Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 737068, 17 pages
doi:10.1155/2010/737068

Research Article

Uniqueness of Limit Cycles for a Class of Cubic Systems with Two Invariant Straight Lines

Xiangdong Xie,[1] Fengde Chen,[2] and Qingyi Zhan[3]

[1] *Department of Mathematics, Ningde Normal University, Ningde, Fujian 352100, China*
[2] *School of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350002, China*
[3] *College of Computer and Information Science, Fujian Agriculture and Forestry University, Fuzhou, Fujian 350002, China*

Correspondence should be addressed to Xiangdong Xie, fjndunlu2007@126.com

Received 1 November 2009; Accepted 12 July 2010

Academic Editor: Binggen Zhang

A class of cubic systems with two invariant straight lines $dx/dt = y(1 - x^2)$, $dy/dt = -x + \delta y + nx^2 + mxy + ly^2 + bxy^2$. is studied. It is obtained that the focal quantities of $O(0,0)$ are, $W_0 = \delta$; if $W_0 = 0$, then $W_1 = m(n+l)$; if $W_0 = W_1 = 0$, then $W_2 = -nm(b+1)$; if $W_0 = W_1 = W_2 = 0$, then O is a center, and it has been proved that the above mentioned cubic system has at most one limit cycle surrounding weak focal $O(0,0)$. This paper also aims to solve the remaining issues in the work of Zheng and Xie (2009).

1. Introduction

The study of the polynomial differential system attracts more and more researchers because of the Hilbert's 16th problem [1–5]. The major problem of the polynomial differential system is to calculate the highest order of focal quantities (also known as focal values, or Lyapunov exponents) at its focal and to decide how many limit cycles surrounding a singular point; the system generated at least under some perturbation of coefficients. All this problem is still open.

There are many papers to study the Kukles system, and many achievements are reached, which include the calculation of the focal quantities and decision of the maximum number or limit cycles of the system. Such as paper [5], Hill et al. had studied a class of cubic differential systems and also brought to our attention that a system used to model predator-prey interactions with intratrophic predation could be transformed so that it can be an example of a system of type (1.1).

In papers [6, 7], the authors consider a class of cubic Kukles systems E_3^1:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \delta y + nx^2 + mxy + ly^2 + bxy^2, \tag{1.1}$$

where $b \neq 0$. Paper [6] has proved that the focal quantities of $O(0,0)$ in (1.1) are $W_0 = \delta$, $W_1 = m(n+l), W_2 = -mnb$, if $W_0 = W_1 = W_2 = 0$, O is a center, and if $W_0 = 0$, system (1.1) has at most one limit cycle surrounding O.

Paper [8] considers a class of cubic systems:

$$\frac{dx}{dt} = y(1+x^2), \quad \frac{dy}{dt} = -x + \delta y + nx^2 + mxy + ly^2 + bxy^2, \tag{1.2}$$

where $b \neq 0, 1$, and has proved that the focal quantities of $O(0,0)$ in (1.2) are, $W_0 = \delta, W_1 = m(n+l), W_2 = -mn(b-1)$, if $W_0 = W_1 = W_2 = 0$, O is a center, and if $W_0 = 0$, system (1.2) has at most one limit cycle surrounding O. But in Paper [8], the case $b = 1$ is not considered.

Recently, in paper [9], the authors consider a class of cubic systems:

$$\frac{dx}{dt} = y(1-x), \quad \frac{dy}{dt} = -x + \delta y + nx^2 + mxy + ly^2 + bxy^2, \tag{1.3}$$

where $b \neq 0$. Paper [9] has proved that the focal quantities of $O(0,0)$ in (1.3) are, $W_0 = \delta, W_1 = m(n+l), W_2 = -mnb$. If $W_0 = W_1 = W_2 = 0$, O is a center, and if $W_0 = 0$, system (1.3) has at most one limit cycle surrounding O.

In this paper, we consider the following cubic system

$$\frac{dx}{dt} = y\left(1-x^2\right), \quad \frac{dy}{dt} = -x + \delta y + nx^2 + mxy + ly^2 + bxy^2. \tag{1.4}$$

It comes from system (1.1) by adding some invariant straight line, so system (1.4) is said to be the accompany system of (1.1), and it is also said that system (1.1) and (1.4) is part of the accompany system. Paper [10] introduces the concept of accompany system, and studies the qualitative property of some accompany system.

