Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 640594, 18 pages
doi:10.1155/2010/640594

*Research Article*

# Nonlinear Modelling and Qualitative Analysis of a Real Chemostat with Pulse Feeding

**Yuan Tian,[1,2] Kaibiao Sun,[3] Andrzej Kasperski,[4] and Lansun Chen[2]**

[1] *School of Information Engineering, Dalian University, Dalian 116622, China*
[2] *School of Mathematical Science, Dalian University of Technology, Dalian 116024, China*
[3] *School of Control Science and Engineering, Dalian University of Technology, Dalian 116024, China*
[4] *Faculty of Mathematics, Computer Science, and Econometrics, Bioinformatics Factory, University of Zielona Gora, Szafrana 4a, 65-516 Zielona Gora, Poland*

Correspondence should be addressed to Kaibiao Sun, sunkb@dlut.edu.cn

Received 2 May 2010; Accepted 19 August 2010

Academic Editor: Francisco Solis



The control of substrate concentration in the bioreactor medium should be due to the substrate inhibition phenomenon. Moreover, the oxygen demand in a bioreactor should be lower than the dissolved oxygen content. The biomass concentration is one of the most important factors which affect the oxygen demand. In order to maintain the dissolved oxygen content in an appropriate range, the biomass concentration should not exceed a critical level. Based on the design ideas, a mathematical model of a chemostat with Monod-type kinetics and impulsive state feedback control for microorganisms of any biomass yield is proposed in this paper. By the existence criteria of periodic solution of a general planar impulsive autonomous system, the conditions for the existence of period-1 solution of the system are obtained. The results simplify the choice of suitable operating conditions for continuous culture systems. It also points out that the system is not chaotic according to the analysis on the existence of period-2 solution. The results and numerical simulations show that the chemostat system with state impulsive control tends to a stable state or a period solution.

## 1. Introduction

Bioreactor control is an active area of research on the continuous microorganism cultivation [1]. Modern control strategies require a mathematical model to check behavior of bioprocess and test its stability. Furthermore, they are necessary to optimize bioprocess and obtain maximal profit. According to different reactions and differential control technologies, many dynamic models concerning the culture of microorganism in the chemostat have been established [2–5]. However, there are a lot of factors affecting the growth and reproduction of the microorganisms in the process of bioreacts. For example, for some aerobic microbes, the dissolved oxygen content in the medium is a key factor to microbial growth. In order to maintain the dissolved oxygen content in an appropriate range, it is easy to prevent the process from the decrease of dissolved oxygen concentration (DOC) in the bioreactor medium below a low level by the monitoring of DOC oscillations. It is necessary because the low level of DOC decreases biomass yield and specific growth rate [6]. On the other hand, with the growth of the microorganisms, the effect of inhibition between the production and other negative effect will occur when the biomass concentration reaches a critical level. For the purpose of continuously culturing microorganisms and decreasing the inhibition effect, it is necessary to keep the biomass concentration lower than a critical level.

Many biological phenomena involve thresholds, bursting rhythm models in, for example, medicine, biology, pharmacokinetics, and frequency modulated systems, that exhibit impulsive effects. Thus, impulsive differential equations appear as a natural description of the observed evolution phenomena resulting from several real-world problems [7]. Many papers have investigated the systems with sudden perturbations which are involving in impulsive differential equations. Authors in [8–11] introduced some impulsive differential equations in population dynamics and obtained some interesting results. The research on the chemostat model with impulsive perturbations was studied by Sun and Chen [4]. Tang and Chen [12] introduced a Lotka-Volterra model with state-dependent impulsion and analyzed the existence and stability of positive period-1 solution. Jiang et al. [13] and Smith [14] have studied the state-dependent models with impulsive state control, where the model has a first integral, and obtained the complete expression of the period of the periodic solution. Jiang et al. [15] and Zeng et al. [16] have also discussed the models concerning integrated pest management (IPM). In the bioprocess, Guo and Chen [17, 18], Sun et al. [19–21], and Tian et al. [22, 23] have studied state-dependent models with impulsive state control by applying the Poincaré principle and Poincaré-Bendixson theorem of the impulsive differential equation, respectively.

This paper aims at proposing a mathematical model of a chemostat with variable yield and feedback control, described by the impulsive differential equation, and studying the dynamics of the bioreact. The rest of this paper is organized as follows. In Section 2 we introduce a chemostat model with Monod's growth rate and impulsive state feedback control for microorganisms of any biomass yield. In Section 3, we obtain the conditions for the existence of positive period-1 solution by the Poincaré-Bendixson theorem. We also point out that the proposed system is not chaotic according to the analysis of period-2 solution. In Section 4, we give the numerical simulations to verify the theoretical results, such as the existence of period-1 solutions, obtained in this paper and discuss the biological essence. Finally in Section 5 we present the conclusions.

## 2. Model Formulation

If the microorganisms' growth proceeds in accordance with Monod-type kinetics, that is, according to dependence

$$\mu(S) = \frac{\mu_{\max} S}{K_S + S}, \tag{2.1}$$

which is commonly used to model a large variety of biochemical reaction [24, 25], then the deterministic model of microbial growth in the chemostat is of the form [26]:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= D(S_F - S) - \frac{1}{Y_{x/S}}\frac{\mu_{\max}S}{K_S + S}x, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mu_{\max}S}{K_S + S}x - Dx, \\
S(0^+) &= S_0, \qquad x(0^+) = x_0,
\end{aligned} \tag{2.2}$$

where $x(t)$ denotes the biomass concentration, $S(t)$ the substrate concentration, $Y_{x/S}$ the biomass yield, $\mu_{\max}$ the maximum specific growth rate, $K_S$ the saturation constant, $S_F$ the concentration of the feed substrate, $D$ the dilution rate of the chemostat, $t$ denotes time, and $x_0$ and $S_0$ denote the initial biomass concentration and substrate concentration in the bioreactor medium; all parameters are positive.

Crooke et al. [27] showed that the biomass yield expression plays an important role for the generation of oscillatory behavior in continuous bioprocess models. Further, Crooke and Tanner [28] have proved that model (2.2) could not exhibit any periodic solution if the biomass yield in the model is constant. On the other hand, in reality, growth yields have not shown a constant pattern [19, 29, 30], so it is necessary to examine the bioprocess for the real biomass yield.

Not all energy produced in catabolic processes is used for cellular material synthesis, part of it (i.e., so-called maintenance energy) is used for maintaining life functions, for that reason dependence of biomass yield on $\mu$ is conditioned physiologically. An effect of existence of maintenance energy is biomass yield dependant on growth rate. One of the most important models of quantification of maintenance energy in microorganism growth balance is Pirt's model [31, 32]. According to those models, for very low substrate concentration the amount of energy obtained from the substrate is not sufficient for maintenance energy. In this case, the energy obtained from the substrate is fully assigned for maintenance energy. For substrate concentration greater than a certain minimum substrate concentration, there occurs a positive rate of biomass growth. If the amount of energy assigned for maintenance energy makes a considerable part of energy produced in catabolic processes, then can be assumed an almost linear dependence $Y_{x/S}$ on specific growth rate. For high concentration of substrate, a high specific growth rate is obtained. In such conditions biomass yield achieves the maximum value ($Y_{x/S_{\max}}$, where $Y_{x/S_{\max}} < 1$) and is practically constant, that is, it does not depend on the substrate concentration. When the amount of substrate essential to ensure maintenance energy is small, the described characteristics of cells can be approximated with sigmoid function, which has high flexibility in adapting to reality.

