Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 293846, 9 pages
doi:10.1155/2010/293846

Research Article

An Existence Result for Second-Order Impulsive Differential Equations with Nonlocal Conditions

Meili Li and Haiqiang Liu

Department of Applied Mathematics, Donghua University, Shanghai 201620, China

Correspondence should be addressed to Meili Li, stylml@dhu.edu.cn

Received 2 May 2010; Accepted 17 June 2010

Academic Editor: Guang Y. Zhang

The Leray-Schauder alternative is used to investigate the existence of solutions for second-order impulsive differential equations with nonlocal conditions in Banach spaces. The results improve some recent results.

1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is a lot richer than the corresponding theory of nonimpulsive differential equations. Many evolutionary processes in nature are characterized by the fact that at certain moments in time an abrupt change of state is experienced. That is the reason for the rapid development of the theory of impulsive differential equations; see the monographs [1, 2].

This paper is concerned with the study on existence of second-order impulsive differential equations with nonlocal conditions of the form

$$\begin{gathered} x''(t) = f\left(t, x(t), x'(t)\right), \quad t \in J = [0, b],\ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, \dots, m, \\ \Delta x'|_{t=t_k} = \overline{I}_k\left(x(t_k), x'(t_k)\right), \quad k = 1, \dots, m, \\ x(0) + g(x) = x_0, \qquad x'(0) = \eta, \end{gathered} \tag{1.1}$$

where the state $x(\cdot)$ takes values in Banach space X with the norm $\|\cdot\|$, $x_0, \eta \in X$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$. f, g, and $I_k, \overline{I}_k$ $(k = 1, 2, \dots, m)$ are given functions to be specified later.

The nonlocal condition is a generalization of the classical initial condition. The first results concerning the existence and uniqueness of mild solutions to Cauchy problems with nonlocal conditions were studied by Byszewski [3]. Recently, theorems about existence, uniqueness and continuous dependence of impulsive differential abstract evolution Cauchy problems with nonlocal conditions have been studied by Fu and Cao [4], Anguraj and Karthikeyan [5], Abada et al. [6], Li and Han [7], and in the references therein.

Up to now there have been very few papers in this direction dealing with the existence of solutions for second-order impulsive differential equations with nonlocal conditions. Our purpose here is to extend the results of first-order impulsive differential equations to second-order impulsive differential equations with nonlocal conditions.

Our main results are based on the following lemma [8].

Lemma 1.1 (Leray-Schauder alternative). *Let S be a convex subset of a normed linear space E and assume that $0 \in S$. Let $G : S \to S$ be a completely continuous operator, and let*

$$\zeta(G) = \{x \in S : x = \lambda G x, \text{for some } 0 < \lambda < 1\}. \tag{1.2}$$

Then either $\zeta(G)$ is unbounded or G has a fixed point.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Denote $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $J' = J \setminus \{t_k\}, k = 1, 2, \ldots, m$. We define the following classes of functions:

$\mathrm{PC}(J, X) = \{x : J \to X : x_k \in C(J_k, X), k = 0, 1, \ldots, m$, and there exist $x(t_k^+), x(t_k^-), k = 1, \ldots, m$ with $x(t_k) = x(t_k^-)\}$,

$\mathrm{PC}^1(J, X) = \{x \in \mathrm{PC}(J, X) : x_k' \in C(J_k, X), k = 0, 1, \ldots, m$, and there exist $x'(t_k^+), x'(t_k^-), k = 1, \ldots, m$ with $x'(t_k) = x'(t_k^-)\}$, where x_k and x_k' represent the restriction of x and x' to J_k, respectively $(k = 0, \ldots, m)$, and $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|$.

Obviously, $\mathrm{PC}(J, X)$ is a Banach space with the norm $\|x\|_{\mathrm{PC}} = \max\{\|x_k\|_{J_k}, k = 0, \ldots, m\}$, and $\mathrm{PC}^1(J, X)$ is also a Banach space with the norm $\|x\|_{\mathrm{PC}^1} = \max\{\|x\|_{\mathrm{PC}}, \|x'\|_{\mathrm{PC}}\}$.

