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Research Article

Complete Convergence for Weighted Sums of ρ^*-Mixing Random Variables

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We obtain the complete convergence for weighted sums of ρ^*-mixing random variables. Our result extends the result of Peligrad and Gut (1999) on unweighted average to a weighted average under a mild condition of weights. Our result also generalizes and sharpens the result of An and Yuan (2008).

1. Introduction

In many stochastic models, the assumption that random variables are independent is not plausible. So it is of interest to extend the concept of independence to dependence cases. One of these dependence structures is ρ^*-mixing.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_n^m$ denote the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots, X_m$. For any $S \subset N$, define $\mathcal{F}_S = \sigma(X_i, i \in S)$. Given two σ-algebras $\mathcal{A}, \mathcal{B}$ in $\mathcal{F}$, put

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\{\operatorname{corr}(X, Y); X \in L_2(\mathcal{A}), Y \in L_2(\mathcal{B})\}, \tag{1.1}$$

where $\operatorname{corr}(X, Y) = (EXY - EXEY)/\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}$. Define the ρ^*-mixing coefficients by

$$\rho_n^* = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T); S, T \subset N \text{ with } \operatorname{dist}(S, T) \geq n\}. \tag{1.2}$$

Obviously, $0 \leq \rho_{n+1}^* \leq \rho_n^* \leq \rho_0^* = 1$. The sequence $\{X_n, n \geq 1\}$ is called ρ^*-mixing (or $\widetilde{\rho}$-mixing) if there exists $k \in N$ such that $\rho_k^* < 1$. Note that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables, then $\rho_n^* = 0$ for all $n \geq 1$.

A number of limit results for ρ^*-mixing sequences of random variables have been established by many authors. We refer to Bradley [1] for the central limit theorem, Bryc and Smoleński [2], Peligrad and Gut [3], and Utev and Peligrad [4] for moment inequalities, Gan [5], Kuczmaszewska [6], and Wu and Jiang [7] for almost sure convergence, and An and Yuan [8], Cai [9], Gan [5], Kuczmaszewska [10], Peligrad and Gut [3], and Zhu [11] for complete convergence.

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [12]. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty \quad \forall \epsilon > 0. \tag{1.3}$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. Hsu and Robbins [12] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdös [13] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz [14] strong law of large numbers.

Theorem 1.1 (Baum and Katz [14]). *Let $p \geq 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{p\alpha-2} P(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| > \epsilon n^{\alpha}) < \infty$ *for all* $\epsilon > 0$,

Peligrad and Gut [3] extended the result of Baum and Katz [14] to ρ^*-mixing random variables.

Theorem 1.2 (Peligrad and Gut [3]). *Let $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{p\alpha-2} P(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| > \epsilon n^{\alpha}) < \infty$ *for all* $\epsilon > 0$.

Cai [9] complemented Theorem 1.2 when $p = 1/\alpha$.

Recently, An and Yuan [8] obtained a complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables.

Theorem 1.3 (An and Yuan [8]). *Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^p = O\left(n^{\delta}\right) \quad \text{for some } 0 < \delta < 1, \tag{1.4}$$

$$\# A_{nk} = \#\left\{1 \le i \le n : |a_{ni}|^p > (k+1)^{-1}\right\} \ge n e^{-1/k} \quad \forall k \ge 1,\ n \ge 1. \tag{1.5}$$

Then the following statements are equivalent:

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{p\alpha-2} P(\max_{1\le j\le n} |\sum_{i=1}^{j} a_{ni} X_i| > \epsilon n^{\alpha}) < \infty$ *for all* $\epsilon > 0$.

Note that the result of An and Yuan [8] is not an extension of Peligrad and Gut's [3] result, since condition (1.4) does not hold for the array with $a_{ni} = 1, 1 \le i \le n, n \ge 1$. An and Yuan [8] proved the implication (i)⇒(ii) under condition (1.4), and proved the converse under conditions (1.4) and (1.5). However, the array satisfying both (1.4) and (1.5) does not exist. Noting that $\# A_{nk}/(k+1) \le \sum_{i=1}^{n} |a_{ni}|^p \le O(n^{\delta})$, we have that $n e^{-1/k} \le \# A_{nk} \le (k+1) O(n^{\delta})$. But, this does not hold when k is fixed and n is large enough.

