{20} + 2l_{11}\mu_1 + l_{02}\mu_1^2. \end{aligned} \tag{4.14}$$

We denoted by

$$\begin{gathered} g_{20} = v_1 B^1(q,q), \qquad g_{11} = v_1 B^1(q,\overline{q}), \qquad g_{02} = v_1 B^1(\overline{q},\overline{q}), \\ h_{20}^1 = (1 - v_1 - \overline{v}_1) g_{20}, \qquad h_{11}^1 = (1 - v_1 - \overline{v}_1) g_{11}, \qquad h_{02}^1 = (1 - v_1 - \overline{v}_1) g_{02}, \\ h_{20}^2 = -(v_1\mu_2 + \overline{v}_1\mu_1) B^1(q,q), \qquad h_{11}^2 = -(v_1\mu_2 + \overline{v}_1\mu_1) B^1(q,\overline{q}), \\ h_{02}^2 = -(v_1\mu_2 + \overline{v}_1\mu_1) B^1(\overline{q},\overline{q}), \end{gathered} \tag{4.15}$$

$$w_{20} = \left(\mu_1^2 I - A\right)^{-1} \begin{pmatrix} h_{20}^1 \\ h_{20}^2 \end{pmatrix}, \qquad w_{11} = (I - A)^{-1} \begin{pmatrix} h_{11}^1 \\ h_{11}^2 \end{pmatrix}, \qquad w_{02} = \left(\mu_2^2 - A\right)^{-1} \begin{pmatrix} h_{02}^1 \\ h_{02}^2 \end{pmatrix}, \tag{4.16}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, A is given by (3.4), and

$$\begin{gathered} r_{20} = B^1(q, w_{20}), \qquad r_{11} = B^1(q, w_{11}), \\ D_0 = D^1(q,q,\overline{q}) = l_{30} + (\mu_1 + 2\mu_2) l_{21} + \mu_2(2\mu_1 + \mu_2) l_{12} + \mu_1\mu_2^2 l_{03}, \\ g_{21} = v_2(r_{20} + 2r_{11} + D_0). \end{gathered} \tag{4.17}$$

Using the normal form for the Neimark-Sacker bifurcation of the dynamic systems with discrete time [6] and (4.15), (4.16), and (4.17), we obtain the following.

Proposition 4.3. (i) *The solution of the system* (2.6) *in the neighborhood of the fixed point* $(y_0, z_0) \in \mathbb{R}^n$ *is given by*

$$
\begin{aligned}
y(t) &= y_0 + \mu_2 x(t) + \mu_1 \overline{x}(t) + \frac{1}{2} w_{20}^2 x(t)^2 + w_{11}^2 x(t)\overline{x}(t) + \frac{1}{2} w_{02}^2 \overline{x}(t)^2, \\
y(t-1) = z(t) &= z_0 + x(t) + \overline{x}(t) + \frac{1}{2} w_{20}^1 x(t)^2 + w_{11}^1 x(t)\overline{x}(t) + \frac{1}{2} w_{02}^1 \overline{x}(t)^2,
\end{aligned}
\tag{4.18}
$$

where $x(t) \in \mathbb{C}$ *is the solution of the following equation:*

$$
x(t+1) = \mu_1 x(t) + \frac{1}{2} g_{20} x(t)^2 + g_{11} x(t)\overline{x}(t) + \frac{1}{2} g_{02} \overline{x}(t)^2 + \frac{1}{2} g_{21} x(t)^2 \overline{x}(t). \tag{4.19}
$$

(ii) *A complex variable transformation exists so that* (4.18) *becomes*

$$
\omega(t+1) = \mu_1 \omega(t) + L_c \omega_t^2 \overline{\omega}(t) + O\left(|\omega(t)|^4\right), \tag{4.20}
$$

where

$$
L_c = \frac{g_{20} g_{11} (\mu_2 - 3 - 2\mu_1)}{2(\mu_1^2 - \mu_1)(\mu_2 - 1)} + \frac{|g_{11}|^2}{(1 - \mu_1)} + \frac{|g_{20}|^2}{2(\mu_1^2 - \mu_2)} + \frac{g_{21}}{2} \tag{4.21}
$$

is the Lyapunov coefficient.