Definition 2.1. A map $f : J \times X \times X \to X$ is said to be an L^1-Carathéodory if

(i) $f : (\cdot, w, v) : J \to X$ is measurable for every $w, v \in X$,

(ii) $f : (t, \cdot, \cdot) : X \times X \to X$ is continuous for almost all $t \in J$,

(iii) for each $i > 0$, there exists $\alpha_i \in L^1(J, R_+)$ such that for almost all $t \in J$

$$\sup_{\|w\|, \|v\| \le i} \|f(t, w, v)\| \le \alpha_i(t). \tag{2.1}$$

Definition 2.2. A function $x \in \mathrm{PC}^1(J, X) \cap C^2(J', X)$ is said to be a solution of (1.1) if x satisfies the equation $x''(t) = f(t, x(t), x'(t))$ a.e. on J', the conditions $\Delta x|_{t=t_k} = I_k(x(t_k))$, $\Delta x'|_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k)), k = 1, \ldots, m$, and $x(0) + g(x) = x_0$, $x'(0) = \eta$.

Lemma 2.3. *If* $x \in PC^1(J, X) \cap C^2(J', X)$ *satisfies*

$$x''(t) = f(t, x(t), x'(t)), \quad t \neq t_k \ (k = 1, 2, \ldots, m), \tag{2.2}$$

then

$$x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s))ds + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right], \quad \forall t \in J, \tag{2.3}$$

$$\begin{aligned} x(t) = x(0) + x'(0)t &+ \sum_{0<t_k<t} \left[x(t_k^+) - x(t_k)\right] + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right](t - t_k) \\ &+ \int_0^t (t-s) f(s, x(s), x'(s))ds, \quad \forall t \in J. \end{aligned} \tag{2.4}$$

Proof. Assume that $t_k < t \leq t_{k+1}$ (here $t_0 = 0, t_{m+1} = b$). Then

$$\begin{aligned} x'(t_1) - x'(0) &= \int_0^{t_1} f(s, x(s), x'(s))ds, \\ x'(t_2) - x'(t_1^+) &= \int_{t_1}^{t_2} f(s, x(s), x'(s))ds, \\ &\vdots \\ x'(t_k) - x'(t_{k-1}^+) &= \int_{t_{k-1}}^{t_k} f(s, x(s), x'(s))ds, \\ x'(t) - x'(t_k^+) &= \int_{t_k^+}^{t} f(s, x(s), x'(s))ds. \end{aligned} \tag{2.5}$$

Adding these together, we get

$$x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s))ds + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right], \tag{2.6}$$

that is, (2.3) holds.

Similarly, we have

$$x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0<t_k<t} \left[x(t_k^+) - x(t_k)\right]. \tag{2.7}$$

Substitution of (2.3) in (2.7) gives

$$x(t) = x(0) + x'(0)t + \sum_{0<t_k<t} \left[x(t_k^+) - x(t_k)\right] + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right](t - t_k)$$
$$+ \int_0^t (t-s) f(s, x(s), x'(s)) ds, \quad \forall t \in J, \tag{2.8}$$

that is, (2.4) holds. □

We assume the following hypotheses:

(H_1) $f : J \times X \times X \to X$ is an L^1-Carathéodory map;

(H_2) $I_k \in C(,X)$, $\overline{I}_k \in C(X \times X, X)$, and there exist constants $d_k, \overline{d}_k$ such that $\|I_k(w)\| \le d_k$, $\|\overline{I}_k(w,v)\| \le \overline{d}_k (k = 1, \ldots, m)$ for every $w, v \in X$;