In this paper, we obtain a new complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables. Our result extends the result of Peligrad and Gut [3], and generalizes and sharpens the result of An and Yuan [8].

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance, $[x]$ denotes the integer part of x, and $a \wedge b = \min\{a, b\}$.

2. Main Result

To prove our main result, we need the following lemma which is a Rosenthal-type inequality for ρ^*-mixing random variables.

Lemma 2.1 (Utev and Peligrad [4]). *Let $\{X_n, n \ge 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $E|X_n|^r < \infty$ for some $r \ge 2$ and all $n \ge 1$. Then there exists a constant $D = D(r, k, \rho_k^*)$ depending only on r, k, and ρ_k^* such that for any $n \ge 1$,*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^r\right) \le D\left\{\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} EX_i^2\right)^{r/2}\right\}, \tag{2.1}$$

where $\rho_k^* < 1$.

Now we state the main result of this paper.

Theorem 2.2. *Let $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \quad \text{for some } q > p. \tag{2.2}$$

If $E|X_1|^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \epsilon n^{\alpha}\right) < \infty \quad \forall \epsilon > 0. \tag{2.3}$$

Conversely, if (2.3) *holds for any array $\{a_{ni}\}$ satisfying* (2.2), *then $E|X_1|^p < \infty$.*

To prove Theorem 2.2, we first prove the following lemma which is the sufficiency of Theorem 2.2 when the array is bounded.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$ and $E|X_1|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $|a_{ni}| \leq 1$ for $1 \leq i \leq n$ and $n \geq 1$. Then* (2.3) *holds.*

Proof. For $1 \leq i \leq n$ and $n \geq 1$, define $X'_{ni} = X_i I(|X_i| \leq n^{\alpha})$. Since $EX_i = 0$ and $\sum_{i=1}^{n} |a_{ni}| \leq n$, we have that

$$\begin{aligned}
n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right| &= n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}EX_i I(|X_i| > n^{\alpha})\right| \\
&\leq n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|E|X_1|I(|X_1| > n^{\alpha}) \\
&\leq n^{1-\alpha}E|X_1|I(|X_1| > n^{\alpha}) \\
&\leq n^{1-p\alpha}E|X_1|^p I(|X_1| > n^{\alpha}) \longrightarrow 0
\end{aligned} \tag{2.4}$$

as $n \to \infty$. Hence for n large enough, we have

$$n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right| < \frac{\epsilon}{2}. \tag{2.5}$$

It follows that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \epsilon n^{\alpha}\right) \\
&\quad\le \sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{i=1}^{n} P(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X'_{ni}\right| > \epsilon n^{\alpha}\right) \\
&\quad\le \sum_{n=1}^{\infty} n^{p\alpha-1} P(|X_1| > n^{\alpha}) + C\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right| > \frac{\epsilon n^{\alpha}}{2}\right) \\
&\quad =: I + CJ.
\end{aligned}
\tag{2.6}
$$

Noting that $\sum_{n=1}^{\infty} n^{p\alpha-1}P(|X_1| > n^{\alpha}) \le CE|X_1|^p < \infty$, we have $I < \infty$. Thus, it remains to show that $J < \infty$.

We have by Markov's inequality and Lemma 2.1 that for any $r \ge 2$,

$$
\begin{aligned}
J &\le \left(\frac{2}{\epsilon}\right)^{r} \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right|^{r} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^2 E\left|X'_{ni}\right|^2\right)^{r/2} + \sum_{i=1}^{n} |a_{ni}|^r E\left|X'_{ni}\right|^r\right\} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2}\left(E|X_1|^2 I(|X_1| \le n^{\alpha})\right)^{r/2} + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1} E|X_1|^r I(|X_1| \le n^{\alpha}) \\
&=: CJ_1 + CJ_2.
\end{aligned}
\tag{2.7}
$$

In the last inequality, we used the fact that $|a_{ni}| \le 1$ for $1 \le i \le n$ and $n \ge 1$.