Now without loss of generality, we may assume that $m \leq 0$ (if $m \geq 0$, let $(x, y, t) \to (x, -y, -t)$, then m change its sign), and may assume that $l \geq 0$ (if $l \leq 0$, let $(x, y, t) \to (-x, y, -t)$, then l change its sign, but m does not change its sign). So we study the system (1.4) with $m \leq 0$, $l \geq 0$.

System (1.4) has a critical point $O(0,0)$ and $N(1/n, 0)$, if $n \neq 0$, and other critical points (if have) lie on the invariant straight line $x = \pm 1$. Now we transform (1.4) into Lienard equation; note that

$$\begin{aligned} \frac{dx}{dt} &= -\left(x^2-1\right)y \equiv f_0(x) - f_1(x)y, \\ \frac{dy}{dt} &= x(nx-1) + (mx+\delta)y + (l+bx)y^2 \equiv g_0(x) + g_1(x)y + g_2(x)y^2. \end{aligned} \tag{1.5}$$

Let $x = x,\ \xi = f_0(x) - f_1(x)y$, then (1.5) can be reduced to

$$\frac{dx}{dt} = \xi, \quad \frac{d\xi}{dt} = -\varphi_0(x) - \varphi_1(x)\xi - \varphi_2(x)\xi^2, \tag{1.6}$$

where $\varphi_0(x) = f_1(x)g_0(x),\ \varphi_1(x) = -g_1(x),\ \varphi_2(x) = (g_2(x) - f_1'(x))/f_1(x)$, and let

$$\begin{gathered} x = x, \quad \xi = ue^{-\int_a^x \varphi_2(s)ds}, \\ x = x, \quad y = u + \int_a^x \varphi_1 e^{\int_a^s \varphi_2(z)dz} ds, \quad \frac{dt}{d\tau} = e^{\int_a^x \varphi_2(s)ds}. \end{gathered} \tag{1.7}$$

System (1.4) can be reduced to

$$\begin{gathered} \frac{dx}{d\tau} = y - \int_a^x \varphi_1(s)e^{\int_a^s \varphi_2(z)dz} ds \equiv y - F(x), \\ \frac{dy}{d\tau} = -\varphi_0 e^{2\int_a^x \varphi_2(s)ds} \equiv -g(x). \end{gathered} \tag{1.8}$$

2. The Problem of the Center or Focal for Critical Point $O(0,0)$

In this section, since we will study the problem of center or focus for critical point $O(0,0)$, we take $a = 0$ in system (1.8). In order to calculate the focal quantities of system (1.4) (or system (1.8)) in O, we need to let $\delta = 0$, and only consider, $|x| \ll 1$, that is $|x-1| = 1-x$, so

$$\int_0^x \varphi_2(s)ds = \int_0^x \frac{l+bs-2s}{s^2-1}ds = \frac{b+l}{2}\ln|x-1| + \frac{b-l}{2}\ln|x+1| - \ln\left|x^2-1\right|. \tag{2.1}$$

So

$$\begin{gathered} F'(x) = f(x) = -(\delta + mx)(1-x)^{(b+l-2)/2}(1+x)^{(b-l-2)/2}, \\ g(x) = x(1-nx)(1-x)^{b+l-1}(1+x)^{b-l-1}. \end{gathered} \tag{2.2}$$

We use method of paper [11], so $f(x),\ g(x)$ need to be written in the power series as follows:

$$\begin{aligned} f(x) &\equiv F'(x) = -mx + mlx^2 - \frac{1}{2}m\left(l^2 - b + 2\right)x^3 + \frac{1}{6}m\left(l^3 + 8l - 3bl\right)x^4 + \cdots \\ &\equiv b_1x + b_2x^2 + b_3x^3 + b_4x^4 \cdots, \\ g(x) &= x - (2l+n)x^2 + \left(2l^2 - b - n + 2nl + 1\right)x^3 \\ &\quad - \frac{1}{3}\left(4l^3 + 8l + 6nl^2 + 3n - 6bl - 3nb\right)x^4 + \cdots \\ &\equiv C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \cdots. \end{aligned} \tag{2.3}$$