(H_3) $g : \mathrm{PC}(J,X) \to X$ is a continuous function and there exists a constant M such that

$$\|g(x)\| \le M, \quad \text{for each } x \in \mathrm{PC}(J,X); \tag{2.9}$$

(H_4) there exists a function $p \in L^1(J, R_+)$ such that

$$\|f(t,w,v)\| \le p(t)\psi(\|w\| + \|v\|), \quad \text{for a.e. } t \in J \text{ and every } w, v \in X, \tag{2.10}$$

where $\psi : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function with

$$(b+1)\int_0^b p(s) ds < \int_c^\infty \frac{ds}{\psi(s)}, \tag{2.11}$$

where

$$c = \|x_0\| + M + (b+1)\|\eta\| + \sum_{k=1}^{m} \left[d_k + (b + 1 - t_k)\overline{d_\mathrm{k}}\right]; \tag{2.12}$$

(H_5) for each bounded $B \subseteq \mathrm{PC}^1(J,X)$ and $t \in J$ the set

$$\left\{x_0 - g(x) + t\eta + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t} \overline{I}_k(x(t_k), x'(t_k))(t - t_k) + \int_0^t (t-s) f(s, x(s), x'(s)) ds : x \in B\right\} \tag{2.13}$$

is relatively compact in X.

3. Main Results

Theorem 3.1. *If the hypotheses* (H_1)–(H_5) *are satisfied, then the second-order impulsive nonlocal initial value problem* (1.1) *has at least one solution on* J.

Proof. Consider the space $B = \mathrm{PC}^1(J, X)$ with norm

$$\|x\|_{\mathrm{PC}^1} = \max\{\|x\|_{\mathrm{PC}}, \|x'\|_{\mathrm{PC}}\}. \tag{3.1}$$

We will now show that the operator G defined by

$$\begin{aligned} Gx(t) = x_0 - g(x) + t\eta + \sum_{0<t_k<t} I_k(x(t_k)) + \sum_{0<t_k<t} \overline{I}_k(x(t_k), x'(t_k))(t - t_k) \\ + \int_0^t (t-s) f(s, x(s), x'(s))ds, \quad t \in J \end{aligned} \tag{3.2}$$

has a fixed point. This fixed point is then a solution of (1.1).

First we obtain a priori bounds for the following equation:

$$\begin{aligned} x(t) = \lambda[x_0 - g(x) + t\eta] + \lambda \sum_{0<t_k<t} I_k(x(t_k)) + \lambda \sum_{0<t_k<t} \overline{I}_k(x(t_k), x'(t_k))(t - t_k) \\ + \lambda \int_0^t (t-s) f(s, x(s), x'(s))ds, \quad t \in J. \end{aligned} \tag{3.3}$$

We have

$$\|x(t)\| \le \|x_0\| + M + b\|\eta\| + \sum_{k=1}^m d_k + \sum_{k=1}^m (b - t_k)\overline{d}_k + b\int_0^t p(s)\varphi(\|x(s)\| + \|x'(s)\|)ds, \quad t \in J. \tag{3.4}$$

Denoting by $\mu(t)$ the right-hand side of the above inequality, we have

$$\begin{aligned} \mu(0) = \|x_0\| + M + b\|\eta\| + \sum_{k=1}^m \left[d_k + (b - t_k)\overline{d}_k\right], \quad \|x(t)\| \le \mu(t), \quad t \in J, \\ \mu'(t) = bp(t)\varphi(\|x(\mathrm{t})\| + \|x'(t)\|), \quad t \in J. \end{aligned} \tag{3.5}$$

But

$$x'(t) = \lambda\eta + \lambda\int_0^t f(s, x(s), x'(s))ds + \lambda \sum_{0<t_k<t} \overline{I}_k(x(t_k), x'(t_k)), \quad t \in J. \tag{3.6}$$

Thus we have

$$\|x'(t)\| \le \|\eta\| + \int_0^t p(s)\varphi(\|x(s)\| + \|x'(s)\|)ds + \sum_{k=1}^{m} \overline{d}_k, \quad t \in J. \tag{3.7}$$

Denoting by $r(t)$ the right-hand side of the above inequality, we have

$$\begin{gathered} r(0) = \|\eta\| + \sum_{k=1}^{m} \overline{d}_k, \quad \|x'(t)\| \le r(t), \quad t \in J, \\ r'(t) = p(t)\varphi(\|x(t)\| + \|x'(t)\|), \quad t \in J. \end{gathered} \tag{3.8}$$

