Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 706483, 12 pages
doi:10.1155/2010/706483

Review Article

A Note on the Modified q-Bernstein Polynomials

Taekyun Kim,[1] Lee-Chae Jang,[2] and Heungsu Yi[3]

[1] *Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Department of Mathematics and Computer Science, Konkuk University, Chungju 138-701, Republic of Korea*
[3] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

Received 20 May 2010; Revised 11 July 2010; Accepted 14 July 2010

Academic Editor: Leonid Shaikhet

We propose the modified q-Bernstein polynomials of degree n which are different q-Bernstein polynomials of Phillips (1997). From these modified q-Bernstein polynomials of degree n, we derive some recurrence formulae for the modified q-Bernstein polynomials.

1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operators in [1]:

$$B_n(f:x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x), \tag{1.1}$$

where $\binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$. Here $B_n(f:x)$ is called the *Bernstein operator of order n for f*. For $k, n \in \mathbb{Z}_+$, the *Bernstein polynomial of degree n* is defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \tag{1.2}$$

where $x \in [0,1]$. For example,

$$B_{0,1}(x) = 1 - x, \qquad B_{1,1}(x) = x,$$
$$B_{0,2}(x) = (1-x)^2, B_{1,2}(x) = 2x(1-x), B_{2,2}(x) = x^2, \ldots. \tag{1.3}$$

Also, $B_{k,n}(x) = 0$, for $k > n$, because $\binom{n}{k} = 0$.

Some people have studied the Bernstein polynomials in the area of approximation theory (see [2] through [3]). Note that for $k \in \mathbb{Z}_+$ and $x \in [0,1]$,

$$\begin{aligned} \frac{t^k e^{(1-x)t} x^k}{k!} &= \frac{x^k}{k!}\left(t^k \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{n!}\right) \\ &= \frac{x^k}{k!} \sum_{\mathrm{n}=0}^{\infty} \frac{(1-x)^n (n+1)\cdots(n+k)}{(n+k)!} t^{n+k} \\ &= \sum_{n=k}^{\infty}\left(\binom{n}{k} x^k (1-x)^{n-k}\right)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!}. \end{aligned} \tag{1.4}$$

Because $B_{k,0}(x) = B_{k,1}(x) = \cdots = B_{k,k-1}(x) = 0$, we obtain the generating function for $B_{k,n}(x)$ as follows:

$$F^{(k)}(t,x) := \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x)\frac{t^n}{n!} \tag{1.5}$$

(see [4, 5]), where $k \in \mathbb{Z}_+$ and $x \in [0,1]$. Notice that

$$B_{k,n}(x) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} & \text{if } n \geq k, \\ 0 & \text{if } n < k, \end{cases} \tag{1.6}$$

for $n, k \in \mathbb{Z}_+$ (see [2]).

Let $0 < q < 1$. Define *the q-number of* x by

$$[x]_q := \frac{1-q^x}{1-q}. \tag{1.7}$$

See [2] through [3] for details and related facts. Note that $\lim_{q\to 1}[x]_q = x$. In [6], Phillips proposed a generalization of the classical Bernstein polynomials based on q-integers. In the last decade some new generalizations of well-known positive linear operators, based on q-integers were introduced and studied by several authors (see [1–13]). Recently, Simsek and Acikgoz have also studied the q-extension of Bernstein-type polynomials [5]. Their q-Bernstein-type polynomials are given by

$$Y_n(k;x:q) = \binom{n}{k}\frac{(-1)^k k!}{(1-q)^{n-k}}\sum_{m,l=0}^{\infty}\sum_{j=0}^{n-k}\binom{k+l-1}{l}\binom{n-k}{k} \times\left(\frac{(-1)^j q^{l+j(1-x)}S(m,k)(x\ln q)^m}{m!}\right), \tag{1.8}$$