We use mark in paper [11]; let $c_{n-2} = C_{n-2}/nC_0, n = 3, 4, \ldots$, and $\beta_m = b_{m+1}/(m+2) - c_m b_1$, then $\beta_1 = b_2/3 - c_1 b_1 = b_2/3 - C_1 b_1/3 = -m(n+l)/3,\ g_3 = 2\beta_1 = -2m(n+l)/3$. If $\delta = g_3 = 0$, then

$$\beta_2 = \frac{b_3}{4} - c_2 b_1 = \frac{b_3}{4} - \frac{C_2 b_1}{4} = \frac{m\left(3l^2 - b + 4nl\right)}{8},$$
$$\beta_3 = \frac{b_4}{5} - c_3 b_1 = \frac{b_4}{5} - \frac{C_3 b_1}{5} = \frac{m}{30}\left(9bl - 7l^3 - 8l - 12nl^2 + 6nb - 6n\right), \tag{2.4}$$

since $\delta = g_3 = 0$, that is, $m(n+l) = 0$, so

$$\beta_2 = \frac{m(nl - b)}{8},$$
$$\beta_3 = \frac{m}{30}\left(3bl - 5nl^2 - 2l\right), \tag{2.5}$$
$$g_5 = -8c_1\beta_2 + 2\beta_3 = -\frac{8}{3}\beta_2 + 2\beta_3 = \frac{2mn}{15}(b+1).$$

From paper [11], if $\delta = 0, g_3 > 0 (g_3 < 0)$, then O is stable(unstable) weak focal of order one. So for system (1.4), $W_0 = \delta$; if $W_0 = 0$, then $W_1 = m(n+l)$, and if $W_1 > 0 (W_1 < 0)$, then O is unstable (stable) weak focal of order one; If $W_0 = W_1 = 0$, that is, $\delta = m(n+l) = 0$, then $W_2 = -mn(b+1)$. (note that g_3 and W_1, g_5 and W_2 have opposite signs). If $W_0 = W_1 = W_2 = 0$ and $b + 1 \neq 0$, we will prove that $O(0,0)$ is a center of system (1.4). Since $W_0 = W_1 = W_2 = 0,\ b+1 \neq 0$, that is, $\delta = m(n+l) = mn = 0, b+1 \neq 0$, so $\delta = m = 0$, or $\delta = n = l = 0$. If $\delta = m = 0$, then system (1.4) can be reduced to

$$\frac{dx}{d\tau} = y\left(1 - x^2\right) \equiv P(x,y), \quad \frac{dy}{d\tau} = -x + nx^2 + ly^2 + bxy^2 \equiv Q(x,y). \tag{2.6}$$

This system is symmetry about x-axis because of $P(x,-y) = -P(x,y),\ Q(x,-y) = Q(x,y)$, so O is a center. If $\delta = n = l = 0$, then system (1.4) can be reduced to

$$\frac{dx}{d\tau} = y\left(1 - x^2\right), \quad \frac{dy}{d\tau} = -x + mxy + bxy^2. \tag{2.7}$$

It is integrable system, so O is a center, hence we have the following theorem.

Theorem 2.1. *For system* (1.4), *let* $b + 1 \neq 0$; *the focus quantities of* $O(0,0)$ *are* $W_0 = \delta$; *if* $W_0 = 0$, *then* $W_1 = m(n+l)$; *if* $W_0 = W_1 = 0$, *then* $W_2 = -nm(b+1)$; *if* $W_0 = W_1 = W_2 = 0$, *then* O *is a center. If* $W_0 > 0 (W_0 < 0)$, *or* $W_0 = 0, W_1 > 0 (W_1 < 0)$, *or* $W_0 = W_1 = 0, W_2 > 0 (W_2 < 0)$, *then* O *is an unstable (stable) critical point. If* $0 < |W_0| \ll |W_1| \ll |W_2| \ll 1, W_0 W_1 < 0, W_1 W_2 < 0$, *then system* (1.4) *has at least two limit cycles surrounding* O.

3. Nonexistence of Limit Cycle Surrounding $O(0,0)$

In this section, we study the nonexistence of limit cycle surrounding the weak focal $O(0,0)$. O is weak focal if and only if $\delta = 0$, so we let $\delta = 0$ in system (1.4).

Lemma 3.1. *If $m = 0$, system* (1.4) *has no limit cycles surrounding O.*

Proof. Since system (1.4) forms a generalized rotated vector field with respect to parameter δ(refer to paper [12, page 241]), and when $\delta = m = 0$, O is a center, so when $m = 0, \delta \neq 0$, system (1.4) has no limit cycles (refer to paper [12, page 244, th. 3.1]).