If $p \ge 2$, then we take large enough r such that $r > \max\{(p\alpha - 1)/(\alpha - 1/2), p\}$. Since $r > (p\alpha - 1)/(\alpha - 1/2)$, we get

$$
J_1 \le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty. \tag{2.8}
$$

Since $r > p$, we also get

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 1} \sum_{i=1}^{n} E|X_1|^r I\left((i-1)^{\alpha} < |X_1| \le i^{\alpha}\right) \\ &= \sum_{i=1}^{\infty} E|X_1|^r I\left((i-1)^{\alpha} < |X_1| \le i^{\alpha}\right) \sum_{n=i}^{\infty} n^{p\alpha - r\alpha - 1} \\ &\le C \sum_{i=1}^{\infty} E|X_1|^r I\left((i-1)^{\alpha} < |X_1| \le i^{\alpha}\right) i^{p\alpha - r\alpha} \\ &\le CE|X_1|^p < \infty. \end{aligned} \tag{2.9}$$

If $p < 2$, then we take $r = 2$. Since $r > p$, (2.9) still holds, and so $J_1 = J_2 < \infty$. □

We next prove the sufficiency of Theorem 2.2 when the array is unbounded.

Lemma 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$ and $E|X_1|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $a_{ni} = 0$ or $|a_{ni}| > 1$, and*

$$\sum_{i=1}^{n} |a_{ni}|^q \le n \quad \text{for some } q > p. \tag{2.10}$$

Then (2.3) *holds.*

Proof. If $p < 2$, then we can take $\delta > 0$ such that $p < p + \delta < \min\{2, q\}$. Since $a_{ni} = 0$ or $|a_{ni}| > 1$, we have that $\sum_{i=1}^{n} |a_{ni}|^{p+\delta} \le \sum_{i=1}^{n} |a_{ni}|^q \le n$. Thus we may assume that (2.10) holds for some $p < q < 2$ when $p < 2$.

Let $S'_{nj} = \sum_{i=1}^{j} a_{ni} X_i I(|a_{ni} X_i| \le n^{\alpha})$ for $1 \le j \le n$ and $n \ge 1$. In view of $EX_i = 0$, we get

$$\begin{aligned} n^{-\alpha} \max_{1 \le j \le n} \left| ES'_{nj} \right| &= n^{-\alpha} \max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni} EX_i I(|a_{ni} X_i| > n^{\alpha}) \right| \\ &\le n^{-\alpha} \sum_{i=1}^{n} E|a_{ni} X_i| I(|a_{ni} X_i| > n^{\alpha}) \\ &\le n^{-p\alpha} \sum_{i=1}^{n} E|a_{ni} X_i|^p I(|a_{ni} X_i| > n^{\alpha}) \\ &\le n^{-p\alpha} \sum_{i=1}^{n} |a_{ni}|^p E|X_1|^p \\ &\le n^{-p\alpha} \left(\sum_{i=1}^{n} |a_{ni}|^q \right)^{p/q} n^{1-p/q} E|X_1|^p \\ &\le n^{1-p\alpha} E|X_1|^p \longrightarrow 0, \end{aligned} \tag{2.11}$$

since $p\alpha > 1$. Hence for n large enough, we have that $n^{-\alpha}\max_{1\le j\le n}|ES'_{nj}| < \epsilon/2$. It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \epsilon n^{\alpha}\right) \\
&\quad\le \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le i\le n}|a_{ni}X_i| > n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|S'_{nj}\right| > \epsilon n^{\alpha}\right) \\
&\quad\le \sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{\alpha}) + C\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|S'_{nj} - ES'_{nj}\right| > \frac{\epsilon n^{\alpha}}{2}\right) \\
&\quad =: I + CJ.
\end{aligned} \tag{2.12}$$

For $1 \le j \le n-1$ and $n \ge 2$, let

$$I_{nj} = \left\{1 \le i \le n : n^{1/q}(j+1)^{-1/q} < |a_{ni}| \le n^{1/q} j^{-1/q}\right\}. \tag{2.13}$$

Then $\{I_{nj}, 1 \le j \le n-1\}$ are disjoint, $\bigcup_{j=1}^{n-1} I_{nj} = \{1 \le i \le n : a_{ni} \ne 0\}$, and $\sum_{j=1}^{k} \# I_{nj} \le k+1$ for $1 \le k \le n-1$, since

$$n \ge \sum_{\{1\le i\le n: a_{ni}\ne 0\}} |a_{ni}|^q = \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^q \ge n\sum_{j=1}^{k}\frac{1}{j+1}\# I_{nj} \ge \frac{n}{(k+1)}\sum_{j=1}^{k}\# I_{nj}. \tag{2.14}$$