In this work, the sigmoid function, that is,

$$Y_{x/S} = \left(a + \exp(-bS)\right)^{-1}, \quad \text{(see Figure 1)} \tag{2.3}$$

which has high flexibility to fit any real biomass yield is used, where $a = 1/Y_{x/S_{\max}}$, $b$ is the cell sensitivity to the substrate under optimal growth conditions (optimal temperature, pH,

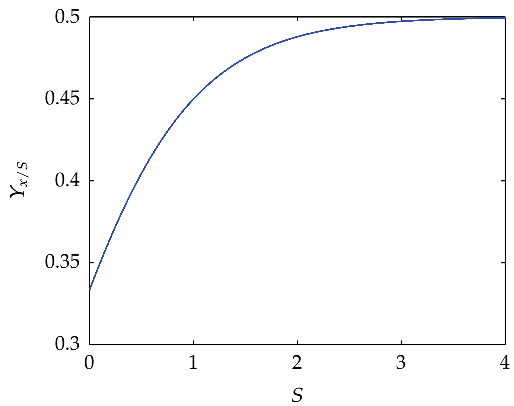



**Figure 1:** The biomass yield for $Y_{x/S_{\max}} = 0.5$ and the cell sensitivity to the substrate equals 1.5 (i.e., $a = 2$, $b = 1.5$).

DOC, and other); $a > 1$ and $b > 0$ are the biological constraints. For the selected known point $(S, Y_{x/S})$, the coefficient $b$ can be calculated as:

$$b = -\frac{\ln\left(Y_{x/S}^{-1} - Y_{x/S_{\max}}^{-1}\right)}{S}. \tag{2.4}$$

Then model (2.2) has the following form:

$$\begin{aligned}
\frac{dS}{dt} &= D(S_F - S) - (a + \exp(-bS))\frac{\mu_{\max} S}{K_S + S}x, \\
\frac{dx}{dt} &= \frac{\mu_{\max} S}{K_S + S}x - Dx, \\
S(0^+) &= S_0, \qquad x(0^+) = x_0.
\end{aligned} \tag{2.5}$$

In particular, when substrate concentration ($S$) is high, $Y_{x/S} = 1/a = Y_{x/S_{\max}}$, and the biomass yield is constant. When the cell sensitivity to the substrate is very high (i.e., $b > 100$), $Y_{x/S}$ is also practically constant for any substrate concentration different from zero. For example for *Saccharomyces cerevisiae* and glucose as the substrate, $b \approx 200$, what means that these microorganisms are very sensitive to glucose [30].

According to the design ideas of the bioreactor, the biomass concentration should not exceed a critical level. When the biomass concentration $x(t)$ in the bioreactor reaches the set level $x_{\text{set}}$ (where $0 < x_{\text{set}} \leq x_{\text{critical}}$ and $x_{\text{critical}}$ is the critical level of biomass concentration in the bioreactor medium), then part of the medium containing biomass and substrate is discharged from the bioreactor, and the next portion of medium of a given substrate concentration is inputted impulsively. Therefore, system (2.2) can be modified as follows by introducing the impulsive state feedback control:

$$\begin{aligned}
&\left.\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= D(S_F - S) - \left(a + \exp(-bS)\right)\frac{\mu_{\max}S}{K_S + S}x,\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mu_{\max}S}{K_S + S}x - Dx,
\end{aligned}\right\} \quad x < x_{\text{set}},\\
&\left.\begin{aligned}
\Delta S &= W_{f_1}(S_F - S) - W_{f_2}S,\\
\Delta x &= -\left(W_{f_1} + W_{f_2}\right)x,
\end{aligned}\right\} \quad x = x_{\text{set}},\\
&S(0^+) = S_0, \qquad x(0^+) = x_0,
\end{aligned} \tag{2.6}$$

where $0 \leq W_{f_1} < 1$ is the part of substrate of a given concentration which is inputted into the bioreactor in each biomass oscillation cycle, and $0 \leq W_{f_2} < 1$ is the part of "clear" medium which is inputted into the bioreactor in each biomass oscillation cycle. In addition, we make the following assumptions on $D$ and $W_{f_1}$: (1) $D < \mu_{\max}$ because all microorganisms are removed from the bioreactor when flow through the bioreactor is too fast, that is, when $D \geq \mu_{\max}$; (2) $W_{f_1}$ is equal to $D$ in value, that is, $|W_{f_1}| = |D|$.

We mainly discuss the dynamics, that is, existence of periodic solution of the model (2.6) in the region $\Omega = \{(S, x) \mid S > 0,\ x > 0\}$ according to the existence criteria of periodic solution of the general impulsive autonomous system in [16]. For convenience, we introduce a new notation $W = W_{f_1} + W_{f_2}$, which will be used in the following discussion.

## 3. The Existence of Positive Periodic Solution of System (2.6)

Before discussing the dynamics of system (2.6), we first consider the qualitative characteristic of system (2.5). Clearly, system (2.5) has a boundary equilibrium $(S_F, 0)$ and a positive equilibrium $(S^\star, x^\star)$ if $K_S < (\mu_{\max} - D)S_F/D$ where

$$S^\star = \frac{DK_S}{\mu_{\max} - D} < S_F, \qquad x^\star = (S_F - S^\star)\left(a + \exp(-bS^\star)\right)^{-1} > 0. \tag{3.1}$$

The Jacobian of system (2.5) at $(S_F, 0)$ is

$$J_{(S_F,0)} = \begin{bmatrix} -D & -\left(a + \exp(-bS_F)\right)\dfrac{\mu_{\max}S_F}{K_S + S_F} \\ 0 & \dfrac{\mu_{\max}S_F}{K_S + S_F} - D \end{bmatrix}, \tag{3.2}$$

then the equilibrium $(S_F, 0)$ is stable if $K_S \geq (\mu_{\max} - D)S_F/D$. In this case, we can conclude that every solution of (2.5) tends to a stable equilibrium $(S_F, 0)$ if $K_S \geq (\mu_{\max} - D)S_F/D$. Then microorganisms are not cultured successfully.

For the case $K_S < (\mu_{\max} - D)S_F/D$, the equilibrium $(S_F, 0)$ is a saddle point and $S^\star > 0$, $x^\star > 0$. Since

$$J_{(S^\star,x^\star)} = \begin{bmatrix} -D - \mu_{\max}\Gamma(S^\star)x^\star & -\dfrac{D(S_F - S^\star)}{x^\star} \\ x^\star \dfrac{\mu_{\max}K_S}{(K_S + S^\star)^2} & 0 \end{bmatrix}, \tag{3.3}$$

where

$$\Gamma(S^\star) = \frac{D(S_F - S^\star)}{\mu_{\max}S^\star x^\star}\frac{K_S}{K_S + S^\star} - b\exp(-bS^\star)\frac{S^\star}{K_S + S^\star}. \tag{3.4}$$

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0, \tag{3.5}$$

where

$$\begin{gathered} p = D\left[\frac{(S^\star)^2 + K_S S_F}{S^\star(K_S + S^\star)} - b\exp(-bS^\star)(S_F - S^\star)\left(a + \exp(-bS^\star)\right)^{-1}\right], \\ q = \frac{\mu_{\max}K_S D(S_F - S^\star)}{(K_S + S^\star)^2} > 0. \end{gathered} \tag{3.6}$$

Denote that

$$\kappa(b) = \left[b\frac{(S_F - S^\star)S^\star(K_S + S^\star)}{(S^\star)^2 + K_S S_F} - 1\right]\exp(-bS^\star). \tag{3.7}$$

If $a > \kappa(b)$, then $p > 0$ and the equilibrium $(S^\star, x^\star)$ is asymptotically stable; else if $a < \kappa(b)$, then $p < 0$ and $(S^\star, x^\star)$ is unstable; else $a = \kappa(b)$, then $p = 0$ and $(S^\star, x^\star)$ is a center. Since $\dot{S} < 0$ for $S > S_F$, then any solution starting from the region $\{(S, x) \mid S \geq S_F\}$ will enter into the region $\{(S, x) \mid S < S_F\}$ eventually, so in the following discussion we assume that $S_0 = S(0) < S_F$.