where $S(m,k)$ are the second-kind stirling number. In [5], we can find some interesting formulae related to q-extension of Bernstein polynomials which are different q-Bernstein polynomials of Phillips. In the conference of Jangjeon Mathematical Society which was held in IRAN (on Feb.2010), Acikgoz and Arci has introduced several-type Bernstein polynomials (see [2]). The Acikgoz paper [2] announced in the conference is actually what motivated us to write this paper. In this paper, we considered the q-extension of Bernstein polynomials which were introduced by Acikgoz at the conference of Jangjeon Mathematical Society on Feb. 2010. First, we consider the q-extension of the generating function of Bernstein polynomials in (1.5). Indeed, this generating function is also treated by Simsek and Acikgoz in a previous paper (see [5]). From this q-extension of the generating function for the Bernstein polynomials, we propose the modified q-Bernstein polynomials of degree n which are different q-Bernstein polynomials of Phillips. By using the properties of the modified q-Bernstein polynomials, we obtain some recurrence formulae for the modified q-Bernstein polynomials of degree n.

2. The Modified q-Bernstein Polynomials

For $0 < q < 1$, consider the q-extension of (1.5) as follows:

$$\begin{aligned} F_q^{(k)}(t,x) &:= \frac{t^k e^{[1-x]_q t}[x]_q^k}{k!} \\ &= \frac{[x]_q^k}{k!}\sum_{n=0}^{\infty}\frac{[1-x]_q^n}{n!}t^{n+k} \\ &= \sum_{n=k}^{\infty}\binom{n}{k}[x]_q^k[1-x]_q^{n-k}\frac{t^n}{n!}, \end{aligned} \tag{2.1}$$

where $k,n \in \mathbb{Z}_+$ and $x \in [0,1]$. Note that $\lim_{q\to 1}F_q^{(k)}(t,x) = F^{(k)}(t,x)$. We define the *modified q-Bernstein polynomials* as follows:

$$F_q^{(k)}(t,x) = \frac{t^k e^{[1-x]_q t}[x]_q^k}{k!} = \sum_{n=0}^{\infty}B_{k,n}(x,q)\frac{t^n}{n!}, \tag{2.2}$$

where $k,n \in \mathbb{Z}_+$ and $x \in [0,1]$.

Remark. This generating function is also introduced by Simsek and Acikgoz in a previous paper (see [5]).

By comparing the coefficients of (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$,*

$$B_{k,n}(x,q) = \begin{cases} \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{cases} \tag{2.3}$$

For $0 \leq k \leq n$, we have

$$\begin{aligned} &[1-x]_q B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q) \\ &\quad = [1-x]_q \binom{n-1}{k} [x]_q^k [1-x]_q^{n-1-k} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} \\ &\quad = \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k} + \binom{n-1}{k-1} [x]_q^k [1-x]_q^{n-k} \\ &\quad = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, \end{aligned} \tag{2.4}$$

and the derivatives of the modified q-Bernstein polynomials of degree n are also polynomials of degree $n-1$, that is,

$$\begin{aligned} \frac{d}{dx} B_{k,n}(x,q) &= \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} \frac{\ln q}{q-1} q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} \left(\frac{-\ln q}{q-1} \right) q^{1-x} \\ &= \frac{\ln q}{q-1} \left\{ \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} q^x - \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} q^{1-x} \right\} \\ &= n \left(q^x B_{k-1,n-1}(x,q) - q^{1-x} B_{k,n-1}(x,q) \right) \frac{\ln q}{q-1}. \end{aligned} \tag{2.5}$$

Therefore, we obtain the following recurrence formulae.

Theorem 2.2 (recurrence formulae for $B_{k,n}(x,q)$). *For $k, n \in \mathbb{Z}_+$ and for $x \in [0,1]$,*

$$\begin{aligned} [1-x]_q B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q) &= B_{k,n}(x,q), \\ \frac{d}{dx} B_{k,n}(x,q) &= n \left(q^x B_{k-1,n-1}(x,q) - q^{1-x} B_{k,n-1}(x,q) \right) \frac{\ln q}{q-1}. \end{aligned} \tag{2.6}$$