Let

$$w(t) = \mu(t) + r(t), \quad t \in J. \tag{3.9}$$

Then

$$\begin{aligned} w(0) &= \mu(0) + r(0) = c, \\ w'(t) &= \mu'(t) + r'(t) \\ &\le bp(t)\varphi(w(t)) + p(t)\varphi(w(t)) \\ &= (b+1)p(t)\varphi(w(t)), \quad t \in J. \end{aligned} \tag{3.10}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{\varphi(s)} \le (b+1)\int_0^b p(s)ds < \int_c^{\infty} \frac{ds}{\varphi(s)}, \quad t \in J. \tag{3.11}$$

This inequality implies that there is a constant K such that

$$w(t) = \mu(t) + r(t) \le K, \quad t \in J. \tag{3.12}$$

Then

$$\begin{gathered} \|x(t)\| \le \mu(t), \quad t \in J, \\ \|x'(t)\| \le r(t), \quad t \in J, \end{gathered} \tag{3.13}$$

and hence

$$\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\} \le K. \tag{3.14}$$

Second, we must prove that the operator $G : B \to B$ is a completely continuous operator.

Let $B_i = \{x \in B : \|x\|_{PC^1} \le i\}$ for some $i \ge 1$. We first show that G maps B_i into an equicontinuous family. Let $x \in B_i$ and $t, \bar{t} \in J$. Then for $0 < t < \bar{t} \le b$, we have

$$\begin{aligned}
\left\|(Gx)(t) - (Gx)\left(\bar{t}\right)\right\| &\le \left\|t\eta - \bar{t}\eta\right\| + \left\|\sum_{t \le t_k < \bar{t}} I_k(x(t_k))\right\| \\
&\quad + \left\|\sum_{0 < t_k < t}\left[(t - t_k) - \left(\bar{t} - t_k\right)\right]\bar{I}_k\left(x(t_k), x'(t_k)\right)\right\| \\
&\quad + \left\|\sum_{t \le t_k < \bar{t}}\left(\bar{t} - t_k\right)\bar{I}_k\left(x(t_k), x'(t_k)\right)\right\| \\
&\quad + \left\|\int_0^t\left[(t - s) - \left(\bar{t} - s\right)\right] f\left(s, x(s), x'(s)\right) ds\right\| \\
&\quad + \left\|\int_t^{\bar{t}}\left(\bar{t} - s\right) f\left(s, x(s), x'(s)\right) ds\right\| \\
&\le \left(\bar{t} - t\right)\|\eta\| + \sum_{t \le t_k < \bar{t}} d_k + \sum_{0 < t_k < t}\left(\bar{t} - t\right)\bar{d}_k + \sum_{t \le t_k < \bar{t}}\left(\bar{t} - t_k\right)\bar{d}_k \\
&\quad + \int_0^t\left(\bar{t} - t\right)\alpha_i(s) ds + \int_t^{\bar{t}}\left(\bar{t} - s\right)\alpha_i(s) ds,
\end{aligned} \tag{3.15}$$

and similarly

$$\begin{aligned}
\left\|(Gx)'(t) - (Gx)'\left(\bar{t}\right)\right\| &\le \left\|\int_0^t f\left(s, x(s), x'(s)\right) ds - \int_0^{\bar{t}} f\left(s, x(s), x'(s)\right) ds\right\| \\
&\quad + \left\|\sum_{0 < t_k < t}\bar{I}_k\left(x(t_k), x'(t_k)\right) - \sum_{0 < t_k < \bar{t}}\bar{I}_k\left(x(t_k), x'(t_k)\right)\right\| \\
&\le \int_t^{\bar{t}}\alpha_i(s) ds + \sum_{t \le t_k < \bar{t}}\bar{d}_k.
\end{aligned} \tag{3.16}$$

The right-hand sides are independent of $x \in B_i$ and tend to zero as $\bar{t} - t \to 0$. Thus G maps B_i into an equicontinuous family of functions. It is easy to see that the family GB_i is uniformly bounded. And from (H_5), we know that $\overline{GB_i}$ is compact. Then by Arzela-Ascoli theorem, we can conclude that the map $G : B \to B$ is compact.