(iii) *If* $l_0 = \mathbb{R}e(e^{-i\theta} L_c) < 0$, *where* $\theta = \arg(\mu_1)$, *then in the neighborhood of the fixed point* (y_0, z_0) *there is a stable limit cycle.*

5. The Lyapunov Exponents

If $a_1 > 0$, then for $a \in (0, a_1)$ or $a \in (a_2, 1)$ the system (2.6) has a complex behavior and it can be established by computing the Lyapunov exponents. We will use the decomposed Jacobi matrix of map (3.1) into a product of an orthogonal matrix Q and an uppertriangular matrix R with positive diagonal elements (called QR algorithm [6]). The determination of the Lyapunov exponents can be obtained by solving the following system:

$$
\begin{gathered}
y(t+1) = a + (b - d + v) y(t) + (d - v) z(t) - w\left(y(t) - z(t)\right)^3 + \frac{c y(t) + m}{1 + \exp\left(z(t) - y(t)\right)}, \\
z(t+1) = y(t), \\
x(t+1) = \arctan\left(-\frac{\cos x(t)}{f_{11} \cos x(t) - f_{12} \sin x(t)}\right), \\
\lambda(t+1) = \lambda(t) + \ln\left(\left|\left(f_{11} - \tan x(t+1)\right) \cos x(t) \cos x(t+1) - f_{12} \sin x(t) \cos x(t+1)\right|\right), \\
\mu(t+1) = \mu(t) + \ln\left(\left|\left(f_{11} - \tan x(t+1) + 1\right) \sin x(t) \cos x(t+1) + f_{12} \cos x(t) \cos x(t+1)\right|\right),
\end{gathered}
\tag{5.1}
$$

with

$$f_{11} = \frac{\partial f_1}{\partial y}\bigl(y(t), z(t)\bigr) = b - d + v - 3w\bigl(y(t) - z(t)\bigr)^2 + \frac{c + \varepsilon\bigl(c + m + cy(t)\bigr)\exp\bigl(z(t) - y(t)\bigr)}{\bigl(1 + \varepsilon\exp\bigl(z(t) - y(t)\bigr)\bigr)^2},$$
$$f_{12} = \frac{\partial f_1}{\partial z}\bigl(y(t), z(t)\bigr) = d - v + 3w\bigl(y(t) - z(t)\bigr)^2 - \frac{\varepsilon\bigl(m + cy(t)\bigr)\exp\bigl(z(t) - y(t)\bigr)}{\bigl(1 + \varepsilon\exp\bigl(z(t) - y(t)\bigr)\bigr)^2}. \tag{5.2}$$

The Lyapunov exponents are

$$L_1 = \lim_{t\to\infty}\frac{\lambda(t)}{t}, \qquad L_2 = \lim_{t\to\infty}\frac{\mu(t)}{t}. \tag{5.3}$$

If one of the two exponents is positive, the system has a chaotic behavior.

6. The Stochastic Difference Equation Associated to Difference Equation (2.6)

Let $(\Omega, F, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, P)$ be a filtered probability space with stochastic basis and the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$. Let $\{\xi(t)\}_{t\in I\mathbb{N}}$ be a real valued independent random variable on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, P)$ with $\mathbb{E}(\xi(t)) = 0$ and $\mathbb{E}(\xi(t))^2 < \infty$.