Let f be a continuous function on $[0,1]$. Then the *modified q-Bernstein operator of order n for f* is defined by

$$B_{n,q}(f : x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q), \tag{2.7}$$

where $0 \le x \le 1$, $n \in \mathbb{Z}_+$. We get from Theorem 2.1 and (2.7) that for $f(x) = x$,

$$\begin{aligned} B_{n,q}(f : x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_q^k[1-x]_q^{n-k} \\ &= [x]_q\Big(1-[1-x]_q[x]_q(q-1)\Big)^{n-1} \\ &= f\Big([x]_q\Big)\Big(1+(1-q)[x]_q[1-x]_q\Big)^{n-1}. \end{aligned} \tag{2.8}$$

We also see from Theorem 2.1 that

$$\begin{aligned} B_{n,q}(1 : x) &= \sum_{k=0}^{n} B_{k,n}(x,q) \\ &= \sum_{k=0}^{n}\binom{n}{k}[x]_q^k[1-x]_q^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}[x]_q^k\Big(1-q^{1-x}[x]_q\Big)^{n-k} \\ &= \Big(1+(1-q)[x]_q[1-x]_q\Big)^{n}. \end{aligned} \tag{2.9}$$

The modified q-Bernstein polynomials are symmetric polynomials in the following sense:

$$B_{n-k,n}(1-x,q) = \binom{n}{n-k}[1-x]_q^{n-k}[x]_q^k = B_{k,n}(x,q). \tag{2.10}$$

Therefore, we get the following theorem.

Theorem 2.3. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$,*

$$\begin{aligned} B_{n-k,n}(1-x,q) &= B_{k,n}(x,q), \\ B_{n,q}(1 : x) &= \Big(1+(1-q)[x]_q[1-x]_q\Big)^{n}. \end{aligned} \tag{2.11}$$

For $\zeta \in \mathbb{C}$, $x \in [0,1]$ and for $n \in \mathbb{Z}_+$, consider

$$\frac{n!}{2\pi i}\oint_C \frac{\left([x]_q\zeta\right)^k}{k!} e^{([1-x]_q\zeta)} \frac{d\zeta}{\zeta^{n+1}}, \tag{2.12}$$

where C is a circle around the origin and integration is in the positive direction. We see from the definition of the modified q-Bernstein polynomials and the basic theory of complex analysis including Laurent series that

$$\oint_C \frac{\left([x]_q\zeta\right)^k}{k!} e^{[1-x]_q\zeta} \frac{d\zeta}{\zeta^{n+1}} = \sum_{m=0}^{\infty}\oint_C \frac{B_{k,m}(x,q)\zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} = 2\pi i\left(\frac{B_{k,n}(x,q)}{n!}\right). \tag{2.13}$$

We get from (2.12) and (2.13) that

$$\frac{n!}{2\pi i}\oint_C \frac{\left([x]_q\zeta\right)^k}{k!} e^{[1-x]_q\zeta} \frac{d\zeta}{\zeta^{n+1}} = B_{k,n}(x,q), \tag{2.14}$$

$$\begin{aligned}\oint_C \frac{\left([x]_q\zeta\right)^k}{k!} e^{[1-x]_q\zeta} \frac{d\zeta}{\zeta^{n+1}} &= \frac{[x]_q^k}{k!}\sum_{m=0}^{\infty}\left(\frac{[1-x]_q^m}{m!}\oint_C \zeta^{m-n-1+k} d\zeta\right) \\ &= 2\pi i\left(\frac{[x]_q^k[1-x]_q^{n-k}}{k!(n-k)!}\right).\end{aligned} \tag{2.15}$$

We also get from (2.12) and (2.15) that

$$\frac{n!}{2\pi i}\oint_C \frac{\left([x]_q\zeta\right)^k}{k!} e^{([1-x]_q\zeta)} \frac{d\zeta}{\zeta^{n+1}} = \binom{n}{k}[x]_q^k[1-x]_q^{n-k}. \tag{2.16}$$

