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Research Article

Permanence of a Discrete Nonlinear Prey-Competition System with Delays

Hongying Lu

School of Mathematics and Quantitative Economics, Dongbei University of Finance and Economics, Dalian, Liaoning 116025, China

Correspondence should be addressed to Hongying Lu, hongyinglu543@163.com

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A discrete nonlinear prey-competition system with m-preys and (n-m)-predators and delays is considered. Two sets of sufficient conditions on the permanence of the system are obtained. One set is delay independent, while the other set is delay dependent.

1. Introduction

In this paper, we investigate the following discrete nonlinear prey-competition system with delays:

$$
\begin{aligned}
x_i(k+1) &= x_i(k)\exp\left[r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j^{\alpha_{ij}}(k) - \sum_{j=1}^{n} b_{ij}(k)x_j^{\beta_{ij}}\left(k-\tau_{ij}(k)\right)\right], \\
&\qquad i = 1,2,\dots,m, \\
x_i(k+1) &= x_i(k)\exp\left[-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)x_j^{\alpha_{ij}}(k) + \sum_{j=1}^{m} b_{ij}(k)x_j^{\beta_{ij}}\left(k-\tau_{ij}(k)\right)\right. \\
&\qquad \left. - \sum_{j=m+1}^{n} a_{ij}(k)x_j^{\alpha_{ij}}(k) - \sum_{j=m+1}^{n} b_{ij}(k)x_j^{\beta_{ij}}\left(k-\tau_{ij}(k)\right)\right], \\
&\qquad i = m+1,2,\dots,n,
\end{aligned}
\tag{1.1}
$$

where $x_i(k)$ $(i = 1,2,\ldots,m)$ is the density of prey species i at kth generation, $x_i(k)$ $(i = m+1,\ldots,n)$ is the density of predator species i at kth generation. In this system, the competition among predator species and among prey species is simultaneously considered. For more background and biological adjustments of system (1.1), we can see [1–5] and the references cited therein.

Throughout this paper, we always assume that for all $i,j = 1,2,\ldots,n$,

(H_1) $r_i(k), a_{ij}(k), b_{ij}(k)$ are all bounded nonnegative sequences and $a_{ii}^l \geq 0, b_{ii}^l \geq 0, a_{ii}^l + b_{ii}^l > 0$. Here, for any bounded sequence $f^u = \sup_{k\in N} f(k)$, $f^l = \inf_{k\in N} f(k)$;

(H_2) $\tau_{ij}(k)$ are bounded nonnegative integer sequences, and α_{ij}, β_{ij} are all positive constants.

By a solution of system (1.1), we mean a sequence $\{x_1(k),\ldots,x_n(k)\}$ which defined for $N = \{0,1,\ldots\}$ and which satisfies system (1.1) for $N = \{0,1,\ldots\}$. Motivated by application of system (1.1) in population dynamics, we assume that solutions of system (1.1) satisfy the following initial conditions:

$$x_{\mathrm{i}}(\theta) = \phi_i(\theta), \quad \theta \in N[-\tau, 0] = [-\tau, -\tau+1, \ldots, 0], \quad \phi_i(0) > 0, \tag{1.2}$$

where $\tau = \max\{\tau_{ij}(k), i,j = 1,2,\ldots,n\}$. The exponential forms of system (1.1) assure that the solution of system (1.1) with initial conditions (1.2) remains positive.

Recently, Chen et al. in [1] proposed the following nonlinear prey-competition system with delays:

$$\begin{aligned}
\dot{x}_i(t) &= x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{n} b_{ij}(t)x_j^{\beta_{ij}}\left(t-\tau_{ij}(t)\right)\right], \quad i = 1,2,\ldots,m, \\
\dot{x}_i(t) &= x_i(t)\left[-r_i(t) + \sum_{j=1}^{m} a_{ij}(t)x_j^{\alpha_{ij}}(t) + \sum_{j=1}^{m} b_{ij}(t)x_j^{\beta_{ij}}\left(t-\tau_{ij}(t)\right)\right. \\
&\qquad \left. - \sum_{j=m+1}^{n} a_{ij}(t)x_j^{\alpha_{ij}}(t) - \sum_{j=m+1}^{n} b_{ij}(t)x_j^{\beta_{ij}}\left(t-\tau_{ij}(t)\right)\right], \quad i = m+1,2,\ldots,n.
\end{aligned} \tag{1.3}$$

By using Gaines and Mawhins continuation theorem of coincidence degree theory and by constructing an appropriate Lyapunov functional, they obtained a set of sufficient conditions which guarantee the existence and global attractivity of positive periodic solutions of the system (1.3). In addition, sufficient conditions are obtained for the permanence of the system (1.3) in [2].