For convenience of notation, let $t = 1/(\alpha - 1/q)$. Since $a_{ni} = 0$ or $|a_{ni}| > 1$, and $\sum_{i=1}^{n}|a_{ni}|^q \le n$, we have $a_{11} = 0$. It follows that

$$\begin{aligned}
I &= \sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1}\sum_{i\in I_{nj}} P(|a_{ni}X_i| > n^{\alpha}) \\
&\le \sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1} P\left(|X_1|^t > n j^{t/q}\right)\# I_{nj} \\
&\le \sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1}\# I_{nj}\sum_{k\ge[n j^{t/q}]} P\left(k < |X_1|^t \le k+1\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=2}^{\infty} n^{p\alpha-2} \sum_{k=n}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) \sum_{j=1}^{(n-1)\wedge[((k+1)/n)^{q/t}]} \#I_{nj} \\
&\leq \sum_{n=2}^{\infty} n^{p\alpha-2} \sum_{k=n}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) n \wedge \left(\left[\left(\frac{k+1}{n}\right)^{q/t}\right] + 1\right) \\
&\leq \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{k=n}^{[n^{1+t/q}]} P\Big(k < |X_1|^t \leq k+1\Big) \left(\left[\left(\frac{k+1}{n}\right)^{q/t}\right] + 1\right) \\
&\quad + \sum_{n=1}^{\infty} n^{p\alpha-1} \sum_{k=[n^{1+t/q}]+1}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) \\
&=: I_1 + I_2.
\end{aligned} \tag{2.15}$$

Since $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we obtain

$$\begin{aligned}
I_1 &\leq C \sum_{n=1}^{\infty} n^{p\alpha-2-q/t} \sum_{k=n}^{[n^{1+t/q}]} P\Big(k < |X_1|^t \leq k+1\Big) k^{q/t} \\
&\leq C \sum_{k=1}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) k^{q/t} \sum_{n=[k^{q/(q+t)}]}^{k} n^{p\alpha-2-q/t} \\
&\leq C \sum_{k=1}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) k^{q/t - q\alpha(q-p)/(q+t)} \\
&\leq CE|X_1|^p < \infty.
\end{aligned} \tag{2.16}$$

We also obtain

$$\begin{aligned}
I_2 &\leq \sum_{k=1}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) \sum_{n=1}^{[k^{q/(t+q)}]} n^{p\alpha-1} \\
&\leq C \sum_{k=1}^{\infty} P\Big(k < |X_1|^t \leq k+1\Big) k^{p(\alpha-1/q)} \leq CE|X_1|^p < \infty.
\end{aligned} \tag{2.17}$$

From $I_1 < \infty$ and $I_2 < \infty$, we have $I < \infty$. Thus, it remains to show that $J < \infty$.

We have by Markov's inequality and Lemma 2.1 that for any $r \geq 2$,

$$
\begin{aligned}
J &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\max_{1\leq j\leq n}\left|S'_{nj} - ES'_{nj}\right|^{r} \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leq n^{\alpha})\right)^{r/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n} E|a_{ni}X_i|^{r} I(|a_{ni}X_i| \leq n^{\alpha}) \\
&=: J_1 + J_2.
\end{aligned}
\tag{2.18}
$$

Observe that for $r \geq q$ and $n > m$,

$$
n \geq \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^{q} \geq n\sum_{j=1}^{n-1}\frac{1}{j+1}\#I_{nj} \geq n(m+1)^{r/q-1}\sum_{j=m}^{n-1}(j+1)^{-r/q}\#I_{nj}. \tag{2.19}
$$

So $\sum_{j=m}^{n-1} j^{-r/q}\#I_{nj} \leq Cm^{-(r/q-1)}$ for $r \geq q$ and $n > m$.

For J_1 and J_2, we proceed with two cases.