Therefore, we see from (2.14) and (2.16) that

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_q^{n-k}. \tag{2.17}$$

Note that

$$\begin{aligned}
&\left(\frac{n-k}{n}\right)B_{k,n}(x,q)+\left(\frac{k+1}{n}\right)B_{k+1,n}(x,q)\\
&\quad=\frac{(n-1)!}{k!(n-k-1)!}[x]_q^k[1-x]_q^{n-k}+\frac{(n-1)!}{k!(n-k-1)!}[x]_q^{k+1}[1-x]_q^{n-k-1}\\
&\quad=\left([1-x]_q+[x]_q\right)B_{k,n-1}(x,q)\\
&\quad=\left(1+[x]_q\left(1-q^{1-x}\right)\right)B_{k,n-1}(x,q)\\
&\quad=\left(1+(1-q)[x]_q[1-x]_q\right)B_{k,n-1}(x,q).
\end{aligned}\tag{2.18}$$

Therefore, we can write the modified q-Bernstein polynomials as a linear combination of polynomials of higher order as follows.

Theorem 2.4. *For $k,n\in\mathbb{Z}_+$ and $x\in[0,1]$,*

$$\left(\frac{n+1-k}{n+1}\right)B_{k,n+1}(x,q)+\left(\frac{k+1}{n+1}\right)B_{k+1,n+1}(x,q)=\left(1+(1-q)[x]_q[1-x]_q\right)B_{k,n}(x,q).\tag{2.19}$$

We easily see from (2.17) that for $n,k\in\mathbb{N}$,

$$\begin{aligned}
\left(\frac{n-k+1}{k}\right)\left(\frac{[x]_q}{[1-x]_q}\right)B_{k-1,n}(x,q)&=\left(\frac{n-k+1}{k}\right)\left(\frac{[x]_q}{[1-x]_q}\right)\binom{n}{k-1}[x]_q^{k-1}[1-x]_q^{n-k+1}\\
&=\frac{n!}{k!(n-k)!}[x]_q^k[1-x]_q^{n-k}\\
&=B_{k,n}(x,q).
\end{aligned}\tag{2.20}$$

Thus, the following corollary holds.

Corollary 2.5. *For $n,k\in\mathbb{N}$ and $x\in[0,1]$,*

$$\left(\frac{n-k+1}{k}\right)\left(\frac{[x]_q}{[1-x]_q}\right)B_{k-1,n}(x,q)=B_{k,n}(x,q).\tag{2.21}$$

Note from the definition of the modified q-Bernstein polynomials and the binomial theorem that for $k, n \in \mathbb{Z}_+$,

$$\begin{aligned}
B_{k,n}(x,q) &= \binom{n}{k}[x]_q^k[1-x]_q^{n-k} \\
&= \binom{n}{k}[x]_q^k\left(1-q^{1-x}[x]_q\right)^{n-k} \\
&= \binom{n}{k}[x]_q^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l q^{l(1-x)}[x]_q^l \\
&= \sum_{l=0}^{n-k}\binom{k+l}{k}\binom{n}{k+l}(-1)^l q^{l(1-x)}[x]_q^{l+k} \\
&= \sum_{j=k}^{n}\binom{n}{k}\binom{n}{j}(-1)^{j-k} q^{(1-x)(j-k)}[x]_q^j.
\end{aligned} \tag{2.22}$$

Therefore, we showed that the following theorem holds.

Theorem 2.6. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$,*

$$B_{k,n}(x,q) = \sum_{j=k}^{n}\binom{j}{k}\binom{n}{j}(-1)^{j-k} q^{(1-x)(j-k)}[x]_q^j. \tag{2.23}$$