On the other hand, though most population dynamics are based on continuous models governed by differential equations, the discrete time models are more appropriate than the continuous ones when the size of the population is rarely small or the population has nonoverlapping generations [3–15]. Therefore, it is reasonable to study discrete time prey-competition models governed by difference equations.

As we know, a more important theme that interested mathematicians as well as biologists is whether all species in a multispecies community would survive in the long run, that is, whether the ecosystems are permanent. In fact, no such work has been done for system (1.1).

The main purpose of this paper is, by developing the analytical technique of [4, 8, 16], to obtain two sets of sufficient conditions which guarantee the permanence of system (1.1).

2. Main Results

Firstly, we introduce a definition and some lemmas which will be useful in the proof of the main results of this section.

Definition 2.1. System (1.1) is said to be permanent, if there are positive constants m and M, such that each positive solution $(x_1(k), \dots, x_n(k))$ of system (1.1) satisfies

$$m \le \liminf_{k \to +\infty} x_i(k) \le \limsup_{k \to +\infty} x_i(k) \le M, \quad i = 1, 2, \dots, n. \tag{2.1}$$

Lemma 2.2 (see [8]). *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \le x(k) \exp\{r(k)(1 - ax(k))\} \tag{2.2}$$

for $k \in [k_1, +\infty)$, where a is a positive constant. Then

$$\limsup_{k \to +\infty} x(k) \le \frac{1}{ar^u} \exp(r^u - 1). \tag{2.3}$$

Lemma 2.3 (see [8]). *Assume that $\{x(k)\}$ satisfies*

$$x(k+1) \ge x(k) \exp\{r(k)(1 - ax(k))\}, \quad k \ge K_0, \tag{2.4}$$

$\limsup_{k \to +\infty} x(k) \le x^*$ *and $x(K_0) > 0$, where a is a constant such that $ax^* > 1$ and $K_0 \in N$. Then*

$$\liminf_{k \to +\infty} x(k) \ge \frac{1}{a} \exp(r^u(1 - ax^*)). \tag{2.5}$$

For system (1.1), we will consider two cases, $a_{ii}^l > 0, b_{ii}^l \ge 0$ and $a_{ii}^l \ge 0, b_{ii}^l > 0$ respectively, and then we obtain Lemmas 2.4–2.6.

Lemma 2.4. *Assume that $a_{ii}^l > 0$. Then for every positive solution $(x_1(k), \dots, x_n(k))$ of system* (1.1) *with initial condition* (1.2), *one has*

$$\limsup_{k \to +\infty} x_i(k) \le M_i, \quad i = 1, 2, \dots, n, \tag{2.6}$$

where

$$M_i = \left(\frac{1}{\alpha_{ii} a_{ii}^l}\right)^{1/\alpha_{ii}} \exp\left[r_i^u - \frac{1}{\alpha_{ii}}\right], \quad i = 1,2,\ldots,m,$$
$$M_i = \left(\frac{1}{\alpha_{ii} a_{ii}^l}\right)^{1/\alpha_{ii}} \exp\left[-r_i^l + \sum_{j=1}^{n} a_{ij}^u M_j^{\alpha_{ij}} + \sum_{j=1}^{n} b_{ij}^u M_j^{\beta_{ij}} - \frac{1}{\alpha_{ii}}\right], \quad i = m+1,\ldots,n. \tag{2.7}$$

Proof. Let $x(k) = (x_1(k),\ldots,x_n(k))$ be any positive solution of system (1.1) with initial condition (1.2), for $i = 1,2,\ldots,m$, it follows from system (1.1) that

$$x_i(k+1) \le x_i(k) \exp\left[r_i(k) - a_{ii}(k) x_i^{\alpha_{ii}}(k)\right], \tag{2.8}$$

thus

$$x_i^{\alpha_{ii}}(k+1) \le x_i^{\alpha_{ii}}(k) \exp\left[\alpha_{ii}\left(r_i(k) - a_{ii}(k) x_i^{\alpha_{ii}}(k)\right)\right]. \tag{2.9}$$