The stochastic difference equation associated to the difference equation (2.6) is given by

$$y_{t+1} = a + dy(t-1) + (b-d)y(t) + v\bigl(y(t) - y(t-1)\bigr) - w(y(t) - y(t-1))^3 + \frac{cy(t) + m}{1 + \varepsilon\exp\bigl(y(t-1) - y(t)\bigr)} + g\bigl(y(t), y(t-1)\bigr)\xi(t). \tag{6.1}$$

The function $g(y(t), y(t-1))$ is the contribution of the fluctuations while it is under certain circumstances it is given by

$$g\bigl(y(t)\bigr) = \alpha\bigl(y(t) - y_0\bigr), \quad \alpha \geq 0 \tag{6.2}$$

or

$$g\bigl(y(t-1), y(t)\bigr) = \alpha\bigl(y(t-1) - y(t)\bigr), \quad \alpha \geq 0. \tag{6.3}$$

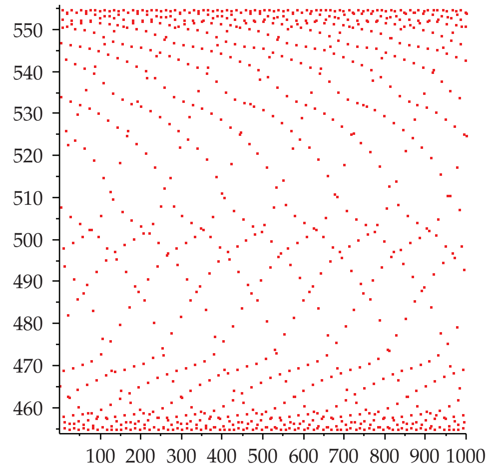

Figure 1: $(t, y(t))$ in the deterministic case for $d = 0$.

From (6.1) and (6.2) the difference equation with multiplicative noise associated to the difference equation (2.6) is given by

$$\begin{aligned} y(t+1) &= a + dz(t) + (b-d)y(t) + v\big(y(t) - z(t)\big) - w\big(y(t) - z(t)\big)^3 \\ &\quad + \frac{cy(t) + m}{1 + \varepsilon \exp\big(z(t) - y(t)\big)} + \alpha\big(y(t) - y_0\big)\xi(t), \\ z(t+1) &= y(t), \end{aligned} \tag{6.4}$$

where y_0 is the fixed point of map (2.6).

Using (6.1) and (6.3) we have

$$\begin{aligned} y(t+1) &= a + dz(t) + (b-d)y(t) + v\big(y(t) - z(t)\big) - w\big(y(t) - z(t)\big)^3 \\ &\quad + \frac{cy(t) + m}{1 + \varepsilon \exp\big(z(t) - y(t)\big)} + \alpha\big(z(t) - y(t)\big)\xi(t), \\ z(t+1) &= y(t). \end{aligned} \tag{6.5}$$

By (2.6), randomizing parameter d equation (6.5) is obtained.

The analysis of the stochastic difference equations (6.4) and (6.5) can be done by using the method from [10–12].

In what follows, we calculate an estimation of the upper (forward 2 th moment) stability exponent (given in [12]) for the stochastic process $(y(t), z(t))_{t\in\mathbb{N}}$.

Let $(y(t), z(t))_{t\in\mathbb{N}}$ be the process that satisfies (6.4). The system of stochastic difference equations (6.4) has the form:

$$\begin{aligned} y(t+1) &= y(t) + f_1\big(y(t), z(t)\big) + f_2\big(y(t)\big)\xi(t), \\ z(t+1) &= z(t) + f_3\big(y(t), z(t)\big), \end{aligned} \tag{6.6}$$

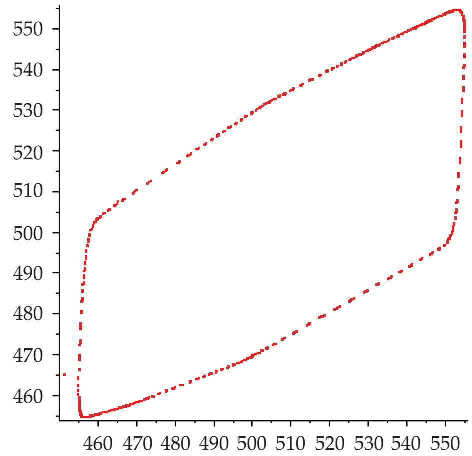

Figure 2: $(y(t-1), y(t))$ in the deterministic case for $d = 0$.