It is possible to write $[x]_q^k$ as a linear combination of the modified q-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. We easily see from the property of the modified q-Bernstein polynomials that

$$\begin{aligned}
\sum_{k=1}^{n}\left(\frac{k}{n}\right)B_{k,n}(x,q) &= \sum_{k=1}^{n}\binom{n-1}{k-1}[x]_q^k[1-x]_q^{n-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}[x]_q^{k+1}[1-x]_q^{n-1-k} \\
&= [x]_q\left([x]_q+[1-x]_q\right)^{n-1},
\end{aligned} \tag{2.24}$$

and that

$$\begin{aligned}
\sum_{k=2}^{n}\frac{\binom{k}{2}}{\binom{n}{2}}B_{k,n}(x,q) &= \sum_{k=2}^{n}\binom{n-2}{k-2}[x]_q^k[1-x]_q^{n-k} \\
&= \sum_{k=0}^{n-2}\binom{n-2}{k}[x]_q^{k+2}[1-x]_q^{n-2-k} \\
&= [x]_q^2\left([x]_q+[1-x]_q\right)^{n-2}.
\end{aligned} \tag{2.25}$$

Continuing this process, we obtain

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q) = [x]_q^j \left([x]_q + [1-x]_q \right)^{n-j}, \tag{2.26}$$

for $j \in \mathbb{N}$. Therefore, we obtain the following theorem.

Theorem 2.7. *For $n, j \in \mathbb{Z}_+$ and $x \in [0,1]$,*

$$\frac{1}{\left([1-x]_q + [x]_q \right)^{n-j}} \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q) = [x]_q^j. \tag{2.27}$$

For $k \in \mathbb{N}$, the *Bernoulli polynomial of order k* is defined by

$$\left(\frac{t}{e^t - 1} \right)^k e^{xt} = \underbrace{\left(\frac{t}{e^t - 1} \right) \times \cdots \times \left(\frac{t}{e^t - 1} \right)}_{k\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \tag{2.28}$$

and $B_n^{(k)} = B_n^{(k)}(0)$ are called the nth *Bernoulli numbers of order k*. It is well known that the *second kind stirling number* is defined by

$$\frac{\left(e^t - 1\right)^k}{k!} := \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!}, \tag{2.29}$$

for $k \in \mathbb{N}$. We note from (2.2) that

$$\begin{aligned}
\frac{\left([x]_q t \right)^k e^{[1-x]_q t}}{k!} &= \frac{[x]_q^k \left(e^t - 1\right)^k}{k!} \left(\frac{t}{e^t - 1} \right)^k e^{[1-x]_q t} \\
&= [x]_q^k \left(\sum_{m=0}^{\infty} S(m,k) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} B_n^{(k)} \left([1-x]_q \right) \frac{t^n}{n!} \right) \\
&= [x]_q^k \sum_{l=0}^{\infty} \left(\sum_{n=0}^{l} \frac{B_n^{(k)} \left([1-x]_q \right) S(l-n,k) l!}{n!(l-n)!} \right) \frac{t^l}{l!}.
\end{aligned} \tag{2.30}$$

We have from (2.2) and (2.30) that

$$B_{k,l}(x,q) = [x]_q^k \sum_{n=0}^{l} \binom{l}{n} B_n^{(k)} \left([1-x]_q \right) S(l-n,k), \tag{2.31}$$

and $B_{k,0}(x,q) = B_{k,1}(x,q) = \cdots = B_{k,k-1}(x,q) = 0$.

Remark. The Equations (2.30) and (2.31) are already known by Simsek and Acikgoz in a previous paper [5, page 7].

Let Δ be the *shift difference operator* defined by $\Delta f(x) = f(x+1) - f(x)$. We see from the iterative method that

$$\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \tag{2.32}$$

for $n \in \mathbb{N}$. We get from (2.29) and (2.32) that

$$\begin{aligned} \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!} &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Delta^k 0^n}{k!} \frac{t^n}{n!}. \end{aligned} \tag{2.33}$$

By comparing the coefficients on both sides above, we have

$$S(n,k) = \frac{\Delta^k 0^n}{k!}, \tag{2.34}$$

for $n, k \in \mathbb{Z}_+$. Thus, we get from (2.31) and (2.34) that

$$B_{k,l}(x,q) = [x]_q^k \sum_{n=0}^{l} \binom{l}{n} B_n^{(k)} \left([1-x]_q \right) \frac{\Delta^k 0^{l-n}}{k!}. \tag{2.35}$$