By Lemma 3.1, we let $m < 0$ in the following.

Now we change (1.4) to lienard equation by (1.8):

$$\begin{aligned} \frac{dx}{d\tau} &= y - \int_0^x \varphi_1(s) e^{\int_0^s \varphi_2(z)dz} ds \equiv y - F(x), \\ \frac{dy}{d\tau} &= -\varphi_0 e^{2\int_0^x \varphi_2(s)ds} \equiv -g(x). \end{aligned} \tag{3.1}$$

Since the limit cycle of (1.4) surrounding $O(0,0)$ must lay in $1 - x^2 > 0, 1 - nx > 0$. Let $f(x) \equiv F'(x)$, then

$$\begin{aligned} f(x) &= -\frac{mx}{1-x^2} e^{\int_0^x ((l+bs)/(s^2-1))ds}, \\ g(x) &= \frac{x(1-nx)}{1-x^2} e^{2\int_0^x ((l+bs)/(s^2-1))ds}, \\ \frac{f}{g} &= -\frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)}, \\ \left(\frac{f}{g}\right)' &= \frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)^2(1-x^2)} W(x), \end{aligned} \tag{3.2}$$

where

$$W(x) = n(b+1)x^2 + (nl - b)x - (n + l). \tag{3.3}$$

Now we define the curve H and L as follows:

$$H : F(u) = F(v); \quad L : \frac{f(u)}{g(u)} = \frac{f(v)}{g(v)}, \quad -1 < v < 0 < u < 1, \ nu < 1, \ nv < 1. \tag{3.4}$$

It easy to see that L, H are continuously differentiable.

Now Let

$$
\begin{aligned}
K = K(u,v) &= \left.\frac{du}{dv}\right|_{L} - \left.\frac{du}{dv}\right|_{H} \\
&= \left(\frac{f(v)}{g(v)}\right)' \Big/ \left(\frac{f(u)}{g(u)}\right)' - \frac{f(v)}{f(u)} \\
&= \frac{e^{-\int_0^v (l+bs/s^2-1)ds}(1-nu)^2(1-u^2)\left[n(b+1)v^2+(nl-b)v-(n+l)\right]}{e^{-\int_0^u (l+bs/s^2-1)ds}(1-nv)^2(1-v^2)\left[n(b+1)u^2+(nl-b)u-(n+l)\right]} \\
&\quad - \frac{v(1-u^2)e^{\int_0^v ((l+bs)/(s^2-1))ds}}{u(1-v^2)e^{\int_0^u ((l+bs)/(s^2-1))ds}}.
\end{aligned}
\tag{3.5}
$$

If H and L intersect in $P(u,v)$, then in P, $f(u)/g(u) = f(v)/g(v)$, that is,

$$
(1-nu)e^{\int_0^u ((l+bs)/(s^2-1))ds} = (1-nv)e^{\int_0^v ((l+bs)/(s^2-1))ds},
\tag{3.6}
$$

so in the intersecting point P of H and L,

$$
K = K(u,v) = \frac{(1-nu)(1-u^2)\left[n(b+1)v^2+(nl-b)v-(n+l)\right]}{(1-nv)(1-v^2)\left[n(b+1)u^2+(nl-b)u-(n+l)\right]} - \frac{v(1-u^2)(1-nu)}{u(1-v^2)(1-nv)},
\tag{3.7}
$$

that is,

$$
K = K(u,v) = \frac{(1-nu)(1-u^2)(v-u)\left[n(b+1)uv+n+l\right]}{u(1-nv)(1-v^2)\left[n(b+1)u^2+(nl-b)u-(n+l)\right]} .
\tag{3.8}
$$

□

Theorem 3.2. *If* $W_0 = 0, b+1 \neq 0$, *and* $W_1W_2 \geq 0$, *then system* (1.4) *or* (3.1) *has no limit cycle surrounding* O.

Proof. Since $W_0 = \delta$, $W_1W_2 = -m^2n(b+1)(n+l)$, and we have supposed that $m < 0$, so $n(b+1)(n+l) \leq 0$. Now we will prove that system (3.1) has no limit cycles under the conditions of $\delta = 0, b+1 \neq 0, n(b+1)(n+l) \leq 0$.