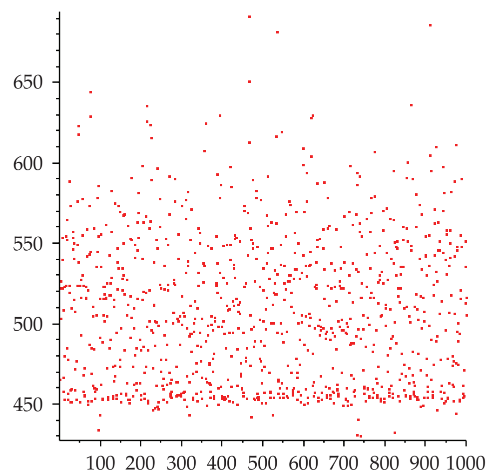

Figure 3: $(t, y(t))$ in the stochastic case for $d = 0$.

where

$$
\begin{aligned}
f_1(y(t), z(t)) &= (b - d + v - 1)y(t) + (d - v)z(t) - w(y(t) - z(t))^3 \\
&\quad + \frac{cy(t) + m}{1 + \varepsilon \exp(z(t) - y(t))}, \\
f_2(y(t)) &= \alpha(y(t) - y_0), \\
f_3(y(t), z(t)) &= y(t) - z(t).
\end{aligned}
\tag{6.7}
$$

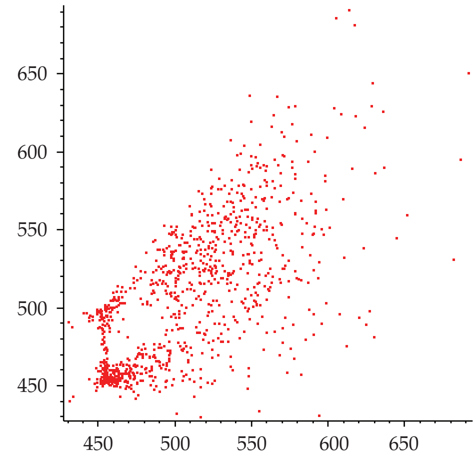

Figure 4: $(y(t-1), y(t))$ in the stochastic case for $d = 0$.

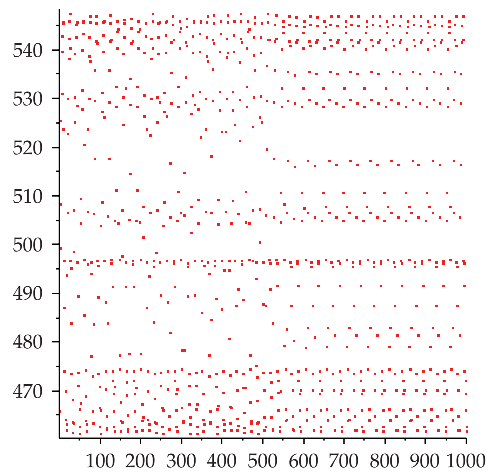

Figure 5: $(t, y(t))$ in the deterministic case for $d = 0.6$.

The upper (forward 2 th moment) stability exponent of the stochastic process $(y(t), z(t))_{t\in\mathbb{N}}$ is defined by [12]

$$\lambda_2 = \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\left(y(t)^2 + z(t)^2\right), \tag{6.8}$$

provided that this limit exists.

Using Theorem 2.2 [12], for (6.5), with f_1, f_2, f_3 given by (6.7) and $w = 0$, we get the following.

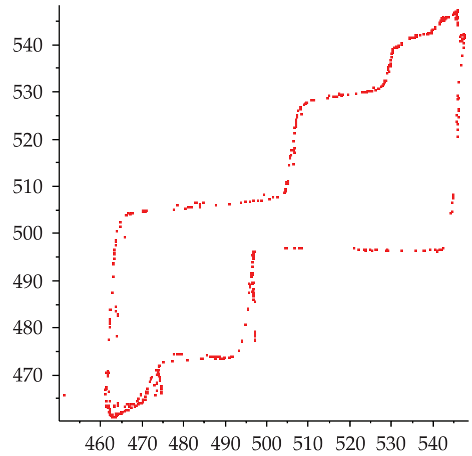

Figure 6: $(y(t-1), y(t))$ in the deterministic case for $d = 0.6$.