Let $(Eh)(x) = h(x+1)$ be the *shift operator*. Then the *q-difference operator* is defined by

$$\Delta_q^n = \Pi_{j=0}^{n-1} \left(E - q^j I \right), \tag{2.36}$$

where I is an identity operator(see [7] through [11]). For $f \in C[0,1]$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{n}{2}} f(n-k), \tag{2.37}$$

where $\binom{n}{k}_q$ is the *Gaussian binomial coefficient* defined by

$$\binom{x}{k}_q = \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!}. \tag{2.38}$$

Let $F_q(t)$ be the generating function of the *q-extension of the second kind stirling number* as follows:

$$F_q(t) := \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}_q q^{\binom{k-j}{2}} e^{[j]_q t} = \sum_{n=0}^{\infty} S(n,k:q)\frac{t^n}{n!}. \tag{2.39}$$

We have from (2.39) that

$$S(n,k:q) = \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^{j} q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n = \frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n, \tag{2.40}$$

where $[k]_q! = [k]_q[k-1]_q\cdots[2]_q[1]_q$. It is not difficult to see that

$$[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S(n,k:q). \tag{2.41}$$

See also [7] through [11] for details and related facts for above. Then, we get from (2.41) and Theorem 2.7 that

$$\sum_{k=0}^{j} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S(j,k:q) = \frac{1}{\left([1-x]_q + [x]_q\right)^{n-j}}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q). \tag{2.42}$$

Therefore, this completes the proof of the following theorem.

Theorem 2.8. *For $n, j \in \mathbb{Z}_+$ and $x \in [0,1]$,*

$$\frac{1}{\left([1-x]_q + [x]_q\right)^{n-j}}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q) = \sum_{k=0}^{j} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S(j,k:q). \tag{2.43}$$

Acknowledgments

The authors express their sincere gratitude to the referees for their valuable suggestions and comments. This paper is supported in part by the Research Grant of Kwangwoon University in 2010.

References

[1] S. Bernstein, “Demonstration du theoreme de Weierstrass, fondee sur le calcul des probabilities,” *Communications of the Kharkov Mathematical Society*, vol. 2, no. 13, pp. 1–2, 1912-1913.

[2] M. Acikgoz and S. Araci, “A study on the integral of the product of several type Bernstein polynomials,” *IST Transaction of Applied Mathematics-Modelling and Simulation*. In press.

[3] N. K. Govil and V. Gupta, "Convergence of q-Meyer-König-Zeller-Durrmeyer operators," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 181–189, 2009.

[4] M. Acikgoz and S. Araci, "On the generating function of the Bernstein polynomials," in *Proceedings of the 8th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM '10)*, AIP, Rhodes, Greece, March 2010.

[5] Y. Simsek and M. Acikgoz, "A new generating function of q-Bernstein-type polynomials and their interpolation function," *Abstract and Applied Analysis*, vol. 2010, Article ID 769095, 12 pages, 2010.

[6] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Annals of Numerical Mathematics*, vol. 4, no. 1–4, pp. 511–518, 1997.

[7] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[8] T. Kim, "New approach to q-Euler polynomials of higher order," *Russian Journal of Mathematical Physics*, vol. 17, no. 2, pp. 218–225, 2010.

[9] T. Kim, "Barnes-type multiple q-zeta functions and q-Euler polynomials," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 25, Article ID 255201, 11 pages, 2010.

[10] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.

[11] T. Kim, S. D. Kim, and D.-W. Park, "On uniform differentiability and q-Mahler expansions," *Advanced Studies in Contemporary Mathematics*, vol. 4, no. 1, pp. 35–41, 2001.

[12] S. Zorlu, H. Aktuglu, and M. A. Özarslan, "An estimation to the solution of an initial value problem via q-Bernstein polynomials," *Journal of Computational Analysis and Applications*, vol. 12, no. 3, pp. 637–645, 2010.

[13] V. Gupta and C. Cristina, "Statistical approximation properties of q-Baskakov-Kantorovich operators," *Central European Journal of Mathematics*. In press.