For the case where $(S^\star, x^\star)$ is stable, all solutions of system (2.6) starting from the region $\{(S, x) \mid 0 < S < S_F, 0 < x < x^\star\}$ will tend to the equilibrium $(S^\star, x^\star)$ and no impulse will occur if $x_{\text{set}} > x^\star$. So in this case, we mainly focus our attentions on the discussion of the following case.

*Assumption 3.1.* Consider that $S^\star > 0$, $0 < x_{\text{set}} < x^\star$, and $(S_0, x_0) \in \Omega_1 = \{(S, x) \mid 0 < x < x_{\text{set}},\ 0 < S < S_F - (a + \exp(-bS_F))x\}$.

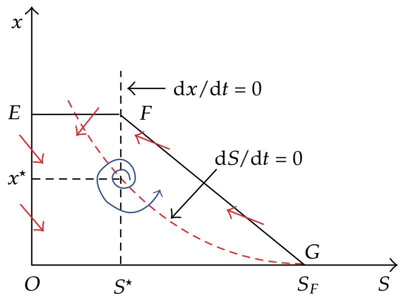



**Figure 2:** The case for $(S^\star, x^\star)$ is unstable and system (2.5) has a limit cycle.

For the case where $(S^\star, x^\star)$ is unstable, it can be shown that there exists one limit cycle in $\Omega$ with an outer boundary $OEFG$ of the Bendixson annular region (see Figure 2), where $\overline{FG}$ is a segment on the line $l_1$ passing the point $(S_F, 0)$ with the slope $-(a + e^{-bS_F})$,

$$l_1 : x + \frac{S}{a + \exp(-bS_F)} - \frac{S_F}{a + \exp(-bS_F)} = 0 \tag{3.8}$$

for that the derivative of $l_1$ along with system (2.5) is

$$\begin{aligned}\frac{\mathrm{d}l_1}{\mathrm{d}t} &= \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(2.5)} + \frac{1}{a + \exp(-bS_F)} \left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{(2.5)} \\ &= x\left(\frac{\mu_{\max} S}{K_S + S} - D\right) + \frac{\left[D(S_F - S) - \left(a + \exp(-bS)\right)\left(\mu_{\max} S/(K_S + S)\right)x\right]}{a + \exp(-bS_F)} \\ &= \frac{S_F - S}{a + \exp(-bS_F)} \frac{\mu_{\max} S}{K_S + S}\left[1 - \frac{a + \exp(-bS)}{a + \exp(-bS_F)}\right] < 0.\end{aligned} \tag{3.9}$$

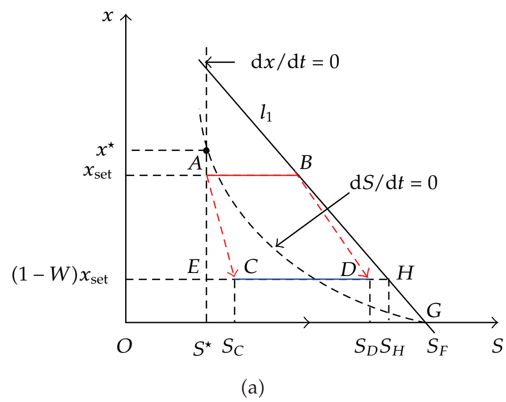
All trajectories starting from the region $\Omega$ tend to the limit cycle. In this case, we mainly focus our attentions on the discussion of the following case.

*Assumption 3.2.* Consider that $S^\star > 0$, $x^\star > 0$ and $(S_0, x_0) \in \Omega_2 = \{(S, x) \mid 0 < x < x^\star,\ 0 < S < S_F - (a + \exp(-bS_F))x\}$.

### *3.1. Existence of Period-1 Solution for* $x_{\text{set}} < x^\star$

In order to apply the existence criteria of period-1 solution, that is, Theorem A.5, we need to construct a closed region $\Omega_P$ such that all the solutions of system (2.6) enter into and retain it. The ideas will be illustrated as follows by using Figure 3.

As shown in Figure 3, the line $x = x_{\text{set}}$ intersects the isoclinal line $\dot{x} = 0$ at the point $A(S^\star, x_{\text{set}})$ and intersects the line $l_1$ at the point $B(S_B, x_{\text{set}})$. The line $x = (1 - W)x_{\text{set}}$ intersects the line $\dot{x} = 0$ at the point $E(S^\star, (1 - W)x_{\text{set}})$ and intersects the line $l_1$ at point $H(S_H, (1 - W)x_{\text{set}})$. Denote $C = C(S_C, (1 - W)x_{\text{set}})$ and $D = D(S_D, (1 - W)x_{\text{set}})$, where $S_C = (1 - W)S^\star + DS_F$ and $S_D = (1 - W)S_B + DS_F$. The impulsive set $M \subseteq \overline{AB} = \{(S, x) \mid S^\star \le S \le S_B,\ x = x_{\text{set}}\}$, and $N = I(M) \subseteq CD = \{(S, x) \mid S_C \le S \le S_D,\ x = (1 - W)x_{\text{set}}\}$.



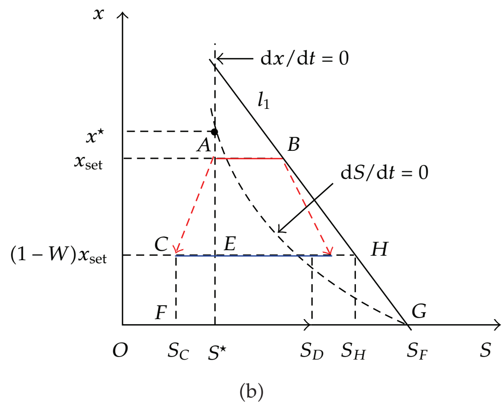




**Figure 3:** Illustration of system (2.6) (a) $S_C \geq S^\star$; (b) $S_C < S^\star$.

Whether the equilibrium $(S^\star, x^\star)$ is stable or not, the following theorem holds.

**Theorem 3.3.** *System* (2.6) *has a period-1 solution under Assumption 3.1.*

*Proof.* Firstly, it can be easily shown that all trajectories of system (2.6) starting from the region $\Omega_1$ must intersect with the segment $\overline{AB}$ and then jump to the segment $\overline{CD}$. Next, we construct the closed region $\Omega_P \subset \Omega_1$ such that all the solutions of system (2.6) starting from $\Omega_1$ enter into and retain in $\Omega_P$. According to the values of $S_C$ and $S^\star$, we will discuss the following two cases.

*Case 1.* Consider that $S_C \geq S^\star$, that is, $W \leq (\mu_{max} - D)S_F/K_S$;

*Case 2.* Consider that $S_C < S^\star$, that is, $W > (\mu_{max} - D)S_F/K_S$.

We first discuss Case 1, where $S_C \geq S^\star$ and the illustration is shown in Figure 3(a).