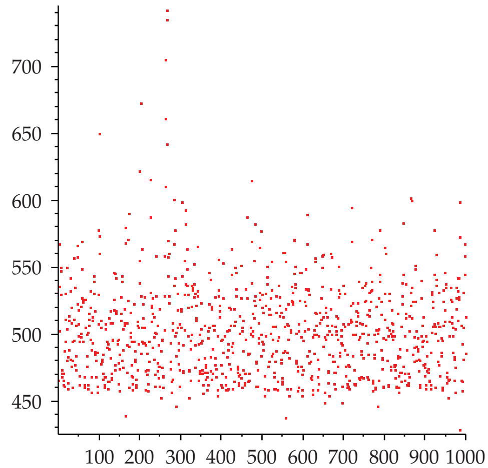

Figure 7: $(t, y(t))$ in the stochastic case for $d = 0.6$.

Proposition 6.1. *Let $(y(t), z(t))$ be the process which satisfies the stochastic difference equation* (6.6) *with $w = 0$. Assume that, for all $t \in \mathbb{N}$, $(y(t), z(t)) \in \mathbb{R}^2$,*

$$\begin{gathered} f_1(y(t), z(t))y(t) + f_3(y(t), z(t))z(t) \le k_1\left(y(t)^2 + z(t)^2\right), \\ f_1(y(t), z(t))^2 + f_3(y(t), z(t))^2 \le k_2\left(y(t)^2 + z(t)^2\right), \\ f_2(y(t), z(t))^2 \le k_3\left(y(t)^2 + z(t)^2\right), \end{gathered} \tag{6.9}$$

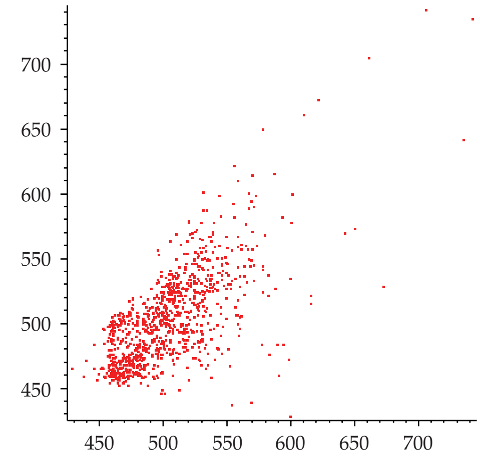

Figure 8: $(y(t-1), y(t))$ in the stochastic case for $d = 0.6$.

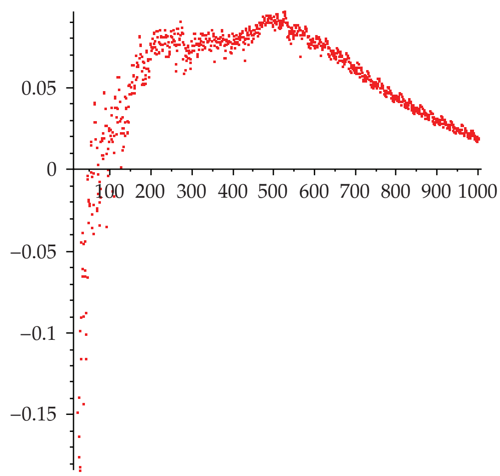

Figure 9: $(t, \lambda(t)/t)$ the Lyapunov exponent in the deterministic case for $d = 0.6$.

where k_1, k_2, k_3 *are finite, deterministic, real numbers. Then*

$$\lambda_2 \leq 2k_1 + k_2 + \sigma^2 k_3, \tag{6.10}$$

with $\mathbb{E}(\xi(t))^2 = \sigma^2$.