Let $u_i(k) = x_i^{\alpha_{ii}}(k)$, we can have

$$\begin{aligned} u_i(k+1) &\le u_i(k) \exp[\alpha_{ii}(r_i(k) - a_{ii}(k) u_i(k))] \\ &\le u_i(k) \exp\left[\alpha_{ii} r_i(k)\left(1 - \frac{a_{ii}^l}{r_i^u} u_i(k)\right)\right]. \end{aligned} \tag{2.10}$$

By applying Lemma 2.2 to (2.10), we obtain

$$\limsup_{k\to+\infty} u_i(k) \le \left(\frac{1}{\alpha_{ii} a_{ii}^l}\right) \exp\left[\alpha_{ii} r_i^u - 1\right] \doteq L_i; \tag{2.11}$$

so, we immediately get

$$\limsup_{k\to+\infty} x_i(k) \le M_i, \quad i = 1,2,\ldots,m. \tag{2.12}$$

For any $\varepsilon > 0$ small enough, it follows from (2.12) that there exists enough large K_1 such that for all $i = 1,2,\ldots,m$ and $k \ge K_1$

$$x_i(k) \le M_i + \varepsilon. \tag{2.13}$$

For $i = m+1,\dots,n$ and $k \geq K_1 + \tau$, (2.13) combining with the i-th equation of system (1.1) leads to

$$x_i(k+1) \leq x_i(k)\exp\left[-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - a_{ii}(k)x_i^{\alpha_{ii}}(k)\right], \tag{2.14}$$

thus

$$\begin{aligned} x_i^{\alpha_{ii}}(k+1) \leq x_i^{\alpha_{ii}}(k)\exp\Bigg[\alpha_{ii}\Bigg(&-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} \\ &+\sum_{j=1}^{m} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - a_{ii}(k)x_i^{\alpha_{ii}}(k)\Bigg)\Bigg]. \end{aligned} \tag{2.15}$$

Similarly, let $u_i(k) = x_i^{\alpha_{ii}}(k)$, we get

$$\begin{aligned} u_i(k+1) &\leq u_i(k)\exp\Bigg[\alpha_{ii}\Bigg(-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} \\ &\qquad +\sum_{j=1}^{m} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - a_{ii}(k)u_i(k)\Bigg)\Bigg] \\ &\leq u_i(k)\exp\Bigg[\alpha_{ii}\Bigg(-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}}\Bigg) \\ &\qquad \times\left(1 - \frac{a_{ii}^l}{-r_i^l + \sum_{j=1}^{m} a_{ij}^u\left(M_j+\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}^u\left(M_j+\varepsilon\right)^{\beta_{ij}}}u_i(k)\right)\Bigg]. \end{aligned} \tag{2.16}$$

By using (2.16), for $i = m+1,\dots,n$, according to Lemma 2.2, it follows that

$$\limsup_{k\to+\infty} u_i(k) \leq \left(\frac{1}{\alpha_{ii}a_{ii}^l}\right)\exp\left[\alpha_{ii}\left(-r_i^l + \sum_{j=1}^{n} a_{ij}^u\left(M_j+\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{n} b_{ij}^u\left(M_j+\varepsilon\right)^{\beta_{ij}}\right) - 1\right]; \tag{2.17}$$

setting $\varepsilon \to 0$ in above inequality, we have

$$\limsup_{k\to+\infty} u_i(k) \le \left(\frac{1}{\alpha_{ii}a_{ii}^l}\right)\exp\left[\alpha_{ii}\left(-r_i^l+\sum_{j=1}^n a_{ij}^u M_j^{\alpha_{ij}}+\sum_{j=1}^n b_{ij}^u M_j^{\beta_{ij}}\right)-1\right] \doteq L_i, \tag{2.18}$$

then

$$\limsup_{k\to+\infty} x_i(k) \le M_i, \quad i=m+1,\ldots,n. \tag{2.19}$$

This completes the proof. □

For convenience, we introduce the following notation.
For $i = 1,2,\ldots,m$

$$\begin{aligned} A_i &= \frac{a_{ii}^u}{r_i^l - \sum_{j=1,j\ne i}^n a_{ij}^u M_j^{\alpha_{ij}} - \sum_{j=1}^n b_{ij}^u M_j^{\beta_{ij}}}, \\ R_i^u &= r_i^u - \sum_{j=1,j\ne i}^n a_{ij}^l M_j^{\alpha_{ij}} - \sum_{j=1}^n b_{ij}^l M_j^{\beta_{ij}}. \end{aligned} \tag{2.20}$$