Next, we show that $G : B \to B$ is continuous. Let $\{u_n\}_0^\infty \sqsubseteq B$ with $u_n \to u$ in B. Then there is an integer q such that $\|u_n(t)\|, \|u_n'(t)\| \le q$ for all n and $t \in J$, so $u_n \in B_q$ and $u \in B_q$. By (H_1), $f(t, u_n(t), u_n'(t)) \to f(t, u(t), u'(t))(n \to \infty)$ for almost all $t \in J$, and since

$\|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\|<2\alpha_q(t)$, we have by the dominated convergence theorem that

$$\begin{aligned}
\|Gu_n-Gu\|_{\mathrm{PC}}=\sup_{t\in J}\Bigg\|&[g(u_n)-g(u)]+\int_0^t(t-s)\left[f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\right]ds\\
&+\sum_{0<t_k<t}I_k(u_n(t_k))-\sum_{0<t_k<t}I_k(u(t_k))\\
&+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(u_n(t_k),u_n'(t_k))-\sum_{0<t_k<t}(t-t_k)\overline{I}_k(u(t_k),u'(t_k))\Bigg\|\\
\le{}&\|g(u_n)-g(u)\|\\
&+\int_0^b(b-s)\|f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\|ds\\
&+\sum_{k=1}^m\|I_k(u_n(t_k))-I_k(u(t_k))\|\\
&+\sum_{k=1}^m(b-t_k)\left\|\overline{I}_k(u_n(t_k),u_n'(t_k))-\overline{I}_k(u(t_k),u'(t_k))\right\|\longrightarrow 0,
\end{aligned}\tag{3.17}$$

$$\begin{aligned}
\|(Gu_n)'-(Gu)'\|_{\mathrm{PC}}=\sup_{t\in J}\Bigg\|&\int_0^t\left[f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\right]ds\\
&+\sum_{0<t_k<t}\overline{I}_k(u_n(t_k),u_n'(t_k))-\sum_{0<t_k<t}\overline{I}_k(u(t_k),u'(t_k))\Bigg\|\\
\le{}&\int_0^b\|\left[f(s,u_n(s),u_n'(s))-f(s,u(s),u'(s))\right]\|ds\\
&+\sum_{k=1}^m\left\|\overline{I}_k(u_n(t_k),u_n'(t_k))-\overline{I}_k(u(t_k),u'(t_k))\right\|\longrightarrow 0.
\end{aligned}\tag{3.18}$$

Thus G is continuous. This completes the proof that G is completely continuous.

Finally, the set $\zeta(G)=\{x\in B: x=\lambda Gx,\ \lambda\in(0,1)\}$ is bounded, as we proved in the first step. As a consequence of Lemma 1.1, we deduce that G has a fixed point $x\in B$ which is a solution of (1.1). □

Acknowledgments

This work is supported by NNSF of China (no. 10971139) and Chinese Universities Scientific Fund (no. B08-1).

References

[1] D. D. Baĭnov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 1989.

[2] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1992.

[3] L. Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 494–505, 1991.

[4] X. Fu and Y. Cao, "Existence for neutral impulsive differential inclusions with nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3707–3718, 2008.

[5] A. Anguraj and K. Karthikeyan, "Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2717–2721, 2009.

[6] N. Abada, M. Benchohra, and H. Hammouche, "Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions," *Journal of Differential Equations*, vol. 246, no. 10, pp. 3834–3863, 2009.

[7] M. Li and M. Han, "Existence for neutral impulsive functional differential equations with nonlocal conditions," *Indagationes Mathematicae*, vol. 20, no. 3, pp. 435–451, 2009.

[8] J. Dugundji and A. Granas, *Fixed Point Theory*, vol. 1 of *Monographie Matematyczne*, PNW, Warsawa, Poland, 1982.