By (3.8), we have $S_B = S_F - (a + \exp(-bS_F))x_{set}$ and $S_H = S_F - (a + \exp(-bS_F))\,(1 - W)x_{set}$. So we have

$$S_D = (1-W)S_B + DS_F = (1-W)\left(S_F - \left(a + \exp(-bS_F)\right)x_{set}\right) + DS_F < S_H. \tag{3.10}$$

From the qualitative characteristic of system (2.6), we know that $(dl_1/dt)|_{(2.6)} < 0$ and $\dot{x}|_{\overline{EH}} > 0$. Besides, $\dot{S} > 0$ for $S = S^\star$. Therefore, we have found a closed region $\Omega_P$, the boundary of which consists of $\overline{AE}, \overline{EH}, \overline{HB}$, and $\overline{BA}$.

For Case 2, it follows from system (2.6) that $x(t) = 0$ and $S(t) = S_F - (S_F - S_0)e^{-Dt}$ for $t \in (0, +\infty)$. Especially, when $S_0 = S_C$, $\overrightarrow{FG}$ is the semitrivial solution of system (2.6). Similar to the discussions of Case 1, we can obtain the closed region $\Omega_P$, the boundary of which consists of $\overline{AE}, \overline{EC}, \overline{CF}, \overline{FG}, \overline{GB}$, and $\overline{BA}$, which can be seen in Figure 3(b).

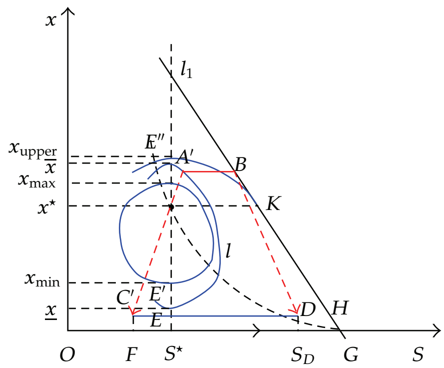



**Figure 4:** Illustration of system (2.6) for the case where $x_{\max} < x_{\text{set}} < x_{\text{upper}}$.

Summarizing Cases 1 and 2 and following from Theorem A.5 that system (2.6) has a period-1 solution. □

### 3.2. Existence of Period-1 Solution for $x_{\text{set}} \geq x^\star$

For the case where $(S^\star, x^\star)$ is unstable, let $\Gamma$ be the limit cycle, $x = x_{\text{upper}}$ be the line tangent with the trajectory starting from the initial point $K(S_K, x^\star)$ and $x = x_{\max}(x_{\max} > x^\star)$, and $x = x_{\min}$ $(x_{\min} < x^\star)$ be the lines tangent with $\Gamma$, where $S_K = S_F - (a + \exp(-bS_F))x^\star$. Then we have the following theorem.

**Theorem 3.4.** *System* (2.6) *has no period-1 solution under Assumption 3.2 and the trajectory tends to the limit cycle if $a < \kappa(b)$ and $x_{set} \geq x_{upper}$, where $\kappa(b)$ is determined by* (3.7).

*Proof.* It can be easily shown that all trajectories starting from the region $\Omega_2$ will not intersect the line $x = x_{\text{set}}$ and tend to the limit cycle eventually. □

For the case where $x_{\max} < x_{\text{set}} < x_{\text{upper}}$, we have the following theorem.

**Theorem 3.5.** *There exists $0 < \underline{x} < x_{\min}$ and $x_{\max} < \overline{x} < x_{upper}$ such that, for $x_{\max} \leq x_{set} \leq \overline{x}$, system* (2.6) *has a period-1 solution for $W > 1 - \underline{x}/x_{set}$ under Assumption 3.2 if $a < \kappa(b)$, where $\kappa(b)$ is determined by* (3.7).

*Proof.* As stated earlier, if $a < \kappa(b)$, then $(S^\star, x^\star)$ is unstable and there exists one limit cycle $\Gamma$. Since $\Gamma$ does not intersect the line $x = 0$, then there exists $0 < \underline{x} < x_{\min}$ such that the trajectory $l$ starting from the initial point $E'(S^\star, \underline{x})$ intersects the line $S = S^\star$ at the point $E''(S^\star, \overline{x})$, where $x_{\max} < \overline{x} < x_{\text{upper}}$. For $x_{\max} \leq x_{\text{set}} \leq \overline{x}$, let $A'$ be the point of the trajectory $l$ intersecting with the line $x = x_{\text{set}}$ and $C'$ be the point of $A'$ after impulsive function $I$. If $W > 1 - \underline{x}/x_{\text{set}}$, the similar to the discussion of Theorem 3.3, we can obtain the closed region $\Omega_P$, the boundary of which consists of $\widehat{A'E'}, \overline{E'E}, \overline{EC'}, \overline{C'F}, \overline{FG}, \overline{GB}$, and $\overline{BA'}$ for $S_{C'} < S^\star$, which can be seen in Figure 4, where $\widehat{A'E'}$ is the part of the trajectory between $A'$ and $E'$. Then by Theorem A.5 system (2.6) has a period-1 solution. Therefore, the trajectory starting from the region $\Omega_2$ either tends to the limit cycle or the period-1 solution. □

For the case where $x^\star \leq x_{\text{set}} \leq x_{\max}$, we have the following theorem.

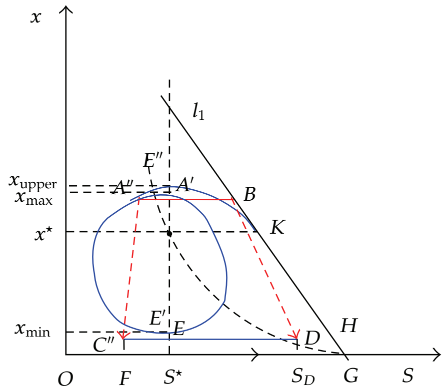



**Figure 5:** Illustration of system (2.6) for the case where $x^\star \le x_{\text{set}} \le x_{\max}$.

**Theorem 3.6.** *System* (2.6) *has a period-1 solution under Assumption 3.2 if* $a < \kappa(b)$, $x^\star \le x_{set} \le x_{\max}$ *and* $W > 1 - \underline{x}/x_{set}$, *where* $\kappa(b)$ *is determined by* (3.7).

*Proof.* Let $E'$ and $E''$ be the points of the limit cycle intersecting with the line $S = S^\star$. Let $A''$ and $A'$ be the points of the limit cycle intersecting with the line $x = x_{\text{set}}$ and $C''$ be the point of $A''$ after impulsive function $I$. It can be easily shown that all trajectories starting from the segment $\overline{C''D}$ will intersect the segment $\overline{A'B}$. Since the equilibrium $(S^\star, x^\star)$ is unstable, then all trajectories starting from the region $\Omega_2$ will intersect the segment $\overline{A''B}$ and jump to the segment $\overline{C''D}$ after at most one impulse. Similar to the discussion of Theorem 3.5, we can obtain the closed region $\Omega_P$, the boundary of which consists of $\widehat{A'E'}, \overline{E'E}, \overline{EC''}, \overline{C''F}, \overline{FG}, \overline{GB}$, and $\overline{BA'}$ for $S_{C''} < S^\star$, which can be seen in Figure 5, where $\widehat{A'E'}$ is the part of the limit cycle between $A'$ and $E'$. Then by Theorem A.5 system (2.6) has a period-1 solution. Therefore, all trajectories starting from the region $\Omega_2$ tend to the period-1 solution. □

### *3.3. Existence of Period-2 Solution*

In the last subsection, we have analyzed the existence of period-1 solution of system (2.6). Next, we will discuss existence of the period-2 solution.