For $i = m+1,\ldots,n$

$$\begin{aligned} A_i &= \frac{a_{ii}^u}{-r_i^u + \sum_{j=1}^m a_{ij}^l m_j^{\alpha_{ij}} + \sum_{j=1}^n b_{ij}^l m_j^{\beta_{ij}} - \sum_{j=m+1,j\ne i}^n a_{ij}^u M_j^{\alpha_{ij}} - \sum_{j=m+1}^n b_{ij}^u M_j^{\beta_{ij}}}, \\ R_i^u &= -r_i^l + \sum_{j=1}^m a_{ij}^u m_j^{\alpha_{ij}} + \sum_{j=1}^m b_{ij}^u m_j^{\beta_{ij}} - \sum_{j=m+1,j\ne i}^n a_{ij}^l M_j^{\alpha_{ij}} - \sum_{j=m+1}^n b_{ij}^l M_j^{\beta_{ij}}. \end{aligned} \tag{2.21}$$

Lemma 2.5. *Assume that* $a_{ii}^l > 0$ *and*

$$\min_{1\le i\le n} L_i A_i > 1, \tag{2.22}$$

hold. Then for any positive solution $(x_1(k),\ldots,x_n(k))$ *of system* (1.1) *with initial condition* (1.2), *one has*

$$\liminf_{k\to+\infty} x_i(k) \ge m_i, \tag{2.23}$$

where

$$m_i = \left(\frac{1}{A_i}\right)^{1/\alpha_{ii}} \exp\left[R_i^u(1-A_iL_i)\right], \quad i=1,2,\ldots,n. \tag{2.24}$$

Proof. Let $x(k) = (x_1(k), \ldots, x_n(k))$ be any positive solution of system (1.1) with initial condition (1.2). From Lemma 2.4, we know that there exists $K_2 > K_1 + \tau$, such that for $i = 1, 2, \ldots, n$ and $k \geq K_2$

$$x_i(k) \leq M_i + \varepsilon. \tag{2.25}$$

For $i = 1, \ldots, m$ and $k \geq K_2 + \tau$, (2.25) combining with the i-th equation of system (1.1) lead to

$$x_i(k+1) \geq x_i(k) \exp\left[r_i(k) - \sum_{j=1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}} - a_{ii}(k)x_i^{\alpha_{ii}}(k)\right], \tag{2.26}$$

thus

$$\begin{aligned} x_i^{\alpha_{ii}}(k+1) \geq x_i^{\alpha_{ii}}(k) \exp\Bigg[\alpha_{ii}\Bigg(& r_i(k) - \sum_{j=1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} \\ & - \sum_{j=1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}} - a_{ii}(k)x_i^{\alpha_{ii}}(k)\Bigg)\Bigg]; \end{aligned} \tag{2.27}$$

let $u_i(k) = x_i^{\alpha_{ii}}(k)$, we can have

$$\begin{aligned} u_i(k+1) &\geq u_i(k) \exp\Bigg[\alpha_{ii}\Bigg(r_i(k) - \sum_{j=1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} \\ &\qquad - \sum_{j=1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}} - a_{ii}(k)u_i(k)\Bigg)\Bigg] \\ &\geq u_i(k) \exp\left[\alpha_{ii} R_{i\varepsilon}(k)\left(1 - A_{i\varepsilon} u_i(k)\right)\right], \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} R_{i\varepsilon}(k) &= r_i(k) - \sum_{j=1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}}, \\ A_{i\varepsilon} &= \frac{a_{ii}^u}{r_i^l - \sum_{j=1, j\neq i}^{n} a_{ij}^u\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=1}^{n} b_{ij}^u\left(M_j + \varepsilon\right)^{\beta_{ij}}}. \end{aligned} \tag{2.29}$$

According to Lemma 2.3, we obtain

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{A_{i\varepsilon}} \exp\left[\alpha_{ii} R^u_{i\varepsilon}(1 - A_{i\varepsilon} L_i)\right], \tag{2.30}$$

where

$$R^u_{i\varepsilon} = r^u_i - \sum_{j=1, j\neq i}^{n} a^l_{ij}\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=1}^{n} b^l_{ij}\left(M_j + \varepsilon\right)^{\beta_{ij}}. \tag{2.31}$$