If the parameters of the model are $a = 250$, $v = 0.1$, $w = 0$, $c = 0.1$, $b = 0.45$, $\varepsilon = 1$, $m = 0.5$, $d = 0.8$, $\sigma = 0.5$, and $\alpha = 1$, then for $k_1 = 15$, $k_2 = 350$, $k_3 = 1$, the inequalities (6.9) are satisfied and $\lambda_2 \leq 380.25$.

A similar result can be obtained for (6.5).

The qualitative analysis of the difference equations (6.4) and (6.5) is more difficult than that in the deterministic case and it will be done in our next papers.

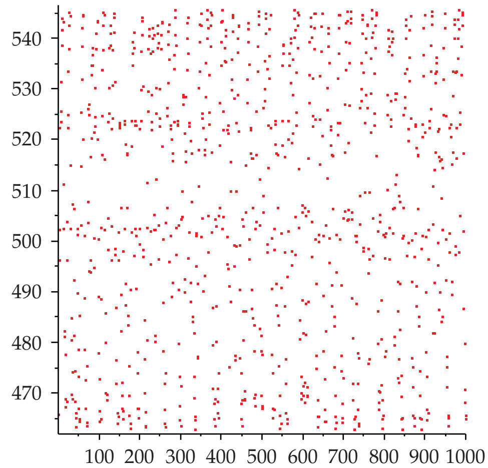

Figure 10: $(t, y(t))$ in the deterministic case for $d = 0.8$.

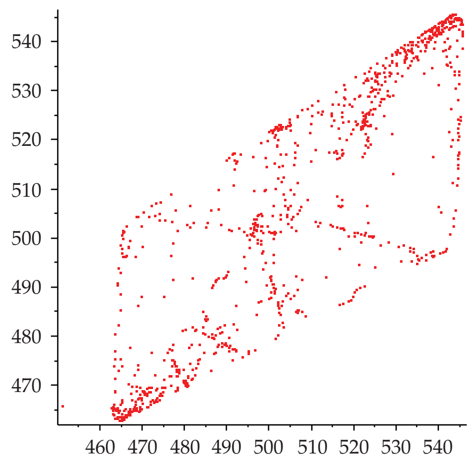

Figure 11: $(y(t-1), y(t))$ in the deterministic case for $d = 0.8$.

7. Numerical Simulation

The numerical simulation is done using a Maple 13 program. We consider different values for the parameters which are used in the real economic processes. We use Box-Muller method for the numerical simulation of (6.4).

For system (6.4) with $\alpha = 1$, $a = 250$, $v = 0.1$, $w = 0$, $c = 0.1$, $b = 0.45$, $\varepsilon = 1$, $m = 0.5$ and the control parameter $d = 0$ we obtain in Figure 1 the evolution of the income in the time domain $(t, y(t))$, in Figure 2 the evolution of the income in the phase space $(y(t-1), y(t))$, in Figure 3 the evolution of the income in the stochastic case, and in Figure 4 the evolution of the income in the phase space in the stochastic case. In the deterministic case the Lyapunov exponent is negative and the system has not a chaotic behavior.

Comparing Figures 1 and 2 with Figures 3 and 4 we observe the solution behaving differently in the deterministic and stochastic cases.

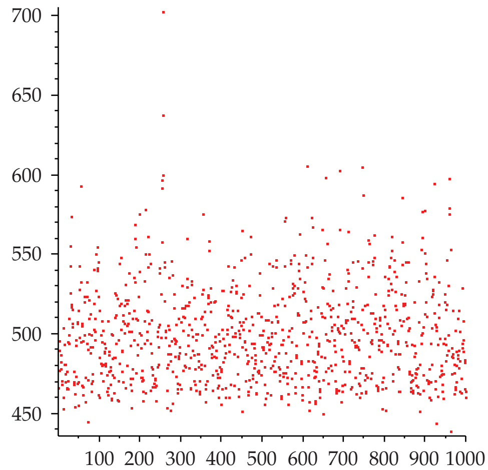

Figure 12: $(t, y(t))$ in the stochastic case for $d = 0.8$.