First supposing $n(b+1) > 0$, $n+l < 0$, we will prove that H and L do not S intersect (S intersect means that H from one side of L astride to another side at intersect point).

(1) If $W(x)$ does not change its sign when $x^2 < 1, 1-nx > 0$, then $(f/g)' \leq 0$ (equal sign only for some x, the same as below), so $f(u)/g(u) < f(v)/g(v)$, for any $-1 < v < 0 < u < 1$, it means that L does not exist, therefore, L and H do not S intersect.

(2) If $W(x)$ change its sign when $x^2 < 1, 1-nx > 0$, and $W'(0) > 0$, then $W(x)$=0 have one or two real roots in $-1 < x < 0, 1-nx > 0$ (If real roots do not exist, then similar to (1), L does not exist); then the curve $y = f(x)/g(x)$ is shown in Figure 1, and the relative position of curve H and L is shown in Figure 2 (If only part of L exists, it does not influence the proof,

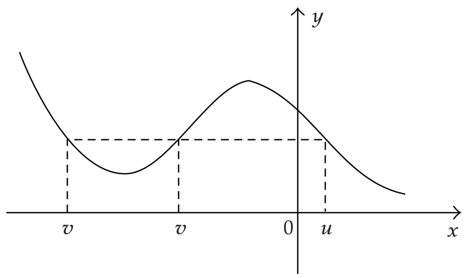

Figure 1: The picture of $y = f(x)/g(x)$ in (2).

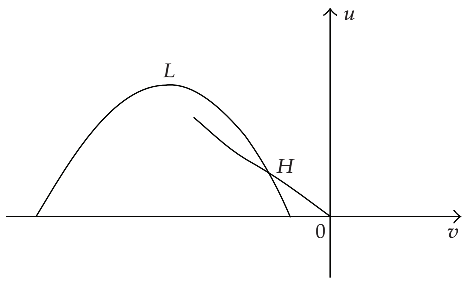

Figure 2: Relative position of L and H in (2).

the same as below). If H and L have an S intersection point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 2

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \leq 0 \tag{3.9}$$

but since $n(b+1)u_1v_1 < 0, b + l < 0, W(u_1) > 0$ (see Figure 1), $1 - nu_1 > 0, 1 - nv_1 > 0$, so

$$\begin{aligned} K_1 \equiv K_1 &\equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \\ &= \frac{(1 - nu_1)(1 - u_1^2)(v_1 - u_1)[n(b+1)u_1v_1 + n + l]}{u_1(1 - nv_1)(1 - v_1^2)W(u_1)} > 0. \end{aligned} \tag{3.10}$$

It is a contradiction, so L and H do not S intersect.

(3) If $W(x)$ change its sign when $x^2 < 1, 1 - nx > 0$, and $W'(0) < 0$, then $W(x) = 0$ have one or two real roots in $0 < x < 1, 1 - nx > 0$, then the curve $y = f(x)/g(x)$ is shown in Figure 3, and the relative position of curves H and L is shown in Figure 4. If H and L have an S intersection point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 4

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \geq 0. \tag{3.11}$$

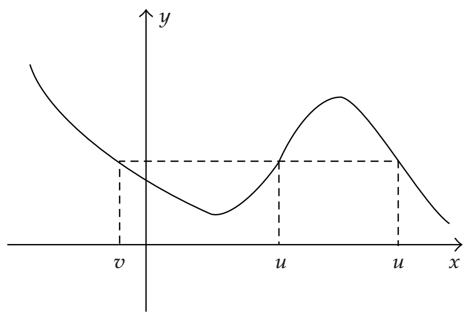

Figure 3: The picture of $y = f(x)/g(x)$ in (3).

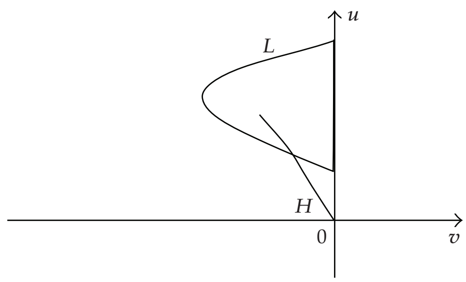

Figure 4: Relative position of L and H in (3).