Setting $\varepsilon \to 0$ in (2.30) leads to

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{A_i} \exp\left[\alpha_{ii} R^u_i(1 - A_i L_i)\right], \tag{2.32}$$

therefore

$$\liminf_{k\to+\infty} x_i(k) \geq m_i, \quad i = 1, 2, \ldots, m. \tag{2.33}$$

For any $\varepsilon > 0$ small enough, it follows from (2.33) that there exists enough large $K_3 > K_2 + \tau$ such that for all $i = 1, \ldots, m$ and $k \geq K_3$

$$x_i(k) \geq m_i - \varepsilon, \tag{2.34}$$

and so, for $i = m+1, \ldots, n$ and $k \geq K_3 + \tau$, it follows from system (1.1) that

$$\begin{aligned} x_i(k+1) \geq{}& x_i(k) \exp\Bigg[-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(m_j - \varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(m_j - \varepsilon\right)^{\beta_{ij}} \\ & - \sum_{j=m+1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=m+1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}} - a_{ii}(k) x_i^{\alpha_{ii}}(k)\Bigg] \\ \geq{}& x_i(k) \exp\left[R_{i\varepsilon}(k)\left(1 - A_{i\varepsilon} x_i^{\alpha_{ii}}(k)\right)\right], \end{aligned} \tag{2.35}$$

where

$$\begin{aligned}
R_{i\varepsilon}(k) &= -r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(m_j - \varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(m_j - \varepsilon\right)^{\beta_{ij}} \\
&\quad - \sum_{j=m+1, j\neq i}^{n} a_{ij}(k)\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=m+1}^{n} b_{ij}(k)\left(M_j + \varepsilon\right)^{\beta_{ij}}, \\
A_{i\varepsilon} &= a_{ii}^{u} \Bigg/ \Bigg\{ -r_i^{u} + \sum_{j=1}^{m} a_{ij}^{l}\left(m_j - \varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{n} b_{ij}^{l}\left(m_j - \varepsilon\right)^{\beta_{ij}} \\
&\quad - \sum_{j=m+1, j\neq i}^{n} a_{ij}^{u}\left(M_j + \varepsilon\right)^{\alpha_{ij}} - \sum_{j=m+1}^{n} b_{ij}^{u}\left(M_j + \varepsilon\right)^{\beta_{ij}} \Bigg\},
\end{aligned} \tag{2.36}$$

by using (2.35), similarly to the analysis of (2.33), for $i = m+1, \ldots, n$

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{A_i} \exp\left[\alpha_{ii} R_i^{u}\left(1 - A_i L_i\right)\right], \tag{2.37}$$

and therefore, we easily get

$$\liminf_{k\to+\infty} x_i(k) \geq m_i, \quad i = m+1, \ldots, n. \tag{2.38}$$

This ends the proof of Lemma 2.5. □

Denote for $i = 1, 2, \ldots, m$

$$\begin{aligned}
\overline{L}_i &= \frac{1}{\beta_{ii} b_{ii}^{l}} \exp\left[\beta_{ii} r_i^{u}(\tau + 1) - 1\right]; \\
\Gamma_i &= r_i^{l} - \sum_{j=1}^{n} a_{ij}^{u} M_j^{\alpha_{ij}} - \sum_{j=1}^{n} b_{ij}^{u} M_j^{\beta_{ij}}; \\
\overline{A}_i &= \frac{b_{ii}^{u} \exp\left[-\beta_{ii}\Gamma_i \tau\right]}{r_i^{l} - \sum_{j=1}^{n} a_{ij}^{u} M_j^{\alpha_{ij}} - \sum_{j=1, j\neq i}^{n} b_{ij}^{u} M_j^{\beta_{ij}}}; \\
\overline{R}_i^{u} &= r_i^{u} - \sum_{j=1}^{n} a_{ij}^{l} M_j^{\alpha_{ij}} - \sum_{j=1, j\neq i}^{n} b_{ij}^{l} M_j^{\beta_{ij}}.
\end{aligned} \tag{2.39}$$