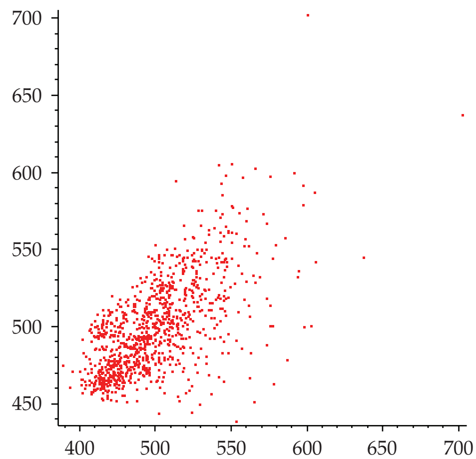

Figure 13: $(y(t-1), y(t))$ in the stochastic case for $d = 0.8$.

For the control parameter $d = 0.6$ Figure 5 displays the evolution of the income in the time domain $(t, y(t))$, Figure 6 the evolution of the income in the phase space $(y(t-1), y(t))$, Figure 7 the evolution of the income in the stochastic case, Figure 8 the evolution of the income in the phase space in the stochastic case, and Figure 9 shows the Lyapunov exponent $(t, \lambda(t)/t)$.

Comparing Figures 5 and 6 with Figures 7 and 8 we observe the solution behaving differently in the deterministic and stochastic cases.

The Lyapunov exponent is positive, therefore the system has a chaotic behavior.

For the control parameter $d = 0.8$, Figure 10 shows the evolution of the income in the time domain $(t, y(t))$, Figure 11 the evolution of the income in the phase space $(y(t-1), y(t))$, Figure 12 the evolution of the income in the stochastic case, Figure 13 the evolution of the income in the phase space in the stochastic case, and Figure 14 shows the Lyapunov exponent $(t, \lambda(t)/t)$.

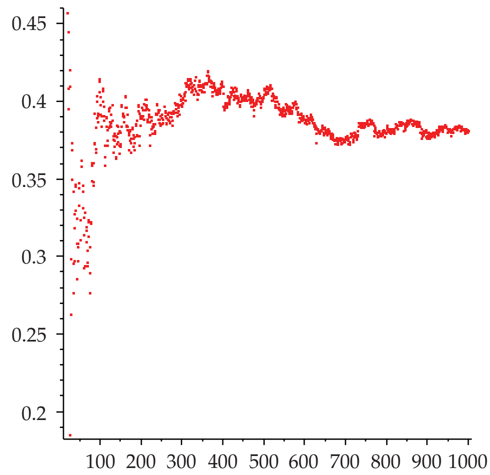

Figure 14: $(t, \lambda(t)/t)$ the Lyapunov exponent in the deterministic case for $d = 0.8$.

Comparing Figures 10 and 11 with Figures 12 and 13 we notice the solution behaving differently in the deterministic and stochastic cases.

The Lyapunov exponent is positive; therefore the system has a chaotic behavior.

Considering a as parameter, we can obtain a Neimark-Sacker bifurcation point or a flip bifurcation point.

8. Conclusion

A dynamic model with discrete time using investment, consumption, sentiment, and saving functions has been studied. The behavior of the dynamic system in the fixed point's neighborhood for the associated map has been analyzed. We have established asymptotic stability conditions for the flip and Neimark-Sacker bifurcations. The QR method is used for determining the Lyapunov exponents and they allows us to decide whether the system has a complex behavior. Also, two stochastic models with multiplicative noise have been associated to the deterministic model. One model was obtained by adding one stochastic control and the other by randomizing the control parameter d. Using a program in Maple 13, we display the Lyapunov exponent and the evolution of the income in the time domain and the phase space, in both deterministic and stochastic cases. A qualitative analysis of the stochastic models will be done in our next papers.

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