Since $n(b+1)u_1v_1 < 0, b+l < 0, 1-nu_1 > 0, 1-nv_1 > 0$, and from the fact that $P_1(u_1, v_1)$ is a intersection point late to O, we have $W(u_1) < 0$, so

$$\begin{aligned} K_1 &\equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \\ &= \frac{(1-nu_1)(1-u_1^2)(v_1-u_1)[n(b+1)u_1v_1+n+l]}{u_1(1-nv_1)(1-v_1^2)W(u_1)} < 0. \end{aligned} \tag{3.12}$$

It is a contradiction, so L and H do not S intersect.

Note that if $W(x)$ change its sign in $x^2 < 1, 1-nx > 0$, then $W'(0) \neq 0$, because if $W'(0) = 0$, then $W(x) = n(b+1)x^2 - (n+l) \geq 0$.

Secondly, supposing $n(b+1) < 0, n+l > 0$, we will prove that H and L do not S intersect.

$(1)'$ If $W(x)$ does not change its sign in $x^2 < 1, 1-nx > 0$, then $(f/g)' \geq 0$, $x^2 < 1, 1-nx > 0$, so $f(u)/g(u) > f(v)/g(v)$, for any $-1 < v < 0 < u < 1$, it means that L does not exist, hence L and H do not S intersect.

$(2)'$ If $W(x)$ change its sign when $x^2 < 1, 1-nx > 0$, and $W'(0) < 0$, then $W(x) = 0$ have one or two real roots in $-1 < x < 0, 1-nx > 0$; then the curve $y = f(x)/g(x)$ is shown

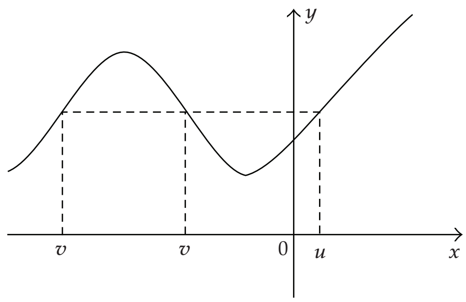

Figure 5: The picture of $y = f(x)/g(x)$ in $(2)'$.

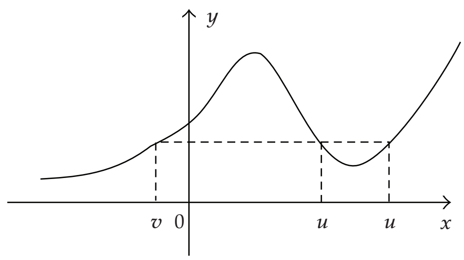

Figure 6: The picture of $y = f(x)/g(x)$ in $(3)'$.

in Figure 5, and the relative position of curves H and L is shown in Figure 2. If H and L have an S intersection point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 2

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \leq 0. \tag{3.13}$$

But since $n(b+1)u_1v_1 > 0, b+l > 0, W(u_1) < 0$ (see Figure 5), $1 - nu_1 > 0,\ 1 - nv_1 > 0$, so

$$\begin{aligned} K_1 &\equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \\ &= \frac{(1-nu_1)(1-u_1^2)(v_1-u_1)[n(b+1)u_1v_1+n+l]}{u_1(1-nv_1)(1-v_1)W(u_1)} > 0. \end{aligned} \tag{3.14}$$

It is a contradiction, so L and H do not S intersect.

$(3)'$ If $W(x)$ change its sign in $x^2 < 1, 1 - nx > 0$, and $W'(0) > 0$, then the curve $y = f(x)/g(x)$ is shown in Figure 6; then the relative position of curves H and L is similar to Figure 4. If H and L have an S intersecting point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 4

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \geq 0 \tag{3.15}$$

but since $W(u_1) > 0$ (see Figure 6), and $n(b+1)u_1v_1 > 0, n+l > 0$, so

$$K_1 \equiv \frac{(1-nu_1)(1-u_1)(v_1-u_1)[nbu_1v_1+n+l]}{u_1(1-nv_1)(1-v_1)W(u_1)} < 0. \tag{3.16}$$

It is a contradiction, so L and H do not S intersect.

Note that if $W(x)$ change its sign in $x^2 < 1, 1-nx > 0$, then $W'(0) \neq 0$. Because if $W'(0) = 0$, then $W(x) = n(b+1)x^2 - (n+l) \leq 0$.