For $i = m+1,\ldots,n$

$$\begin{aligned}
\Upsilon_i &= -r_i^l + \sum_{j=1}^{m} a_{ij}^u M_j^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}^u M_j^{\beta_{ij}};\\
\overline{L}_i &= \frac{1}{\beta_{ii} b_{ii}^l} \exp\left[\beta_{ii}\Upsilon_i(\tau+1)-1\right];\\
\Gamma_i &= -r_i^u + \sum_{j=1}^{m} a_{ij}^l m_j^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}^l m_j^{\beta_{ij}} - \sum_{j=m+1}^{n} a_{ij}^u M_j^{\alpha_{ij}} - \sum_{j=m+1}^{n} b_{ij}^u \left(M_j+\varepsilon\right)^{\beta_{ij}};\\
\overline{A}_i &= \frac{b_{ii}^u \exp\left[-\beta_{ii}\Gamma_i\tau\right]}{-r_i^u + \sum_{j=1}^{m} a_{ij}^l m_j^{\alpha_{ij}} + \sum_{j=1}^{n} b_{ij}^l m_j^{\beta_{ij}} - \sum_{j=m+1}^{n} a_{ij}^u M_j^{\alpha_{ij}} - \sum_{j=m+1, j\neq i}^{n} b_{ij}^u m_j^{\beta_{ij}}};\\
\overline{R}_i^u &= -r_i^l + \sum_{j=1}^{m} a_{ij}^u m_j^{\alpha_{ij}} + \sum_{j=1}^{n} b_{ij}^u m_j^{\beta_{ij}} - \sum_{j=m+1}^{n} a_{ij}^l M_j^{\alpha_{ij}} - \sum_{j=m+1, j\neq i}^{n} b_{ij}^l m_j^{\beta_{ij}}.
\end{aligned} \tag{2.40}$$

Lemma 2.6. *Assume that* $b_{ii}^l > 0$ *and*

$$\left(\overline{H}_3\right) \min_{1\le i\le n} \overline{L}_i \overline{A}_i > 1, \tag{2.41}$$

hold. Then for any positive solution $(x_1(k),\ldots,x_n(k))$ *of system* (1.1) *with initial condition* (1.2), *one has*

$$\overline{m}_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le \overline{M}_i, \quad i=1,2,\ldots,n. \tag{2.42}$$

where

$$\overline{M}_i = \overline{L}_i^{1/\beta_{ii}}, \quad \overline{m}_i = \left(\frac{1}{\overline{A}_i}\right)^{1/\beta_{ii}} \exp\left[\overline{R}_i^u\left(1-\overline{A}_i\overline{L}_i\right)\right]. \tag{2.43}$$

Proof. Let $x(k) = (x_1(k),\ldots,x_n(k))$ be any positive solution of system (1.1) with initial condition (1.2), for $i=1,2,\ldots,m$, it follows from system (1.1) that

$$x_i(k+1) \le x_i(k)\exp\left[r_i(k)\right] \le x_i(k)\exp\left[r_i^u\right], \tag{2.44}$$

$$x_i(k+1) \le x_i(k)\exp\left[r_i(k) - b_{ii}(k)x_i^{\beta_{ii}}(k-\tau_{ii}(k))\right], \tag{2.45}$$

It follows from (2.44)that

$$\prod_{j=k-\tau_{ii}(k)}^{k-1} \frac{x_i(j+1)}{x_i(j)} \le \prod_{j=k-\tau_{ii}(k)}^{k-1} \exp\left[r_i^u\right] \le \exp\left[r_i^u\tau\right], \tag{2.46}$$

and hence

$$x_i(k-\tau_{ii}(k)) \geq x_i(k)\exp\left[-r_i^u\tau\right], \tag{2.47}$$

which, together with (2.45), produces

$$\begin{aligned} x_i(k+1) &\leq x_i(k)\exp\left[r_i(k)-b_{ii}(k)\exp\left[-\beta_{ii}r_i^u\tau\right]x_i^{\beta_{ii}}(k)\right] \\ &\leq x_i(k)\exp\left[r_i(k)\left(1-\frac{b_{ii}^l\exp\left[-\beta_{ii}r_i^u\tau\right]}{r_i^u}x_i^{\beta_{ii}}(k)\right)\right], \end{aligned} \tag{2.48}$$

similar to the analysis of (2.11) and (2.12), for $i=1,2,\ldots,m$

$$\limsup_{k\to+\infty} u_i(k) \leq \overline{L}_i, \tag{2.49}$$

and thus, we immediately get

$$\limsup_{k\to+\infty} x_i(k) \leq \overline{M}_i, \quad i=1,2,\ldots,m. \tag{2.50}$$