(i) If $p \geq 2$, then we take r large enough such that $r > \max\{(p\alpha-1)/(\alpha-1/2), q\}$. Then we obtain that

$$
\begin{aligned}
J_1 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{r/2} \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n}|a_{ni}|^{q}\right)^{r/2} \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty.
\end{aligned}
\tag{2.20}
$$

The second inequality follows by the fact that $a_{ni} = 0$ or $|a_{ni}| > 1$.

Noting that $a_{11} = 0$, we also obtain that

$$\begin{aligned}
J_2 &= \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2} \sum_{j=1}^{n-1} \sum_{i \in I_{nj}} E|a_{ni}X_i|^r I(|a_{ni}X_i| \le n^{\alpha}) \\
&\le \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{j=1}^{n-1} j^{-r/q} \# I_{nj} E|X_1|^r I\left(|X_1|^t \le n(j+1)^{t/q}\right) \\
&\le \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{j=1}^{n-1} j^{-r/q} \# I_{nj} \sum_{0 \le k \le [n(j+1)^{t/q}]} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \\
&= \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{j=1}^{n-1} j^{-r/q} \# I_{nj} \sum_{k=0}^{2n} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \\
&\quad + \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{j=1}^{n-1} j^{-r/q} \# I_{nj} \sum_{k=2n+1}^{[n(j+1)^{t/q}]} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \\
&=: J_3 + J_4.
\end{aligned} \tag{2.21}$$

Since $p\alpha - r\alpha - 2 + r/q < q\alpha - r\alpha - 2 + r/q = -(r-q)(\alpha - 1/q) - 1 < -1$ and $q > p$, we have that

$$\begin{aligned}
J_3 &= \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=0}^{2n} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \sum_{j=1}^{n-1} j^{-r/q} \# I_{nj} \\
&\le C \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=0}^{2n} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \\
&\le C \sum_{k=1}^{\infty} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \sum_{n=[k/2]}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \\
&\le C \sum_{k=1}^{\infty} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) k^{p\alpha - r\alpha - 1 + r/q} \\
&\le C \sum_{k=1}^{\infty} P\left(k < |X_1|^t \le k+1\right) k^{p\alpha - 1} \\
&\le C E|X_1|^p < \infty.
\end{aligned} \tag{2.22}$$

Since $1/t + 1/q - \alpha = 0$ and $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we also have that

$$\begin{aligned}
J_4 &\le \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=2n+1}^{[n^{(q+t)/q}]} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \sum_{j=[(k/n)^{q/t}]-1}^{n-1} j^{-r/q} \# I_{nj} \\
&\le C \sum_{n=2}^{\infty} n^{p\alpha - r\alpha - 2 + r/q} \sum_{k=2n+1}^{[n^{(q+t)/q}]} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) \left(\left[\left(\frac{k}{n}\right)^{q/t} \right] - 1 \right)^{-(r/q-1)} \\
&\le C \sum_{k=5}^{\infty} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) k^{-(r-q)/t} \sum_{n=[k^{q/(q+t)}]}^{[k/2]} n^{p\alpha - 2 - q/t} \\
&\le C \sum_{k=5}^{\infty} E|X_1|^r I\left(k < |X_1|^t \le k+1\right) k^{-(r-q)/t - (\alpha - 1/q)(q-p)} \\
&\le C E|X_1|^p < \infty.
\end{aligned} \tag{2.23}$$

From $J_3 < \infty$ and $J_4 < \infty$, we have $J_2 < \infty$.

(ii) If $p < 2$, then we take $r = 2$. As noted above, we may assume that $p < q < 2$. Since $r > q$, as in the case $p \ge 2$, we have $J_1 = J_2 \le CE|X_1|^p < \infty$. □

We now prove Theorem 2.2 by using Lemmas 2.3 and 2.4.

Proof of Theorem 2.2.