Suppose that $(\widehat{S}, \widehat{x})$ is a period-1 solution of system (2.6). Then we have $(\widehat{S}_0, \widehat{x}_0) \in N \subseteq \overline{CD}$ and $(\widehat{S}_1, \widehat{x}_1) \in M \subseteq \overline{AB}$, where $\widehat{S}_0 = (1-W)\widehat{S}_1 + DS_F$. Let $(S, x)$ be an arbitrary solution of system (2.6). Denote the first intersection point of the trajectory to the impulsive set $M$ by $(S_1, x_{\text{set}})$, and the corresponding consecutive points are $(S_2, x_{\text{set}})$, $(S_3, x_{\text{set}}), \ldots$, respectively. Consequently, under the effect of impulsive function $I$, the corresponding points after pulse are $(S_1^+, (1-W)x_{\text{set}})$, $(S_2^+, (1-W)x_{\text{set}})$, $(S_3^+, (1-W)x_{\text{set}})$, .... By the qualitative analysis of system (2.6), we know that $\dot{x} > 0$ for $S > S^\star$ and $\dot{x} < 0$ for $S < S^\star$. We will consider the following two cases according to the values $\widehat{S}_0$ and $S^\star$.

*Case 1* ($\widehat{S}_0 > S^\star$). In this case, the period-1 solution lies in the region $\{(S, x) \mid S^\star < S < S_F, (1-W)x_{\text{set}} \le x \le x_{\text{set}}\}$. The trajectory of the solution $(S, x)$ jumps to the point $(S_1^+, (1-W)x_{\text{set}})$ from the point $(S_1, x_{\text{set}})$. Without loss of generality, we suppose that $S_1 > \widehat{S}_1$. By the dynamics of system (2.6), we have the following two sequences according to $(S_2, x_{\text{set}})$ lying on the right

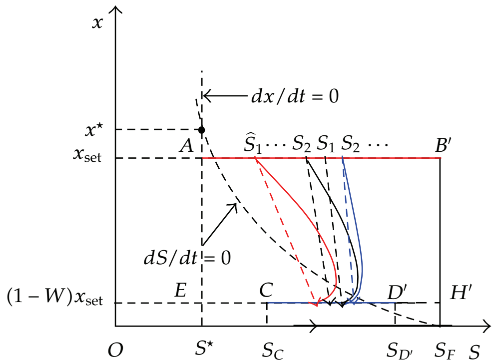



**Figure 6:** The trajectory of the solution $(S, x)$ for the case where $\widehat{S}_0 > S^\star$.

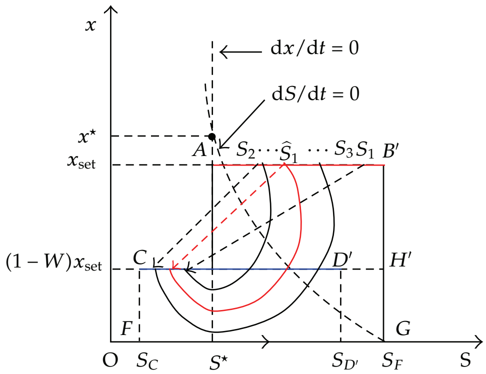



**Figure 7:** The trajectory of the solution $(S, x)$ under sequence (c).

or left of $(S_1, x_{\text{set}})$, as shown in Figure 6:

(a) $\widehat{S}_1 \leq S_1 \leq S_2 \leq S_3 \leq \cdots$

or

(b) $S_1 \geq S_2 \geq S_3 \geq \cdots \geq \widehat{S}_1$.

It is known from the sequence (b) that the trajectory tends to the period-1 solution. Since the sequences are monotone, it follows by Definition A.4 that period-2 solution does not exist in this case.

*Case 2* ($\widehat{S}_0 \leq S^\star$). Similarly, we assume that $S_1 > \widehat{S}_1$. By the dynamics of system (2.6), we have the following property: $(S_{2k}, x_{\text{set}})$ $(k = 1, 2, \ldots)$ lie on the left of $(\widehat{S}_1, x_{\text{set}})$ and $(S_{2k-1}, x_{\text{set}})$ $(k = 1, 2, \ldots)$ lie on the right of $(\widehat{S}_1, x_{\text{set}})$. For example, if $S_1 < (S^\star - DS_F)/(1 - W)$ and $(S_3, x_{\text{set}})$ lies on the left or right of $(S_1, x_{\text{set}})$, the we have the following sequence as shown in Figure 7:

(c) $S_2 \leq S_4 \leq S_6 \leq \cdots \leq \widehat{S}_1 \leq \cdots \leq S_5 \leq S_3 \leq S_1$.

From the above sequence, we know that it is possible that there exists period-2 solution. For example, when $S_1 = S_3$, the period-2 solution will occur.

From the proof of Proposition 3.3 in [5], it can also be obtained that there is no order $k$ solution ($k \geq 3$) in system (2.6) and then the system is not chaotic.

## 4. Simulations and Discussion

We have analyzed theoretically the feedback control for microorganisms of any biomass yield and Monod-type kinetics. The results are new and significant, which not only provide the possibility of a check of system dynamic property, that is, the existence of period-1 solution for different microorganisms and several parameters, but also provide a possibility of making simulation of real process according to the mathematical models determined in the article. To verify the received results, the numerical simulations of system (2.6) are shown. Because of the large practical importance, the case where $(S^\star, x^\star)$ is stable is presented and discussed. Moreover, galactose as the substrate and the microorganisms with $a = 2$ (i.e., $Y_{x/S_{\max}} = 1/2 = 0.5[g/g]$), $b = 1.5$, $\mu_{\max} = 0.3[1/h]$, and $K_S = 2[g/l]$ are used for the demonstration of system behavior. In order to ensure the existence of positive equilibrium, we set $S_F = 6[g/l]$, and $D = 0.1[1/h]$. Then we have $(\mu_{\max} - D)S_F/K_S = 0.6$, $S^\star = 1[g/l]$ and $\kappa(b) \simeq 0.16$. Because $\kappa(b) < a$, $(S^\star, x^\star) = (1, 2.25)$ is a stable focus. Next, we check and show the influence of $x_{\text{set}}$ and $W_{f_2}$ changes on the existence of period-1 solution.

Firstly, we set $x_{\text{set}} = 2[g/l]$. The time series and phase diagram for system (2.6) with $W_{f_2} = 0.1$ is presented in Figure 8, from which we can see that the trajectory tends to be periodic.

Secondly, we change the value of $W_{f_2}$. The phase diagram for system (2.6) with $W_{f_2} = 0.2, 0.4, 0.6$ is displayed in Figure 9. From Figure 9, we can see that the existence of period-1 solution does not depend on the value of $W_f$ in this case, but the value of $W_{f_2}$ affects the position and tendency of the period-1 solution.

Thirdly, we set $x_{\text{set}} = 2.5[g/l]$. The time series and phase diagram is presented in Figure 10. It can be easily seen from Figure 10 that no impulse occurs when $x_{\text{set}} > x^\star$ and the trajectory tends to the stable node $(1.5, 1.2)$.

Therefore, the numerical simulations are consistent with the theorems obtained and presented in Section 3. A potential application area of the chemostat with feedback control is the commercial and industrial biomass production. In such a chemostat, the microorganisms always keep the suitable growth rate and the biomass concentration should be controlled to a given set level for which the dissolved oxygen concentration is considered as optimal. Therefore, in order to eliminate the negative effects such as the insufficiency of dissolved oxygen and decrease of the inhibition effect, one can use the chemostat models with impulsive state feedback control.