For any $\varepsilon>0$ small enough, it follows from (2.50) that there exists enough large $\overline{K}_1$ such that for all $i=1,2,\ldots,m$ and $k\geq\overline{K}_1$

$$x_i(k) \leq \overline{M}_i+\varepsilon. \tag{2.51}$$

For $i=m+1,\ldots,n$ and $k\geq\overline{K}_1+\tau$, (2.51) combining with the i-th equation of system (1.1) lead to

$$\begin{aligned} x_i(k+1) &\leq x_i(k)\exp\left[-r_i^l+\sum_{j=1}^{m}a_{ij}^u\left(M_j+\varepsilon\right)^{\alpha_{ij}}+\sum_{j=1}^{m}b_{ij}^u\left(M_j+\varepsilon\right)^{\beta_{ij}}\right] \\ &= x_i(k)\exp[\Upsilon_{i\varepsilon}], \end{aligned} \tag{2.52}$$

$$\begin{aligned} x_i(k+1) \leq x_i(k)\exp\Bigg[&-r_i(k)+\sum_{j=1}^{m}a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} \\ &+\sum_{j=1}^{m}b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}}-b_{ii}(k)x_i^{\beta_{ii}}(k-\tau_{ii}(k))\Bigg], \end{aligned} \tag{2.53}$$

from (2.53), similar to the argument of (2.44) and (2.47), for $k\geq\overline{K}_1+\tau$, we have

$$x_i(k-\tau_{ii}(k)) \geq x_i(k)\exp[-\Upsilon_{i\varepsilon}\tau], \tag{2.54}$$

substituting (2.54) into (2.53), we get

$$x_i(k+1) \le x_i(k)\exp\left[-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - b_{ii}(k)\exp\left[-\beta_{ii}\Upsilon_{i\varepsilon}\tau\right]x_i^{\beta_{ii}}(k)\right], \tag{2.55}$$

similar to the analysis of (2.18) and (2.19), for $i = m+1,\dots,n$

$$\limsup_{k\to+\infty} u_i(k) \le \overline{L}_i, \tag{2.56}$$

then,

$$\limsup_{k\to+\infty} x_i(k) \le \overline{M}_i, i = m+1,\dots,n. \tag{2.57}$$

For any $\varepsilon > 0$ small enough, it follows from (2.51) and (2.57) that there exists enough large $\overline{K}_2 > \overline{K}_1 + \tau$ such that for all $i = 1,2,\dots,n$ and $k \ge \overline{K}_2$

$$x_i(k) \le \overline{M}_i + \varepsilon. \tag{2.58}$$

Hence, for $i = 1,2,\dots,m$, and $k \ge \overline{K}_2 + \tau$, it follows from system (1.1) that

$$\begin{aligned} x_i(k+1) &\ge x_i(k)\exp\left[r_i^l - \sum_{j=1}^{n} a_{ij}^u\left(M_j+\varepsilon\right)^{\alpha_{ij}} - \sum_{j=1}^{n} b_{ij}^u\left(M_j+\varepsilon\right)^{\beta_{ij}}\right] \\ &= x_i(k)\exp[\Gamma_{i\varepsilon}], \end{aligned} \tag{2.59}$$

$$x_i(k+1) \ge x_i(k)\exp\left[r_i(k) - \sum_{j=1}^{n} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} - \sum_{j=1, j\ne i}^{n} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - b_{ii}(k)x_i^{\beta_{ii}}(k-\tau_{ii}(k))\right], \tag{2.60}$$

from(2.59), similar to the argument of (2.44) and (2.47), for $k \ge \overline{K}_2 + \tau$, we have

$$x_i(k-\tau_{ii}(k)) \le x_i(k)\exp[-\Gamma_{i\varepsilon}\tau], \tag{2.61}$$

and this combined with (2.60) gives

$$x_i(k+1) \geq x_i(k) \exp\left[r_i(k) - \sum_{j=1}^{n} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} - \sum_{j=1, j\neq i}^{n} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - b_{ii}(k)\exp\left[-\beta_{ii}\Gamma_{i\varepsilon}\tau\right] x_i^{\beta_{ii}}(k)\right]. \tag{2.62}$$