Sufficiency. Without loss of generality, we may assume that $\sum_{i=1}^{n} |a_{ni}|^q \le n$ for some $q > p$. For $n \ge 1$, let

$$A_n = \{1 \le i \le n : |a_{ni}| \le 1\}, \qquad B_n = \{1 \le i \le n : |a_{ni}| > 1\}, \tag{2.24}$$

and let $a'_{ni} = a_{ni}$ if $i \in A_n$, $a'_{ni} = 0$ otherwise, and $a''_{ni} = a_{ni}$ if $i \in B_n$, $a''_{ni} = 0$ otherwise. Then

$$\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| \le \max_{1 \le j \le n} \left| \sum_{i=1}^{j} a'_{ni} X_i \right| + \max_{1 \le j \le n} \left| \sum_{i=1}^{j} a''_{ni} X_i \right|. \tag{2.25}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \epsilon n^{\alpha} \right) &\le \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a'_{ni} X_i \right| > \frac{\epsilon n^{\alpha}}{2} \right) \\
&\quad + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} a''_{ni} X_i \right| > \frac{\epsilon n^{\alpha}}{2} \right) \\
&=: I + J.
\end{aligned} \tag{2.26}$$

By Lemma 2.3, we have $I < \infty$. By Lemma 2.4, we have $J < \infty$. Hence (2.3) holds.

Necessity. Choose, for each $n \geq 1, a_{n1} = \cdots = a_{nn} = 1$. Then $\{a_{ni}\}$ satisfies (2.2). By (2.3), we obtain that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \epsilon n^{\alpha}\right) < \infty \quad \forall\epsilon > 0, \tag{2.27}$$

which implies that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|X_j\right| > \epsilon n^{\alpha}\right) < \infty \quad \forall\epsilon > 0. \tag{2.28}$$

Observe that

$$\begin{aligned}
\infty &> \sum_{i=1}^{\infty} \sum_{n=2^{i-1}+1}^{2^i} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|X_j\right| > \epsilon n^{\alpha}\right) \\
&\geq \begin{cases} \displaystyle\sum_{i=1}^{\infty} \left(2^{i-1}\right)^{p\alpha-2} 2^{i-1} P\left(\max_{1\leq j\leq 2^{i-1}}\left|X_j\right| > \epsilon\left(2^i\right)^{\alpha}\right) & \text{if } p\alpha \geq 2, \\ \displaystyle\sum_{i=1}^{\infty} \left(2^{i}\right)^{p\alpha-2} 2^{i-1} P\left(\max_{1\leq j\leq 2^{i-1}}\left|X_j\right| > \epsilon\left(2^i\right)^{\alpha}\right) & \text{if } 1 < p\alpha < 2, \end{cases} \\
&\geq \begin{cases} \displaystyle\sum_{i=1}^{\infty} P\left(\max_{1\leq j\leq 2^{i-1}}\left|X_j\right| > \epsilon\left(2^i\right)^{\alpha}\right) & \text{if } p\alpha \geq 2, \\ \displaystyle 2^{p\alpha-2} \sum_{i=1}^{\infty} P\left(\max_{1\leq j\leq 2^{i-1}}\left|X_j\right| > \epsilon\left(2^i\right)^{\alpha}\right) & \text{if } 1 < p\alpha < 2. \end{cases}
\end{aligned} \tag{2.29}$$

Hence we have that for any $\epsilon > 0, P(\max_{1\leq j\leq 2^{i-1}}|X_j| > \epsilon(2^i)^{\alpha}) \to 0$ as $i \to \infty$, and so $P(\max_{1\leq j\leq n}|X_j| > n^{\alpha}) \to 0$ as $n \to \infty$. The rest of the proof is same as that of Peligrad and Gut [3] and is omitted.

□

Remark 2.5. Taking $a_{ni} = 1$ for $1 \leq i \leq n$ and $n \geq 1$, we can immediately get Theorem 1.2 from Theorem 2.2. If the array $\{a_{ni}\}$ satisfies (1.4), then it satisfies (2.2): taking q such that $p < q < p/\delta$, we have

$$\sum_{i=1}^{n}|a_{ni}|^q \leq \max_{1\leq i\leq n}|a_{ni}|^{q-p}\sum_{i=1}^{n}|a_{ni}|^p \leq Cn^{\delta(q-p)/p}n^{\delta} \leq Cn. \tag{2.30}$$

So the implication (i)⇒(ii) of Theorem 1.3 follows from Theorem 2.2. As noted after Theorem 1.3, the implication (ii)⇒(i) of Theorem 1.3 is not true. Therefore, our result extends the result of Peligrad and Gut [3] to a weighted average, and generalizes and sharpens the result of An and Yuan [8].

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