## 5. Conclusions

The article established the mathematical model of a chemostat with variable biomass yield and feedback control in maintaining the biomass concentration in a desired range. The flexible sigmoid function was proposed in describing the dependence of the biomass yield on the substrate concentration. It was shown that the stability of the bioprocess (i.e., the existence of the positive period-1 solution) depends on the biomass yield and the microorganisms growth rate. It was also shown that for the system there may exist a period-2 solution, but not having period-$k$ ($k \geq 3$), then it is not chaos. The existence of period-1 solution indicates that, in the biomass production, a stable output of the biomass can be achieved. The key to the

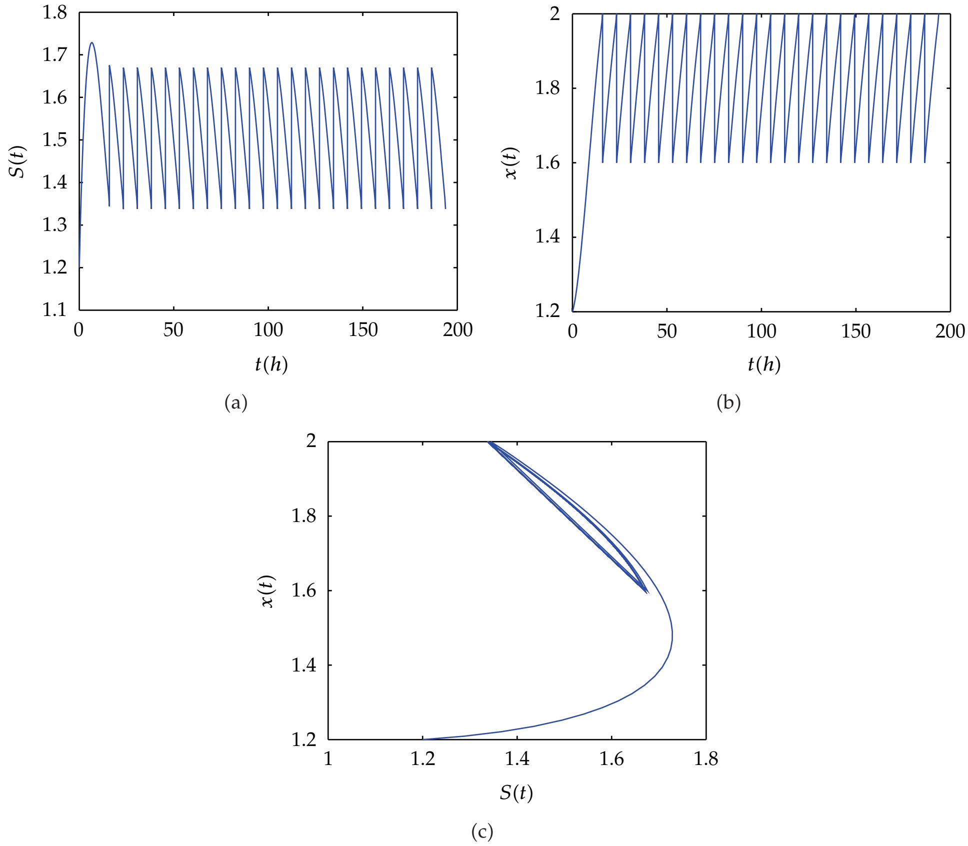



**Figure 8:** The time series and phase diagram for system (2.6) starting from initial point $(S_0, x_0) = (1.2, 1.2)$ with $a = 2$, $b = 1.5$, $x_{set} = 2[g/l]$, and $W_{f_2} = 0.1$.

production is to given a suitable feedback state (i.e., $x_{set}$) and the control parameter (i.e., $D$ and $W_{f_2}$) according to the practice. In addition, the appropriate initial biomass and substrate concentration should also be considered such that the culture achieves the periodic state as soon as possible. The results also provide a possibility of making simulation of real process according to the mathematical models and the parameters determined in this paper.

## Appendix

Before introducing the existence criteria, we give the following definitions to understand the results.

*Definition A.1* (Lakshmikantham et al. [7]). A triple $(X, \pi, \mathbb{R}^+)$ is said to be a semidynamical system if $X$ is a metric space, $\mathbb{R}^+$ is the set of all nonnegative reals, and $\pi: X \times \mathbb{R}^+ \rightarrow X$ is a continuous function such that

(i) $\pi(x, 0) = x$ for all $x \in X$;

(ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{R}^+$.

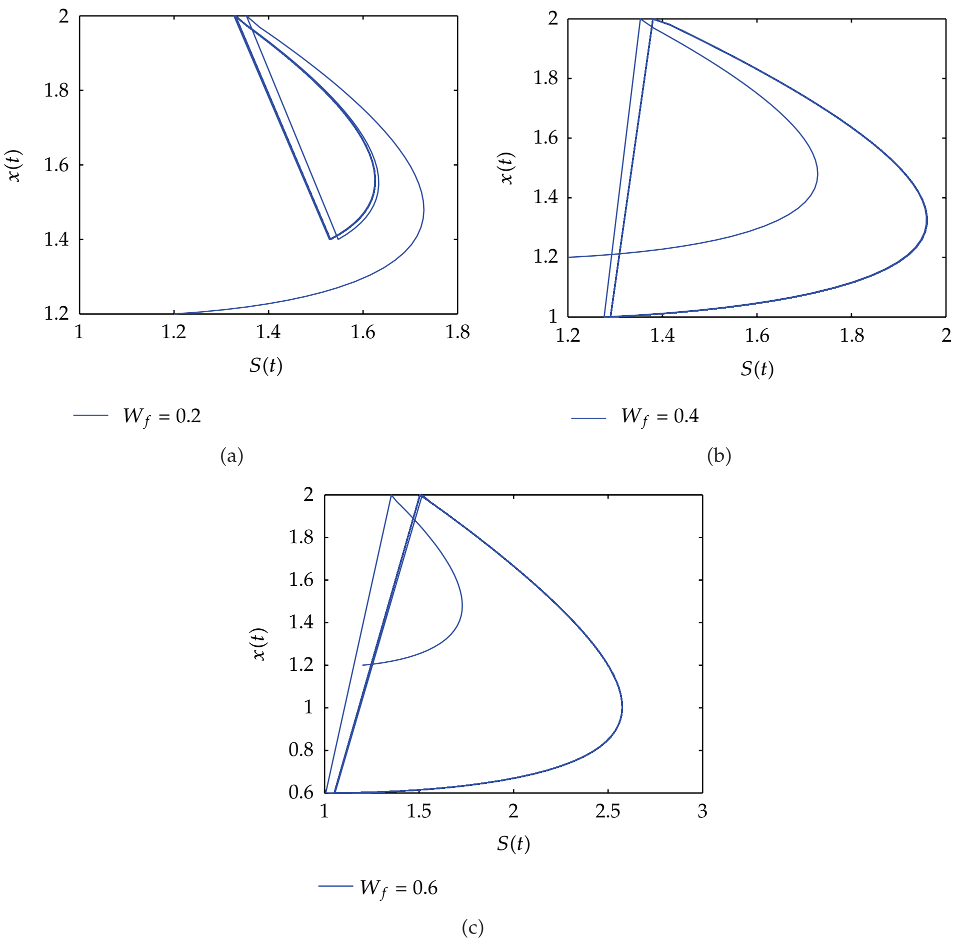



**Figure 9:** The phase diagram for system (2.6) starting from initial point $(S_0, x_0) = (1.2, 1.2)$ with $a = 2$, $b = 1.5$, $x_{\text{set}} = 2[g/l]$, and $W_{f_2} = 0.2, 0.4, 0.6$.