Similar to the argument of (2.32) and (2.33), for $k \geq \overline{K}_2 + \tau$, we obtain

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{\overline{A}_i} \exp\left[\beta_{ii}\overline{R}_i^u\left(1 - \overline{A}_i\overline{L}_i\right)\right], \tag{2.63}$$

then

$$\liminf_{k\to+\infty} x_i(k) \geq \overline{m}_i, \quad i = 1, 2, \ldots, m. \tag{2.64}$$

For any $\varepsilon > 0$ small enough, it follows from (2.63) that there exists enough large $\overline{K}_3 > \overline{K}_2 + \tau$ such that for all $i = 1, \ldots, m$ and $k \geq \overline{K}_3$

$$x_i(k) \geq \overline{m}_i - \varepsilon, \tag{2.65}$$

and so, for $i = m+1, \ldots, n$ and $k \geq K_3 + \tau$, it follows from system (1.1) that

$$\begin{aligned} x_i(k+1) &\geq x_i(k) \exp\left[-r_i^u + \sum_{j=1}^{m} a_{ij}^l\left(m_j-\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}^l\left(m_j-\varepsilon\right)^{\beta_{ij}} - \sum_{j=m+1}^{n} a_{ij}^u\left(M_j+\varepsilon\right)^{\alpha_{ij}} - \sum_{j=m+1}^{n} b_{ij}^u\left(M_j+\varepsilon\right)^{\beta_{ij}}\right] = x_i(k)\exp\left[\Gamma_{i\varepsilon}\right], \\ x_i(k+1) &\geq x_i(k) \exp\left[-r_i(k) + \sum_{j=1}^{m} a_{ij}(k)\left(m_j-\varepsilon\right)^{\alpha_{ij}} + \sum_{j=1}^{m} b_{ij}(k)\left(m_j-\varepsilon\right)^{\beta_{ij}} - \sum_{j=m+1}^{n} a_{ij}(k)\left(M_j+\varepsilon\right)^{\alpha_{ij}} \right. \\ &\qquad \left. - \sum_{j=m+1, j\neq i}^{n} b_{ij}(k)\left(M_j+\varepsilon\right)^{\beta_{ij}} - b_{ii}(k) x_i^{\beta_{ii}}\left(k - \tau_{ii}(k)\right)\right]. \end{aligned} \tag{2.66}$$

Similar to the argument of (2.61) and (2.62), for $k \geq \overline{K}_3 + \tau$, we have

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{\overline{A}_i} \exp\left[\beta_{ii}\overline{R}_i^u\left(1 - \overline{A}_i\overline{L}_i\right)\right], \tag{2.67}$$

then

$$\liminf_{k\to+\infty} x_i(k) \geq \overline{m}_i, \quad i = m+1,\ldots,n. \tag{2.68}$$

This ends the proof of Lemma 2.6. □

Denote (H_3)

$$a_{ii}^l > 0, \quad \min_{1\leq i\leq n} L_i A_i > 1, \tag{2.69}$$

or

$$b_{ii}^l > 0, \quad \min_{1\leq i\leq n} \overline{L}_i \overline{A}_i > 1, \tag{2.70}$$

Our main result in this paper is the following theorem about the permanence of system (1.1).

Theorem 2.7. *Assume that* $(H_1), (H_2)$, *and* (H_3) *hold, then system* (1.1) *is permanent.*

Proof. Let $x(k) = (x_1(k),\ldots,x_n(k))$ be any positive solution of system (1.1) with initial condition (1.2). Suppose $M = \max_{i=1,\ldots,n}\{M_i, \overline{M}_i\}, m = \min_{i=1,\ldots,n}\{m_i, \overline{m}_i\}$. By Lemmas 2.4–2.6, if system (1.1) satisfies $(H_1), (H_2)$, and (H_3), then we have

$$m \leq \liminf_{k\to+\infty} x_i(k) \leq \limsup_{k\to+\infty} x_i(k) \leq M, \quad i = 1,2,\ldots,n. \tag{2.71}$$

The proof is completed. □

In this paper, we study a discrete nonlinear predator-prey system with m-preys and (n-m)-predators and delays, which can be seen as the modification of the traditional Lotka-Volterra prey-competition model. From our main results, Theorem 2.7 gives two sets of sufficient conditions on the permanence of the system (1.1). One set is delay independent, while the other set is delay dependent.

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