We denote sometimes a semidynamical system $(X, \pi, \mathbb{R}^+)$ by $(X, \pi)$. For any $x \in X$, the function $\pi_x\colon \mathbb{R}^+ \to X$ defined by $\pi_x(t) = \pi(x, t)$ is continuous and we call $\pi_x$ the trajectory of $x$. The set

$$C^+(x) = \{\pi(x,t) \mid t \in \mathbb{R}^+\} \tag{A.1}$$

is called the positive orbit of $x$. For any subset $M$ of $X$, we let

$$M^+(x) = C^+(x) \cap M - \{x\}, \qquad M^- = G(x) \cap M - \{x\}, \tag{A.2}$$

where

$$G(x) = \cup\{G(x,t) \mid t \in \mathbb{R}^+\}, \qquad G(x,t) = \{y \mid \pi(y,t) = x\} \tag{A.3}$$

is the attainable set of $x$ at $t \in \mathbb{R}^+$. Finally, we set $M(x) = M^+(x) \cup M^-(x)$.

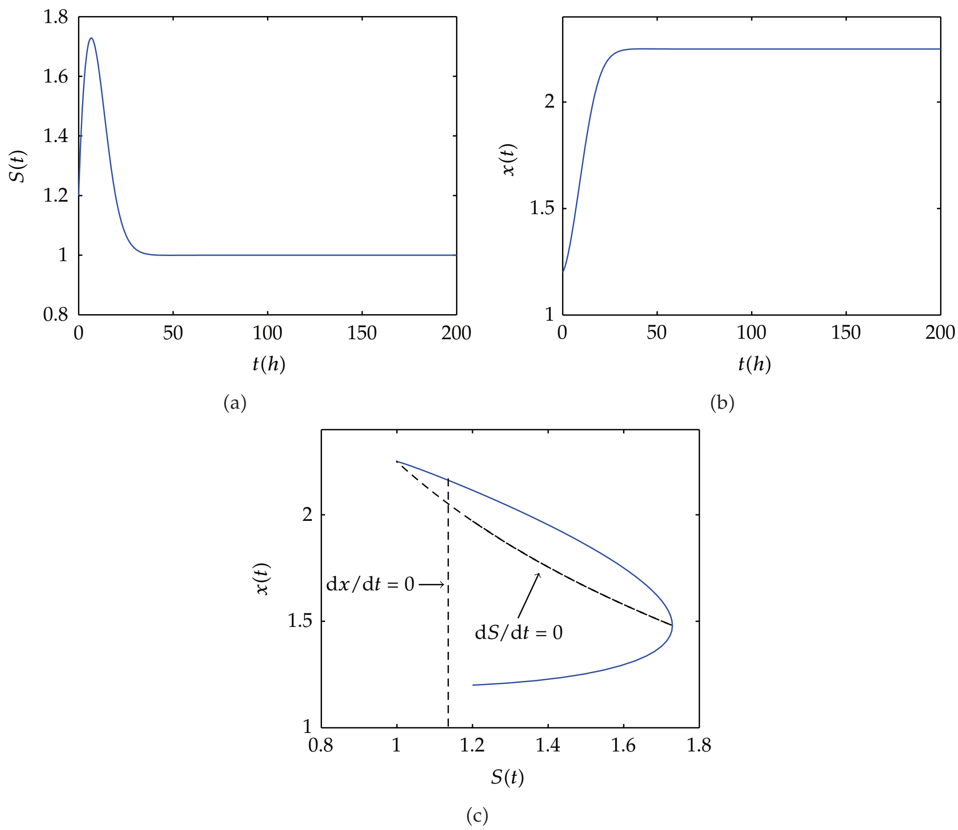



**Figure 10:** The time series and phase diagram for system (2.6) starting from initial point $(S_0, x_0) = (1.2, 1.2)$ with $a = 2$, $b = 1.5$, and $x_{\text{set}} = 2.5[g/l]$.

*Definition A.2* (Lakshmikantham et al. [7]). An impulsive semidynamical system $(X, \pi; M, I)$ consists of a semidynamical system $(X, \pi)$ together with a nonempty closed subset $M$ of $X$ and a continuous function $I : M \to X$ such that the following properties hold:

(i) no point $x \in X$ is a limit point of $M(x)$;

(ii) $[t \mid G(x,t) \cap M \neq \varnothing]$ is a closed subset of $\mathbb{R}^+$.

Throughout this paper we will write $N = I(M) = \{y \in X \mid y = I(x),\ x \in M\}$ and for any $x \in X$, $I(x) = x^+$. We call $M$ the set of impulses, $I$ the impulsive function.

We define a function $\Phi : X \to \mathbb{R}^+ \cup \{\infty\}$ as following:

$$\Phi(x) = \begin{cases} \infty, & \text{if } M^+(x) = \varnothing, \\ s, & \text{if } \pi(x,t) \notin M \quad \text{for } 0 < t < s,\ \pi(x,s) \in M, \end{cases} \tag{A.4}$$

here we call $s$ the time without impulse of $x$, that is to say $s$ is the first time when $\pi(x,0)$ hits $M$.

*Definition A.3* (Lakshmikantham et al. [7]). Let $(X, \pi; M, I)$ be an impulsive semidynamical system and let $x \in X$ and $x \notin M$. The trajectory of $x$ is a function $\tilde{\pi}_x$ defined on subset $[0, s)$ of $\mathbb{R}^+$ ($s$ may be $\infty$) to $X$ inductively as following:

$$\tilde{\pi}_x(t) = \pi\left(x_{n-1}^+, t\right), \qquad \tau_{n-1} \leq t < \tau_n, \tag{A.5}$$

where $\{x_n\}$ satisfy $\pi(x_{n-1}^+, \Phi(x_{n-1}^+)) = x_n$. $\tau_n$ is the sequence of time of impulses relative to $\{x_n\}$, $\tau_n = \sum_{k=0}^{n-1} \Phi(x_k^+)$.

*Definition A.4* (Lakshmikantham et al. [7]). A trajectory $\tilde{\pi}_x$ is said to be periodic of period $\tau$ and order $k$ if there exist positive integers $m \geq 1$ and $k \geq 1$ such that $k$ is the smallest integer for which $x_m^+ = x_{m+k}^+$ and $\tau = \sum_{i=m}^{M+k-1} \Phi(x_i^+)$.

For the following general autonomous impulsive differential equations:

$$\begin{aligned} \dot{S} &= P(S, x), \\ \dot{x} &= R(S, x), \end{aligned} \quad (S, x) \notin M,$$
$$\begin{aligned} \Delta S &= I_1(S, x), \\ \Delta x &= I_2(S, x), \end{aligned} \quad (S, x) \in M. \tag{A.6}$$

Here $(S, x) \in \mathbb{R}^2$, and $P, R, I_1$, and $I_2$ are all functions mapping $\mathbb{R}^2$ into $\mathbb{R}$, $M \subset \mathbb{R}^2$ is the set of impulse, and we assume that

($H1$) $P(S, x)$ and $R(S, x)$ are all continuous with respect to $(S, x) \in \mathbb{R}^2$;

($H2$) $M \subset \mathbb{R}^2$ is a line, $I_1(S, x)$ and $I_2(S, x)$ are linear functions of $S$ and $x$.

For each point $A(S, x) \in M$, we define $I : \mathbb{R}^2 \to \mathbb{R}^2$:

$$I(A) = A^+ = (S^+, x^+) \in R^2, \qquad S^+ = S + I_1(S, x), \qquad x^+ = x + I_2(S, x). \tag{A.7}$$

Obviously, $N = I(M)$ is also a line of $\mathbb{R}^2$ or a subset of a line, and we assume throughout this section that $N \cap M = \emptyset$. From Definition A.2 we know system (A.6) is an impulsive semidynamical system. The following theorem gives the conditions on which system (A.6) has a periodic solution of order one defined by Definition A.4.

**Theorem A.5** (Analogue of Zeng Theorem [16]). *If system* (A.6) *satisfies assumptions H1 and H2, and, there exist a bounded closed simply connected region* $\Omega$ *which has the following properties:*

(i) *there is no singularity in it and the boundary* $\partial\Omega$ *of* $\Omega$ *is composed of two parts:* $L_1$ *and* $L_2$*;*

(ii) $L_1 = \Omega \cap M$ *cannot be tangent with trajectories of system* (A.6) *except at endpoints and* $I(L_1) \subset \Omega$*;*

(iii) *trajectories with initial point in* $L_2$ *will enter into interior of* $\Omega$*,*

*then there must exist a period-1 solution of system* (A.6) *in region* $\Omega$*.*

## Acknowledgments

This paper is supported by the National Natural Science Foundation of China (no. 60774049, Major Research Plan no. 90818025), Specialized Research Fund for the Doctoral Program of Higher Education of China (no. 20090041110003), and the Fundamental Research Funds for the Central Universities.

## References

[1] Y. Zhao and S. Skogestad, "Comparison of various control configurations for continuous bioreactors," *Industrial and Engineering Chemistry Research*, vol. 36, no. 3, pp. 697–705, 1997.
[2] Y. Kuang, "Limit cycles in a chemostat-related model," *SIAM Journal on Applied Mathematics*, vol. 49, no. 6, pp. 1759–1767, 1989.
[3] D.-F. Fu, W.-B. Ma, and S.-G. Ruan, "Qualitative analysis of a chemostat model with inhibitory exponential substrate uptake," *Chaos, Solitons & Fractals*, vol. 23, no. 3, pp. 873–886, 2005.
[4] S.-L. Sun and L.-S. Chen, "Dynamic behaviors of Monod type chemostat model with impulsive perturbation on the nutrient concentration," *Journal of Mathematical Chemistry*, vol. 42, no. 4, pp. 837–847, 2007.
[5] P. De Leenheer and H. Smith, "Feedback control for chemostat models," *Journal of Mathematical Biology*, vol. 46, no. 1, pp. 48–70, 2003.
[6] A. Kasperski, "Modelling of cells bioenergetics," *Acta Biotheoretica*, vol. 56, no. 3, pp. 233–247, 2008.
[7] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1989.
[8] S. Sun and L. Chen, "Permanence and complexity of the eco-epidemiological model with impulsive perturbation," *International Journal of Biomathematics*, vol. 1, no. 2, pp. 121–132, 2008.
[9] X.-Z. Meng and L.-S. Chen, "Permanence and global stability in an impulsive Lotka-Volterra $n$-species competitive system with both discrete delays and continuous delays," *International Journal of Biomathematics*, vol. 1, no. 2, pp. 179–196, 2008.
[10] S. Tang and L. Chen, "Density-dependent birth rate, birth pulses and their population dynamic consequences," *Journal of Mathematical Biology*, vol. 44, no. 2, pp. 185–199, 2002.
[11] S.-Y. Tang and L.-S. Chen, "The effect of seasonal harvesting on stage-structured population models," *Journal of Mathematical Biology*, vol. 48, no. 4, pp. 357–374, 2004.
[12] S.-Y. Tang and L.-S. Chen, "Modelling and analysis of integrated pest management strategy," *Discrete and Continuous Dynamical Systems Series B*, vol. 4, no. 3, pp. 759–768, 2004.
[13] G.-R. Jiang, Q.-S. Lu, and L.-N. Qian, "Chaos and its control in an impulsive differential system," *Chaos, Solitons & Fractals*, vol. 34, no. 4, pp. 1135–1147, 2007.
[14] R. Smith, *Impulsive differential equations with applications to selfcycling fermentation*, Ph.D. thesis of Philosophy, McMaster University, Ontario, Canada, 2001.
[15] G.-R. Jiang, Q.-S. Lu, and L.-N. Qian, "Complex dynamics of a Holling type II prey-predator system with state feedback control," *Chaos, Solitons & Fractals*, vol. 31, no. 2, pp. 448–461, 2007.
[16] G.-Z. Zeng, L.-S. Chen, and L.-H. Sun, "Existence of periodic solution of order one of planar impulsive autonomous system," *Journal of Computational and Applied Mathematics*, vol. 186, no. 2, pp. 466–481, 2006.
[17] H. Guo and L.-S. Chen, "Periodic solution of a turbidostat system with impulsive state feedback control," *Journal of Mathematical Chemistry*, vol. 46, no. 4, pp. 1074–1086, 2009.
[18] H. Guo and L.-S. Chen, "Periodic solution of a chemostat model with Monod growth rate and impulsive state feedback control," *Journal of Theoretical Biology*, vol. 260, no. 4, pp. 502–509, 2009.
[19] K.-B. Sun, Y. Tian, L.-S. Chen, and A. Kasperski, "Nonlinear modelling of a synchronized chemostat with impulsive state feedback control," *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 227–240, 2010.
[20] K.-B. Sun, Y. Tian, L.-S. Chen, and A. Kasperski, "Universal modelling and qualitative analysis of an impulsive bioprocess," *Computers & Chemical Engineering*. In press.
[21] K.-B. Sun, A. Kasperski, Y. Tian, and L.-S. Chen, "New approach to the nonlinear analysis of a chemostat with impulsive state feedback control," *International Journal of Chemical Reactor Engineering*, vol. 8, no. , p. A99, 2010.

[22] Y. Tian, L.-S. Chen, and A. Kasperski, "Modelling and simulation of a continuous process with feedback control and pulse feeding," *Computers and Chemical Engineering*, vol. 34, no. 6, pp. 976–984, 2010.

[23] Y. Tian, K.-B. Sun, L.-S Chen, and A. Kasperski, "Studies on the dynamics of a continuous bioprocess with impulsive state feedback control," *Chemical Engineering Journal*, vol. 157, no. 2-3, pp. 558–567, 2010.

[24] J. R. Lobry, J. P. Flandrois, G. Carret, and A. Pave, "Monod's bacterial growth model revisited," *Bulletin of Mathematical Biology*, vol. 54, no. 1, pp. 117–122, 1992.

[25] J. Alvarez-Ramirez, J. Alvarez, and A. Velasco, "On the existence of sustained oscillations in a class of bioreactors," *Computers and Chemical Engineering*, vol. 33, no. 1, pp. 4–9, 2009.

[26] A. Novick and L. Szilard, "Description of the chemostat," *Science*, vol. 112, no. 2920, pp. 715–716, 1950.

[27] P. S. Crooke, C.-J. Wei, and R. D. Tanner, "The effect of specific growth rate and yield expression on the existence of oscillatory behavior of a continuous fermentation model," *Chemical Engineering Communications*, vol. 6, no. 6, pp. 333–347, 1980.

[28] P. S. Crooke and R. D. Tanner, "Hopf bifurcations for a variable yield continuous fermentation model," *International Journal of Engineering Science*, vol. 20, no. 3, pp. 439–443, 1982.

[29] A. Apuilera and T. Benitez, "Relationship between growth, iermentation, and respiration rates in Saccaromvces cerevisiae: a study based on the analysis of the yield $Y_{PS}$," *Biotechnology and Bioengineering*, vol. 32, pp. 240–244, 1988.

[30] F. Menkel and A. J. Knights, "A biological approach on modelling a variable biomass yield," *Process Biochemistry*, vol. 30, no. 6, pp. 485–495, 1995.

[31] S. P. Tsai and Y. H. Lee, "A model for energy-sufficient culture growth," *Biotechnology and Bioengineering*, vol. 35, no. 2, pp. 138–145, 1990.

[32] K. Schugerl and K. H. Bellgardt, *Bioreaction Engineering: Modeling and Control*, Springer, Berlin, Germany, 2000.