As far, we have proved that H and L have no S intersection if $W_1W_2 > 0$. According to the proof that $f(u)/g(u) = f(v)/g(v)$ and $F(u) = F(v)$ has no intersection to $F(u) = F(v), G(u) = G(v)$ has no intersection in paper [13]; it can be extended to that if $f(u)/g(u) = f(v)/g(v)$ and $F(u) = F(v)$ has no intersection, then $F(u) = F(v), G(u) = G(v)$ has no S intersection. Furthermore, based on the proof of paper [14], if $F(u) = F(v), G(u) = G(v)$ has no S intersection, then the corresponding Lienard equation has no limit cycle.

Now, by [12, 15] we have proved that when $W_0 = 0, b+1 \neq 0, W_1W_2 > 0$, system (1.4) or (3.1) has no limit cycles surrounding O.

Finaly, we consider the case: $W_0 = 0, b+1 \neq 0, W_1W_2 = 0$, that is $n(n+l) = 0$. If $\delta = 0, n = 0, n+l = 0$, then $O(0,0)$ is a center of (1.4); If $\delta = 0, n = 0, n+l \neq 0$, or $\delta = 0, n \neq 0, n+l = 0$, then $L \neq H$ (If $L \equiv H$, then by (3.8), $K(u,v) \equiv 0$). If when $\delta = 0$, $n = 0$, $n+l \neq 0$, or $\delta = 0, n \neq 0, n+l = 0$, L and H S have an intersecting point, then when $0 < |n| \ll 1$, or $0 < |n+l| \ll 1$ this S intersecting point also exists; it is a contradiction to (1)–(3) and (1)′–(3)′, so when $W_0 = 0, b+1 \neq 0, W_1W_2 = 0$, system (1.4) or (3.1) has no limit cycle surrounding O; this completes the proof of Theorem 3.2. □

Lemma 3.3. *If* $W_0 = 0, b+1 = 0, n+l \neq 0$, *system* (1.4) *or* (3.1) *has no limit cycle surrounding* O.

Proof. We consider five cases in the following:

(1) If $nl - b = nl + 1 = 0$, then by (3.3), $W(x) = n + l \neq 0$, so

$$\left(\frac{f}{g}\right)' = \frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)^2(1-x^2)} W(x) \neq 0; \tag{3.17}$$

hence $f(u)/g(u) \neq f(v)/g(v)$, for any $-1 < v < 0 < u < 1$; it means that L does not exist; therefore, L and H do not S intersect.

(2) If $nl + 1 > 0, n+l > 0$, by (3.3), $W(x) = 0$, $x^2 < 1$, $1 - nx > 0$ has at most one positive real root, then the curve $y = f(x)/g(x)$ is similar to Figure 6 (but L has at most one extremal point, and part of L exist; it does not influence the proof, the same as below), and the relative position of curve H and L is similar to Figure 4 (only below half of L exists, the same as below). If H and L have an S intersection point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 4

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \geq 0 \tag{3.18}$$

but since $n(b+1)u_1v_1 = 0, b+l > 0, W(u_1) > 0$ (see Figure 6), $1 - nu_1 > 0,\ 1 - nv_1 > 0$, so

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H = \frac{(1-nu_1)(1-u_1^2)(v_1-u_1)[n(b+1)u_1v_1+n+l]}{u_1(1-nv_1)(1-v_1^2)W(u_1)} < 0. \tag{3.19}$$

It is a contradiction, so L and H do not S intersect.

(3) The case $nl+1<0, n+l<0$ is similar to (2); it is easy to prove that L and H do not S intersect.

(4) If $nl+1>0, n+l<0$, then $W(x)=0, x^2<1, 1-nx>0$ has at most one negative real root, then the curve $y = f(x)/g(x)$ is similar to Figure 1, and the relative position of curves H and L is similar to Figure 2. If H and L have an S intersection point $P_1(u_1, v_1)$ (the first S intersecting point from O), then by Figure 2

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H \le 0 \tag{3.20}$$

but since $n(b+1)u_1v_1 = 0, b+l<0, W(u_1)>0, 1-nu_1>0, 1-nv_1>0$, so

$$K_1 \equiv \left.\frac{du}{dv}\right|_L - \left.\frac{du}{dv}\right|_H = \frac{(1-nu_1)(1-u_1^2)(v_1-u_1)[n(b+1)u_1v_1+n+l]}{u_1(1-nv_1)(1-v_1)W(u_1)} > 0. \tag{3.21}$$

It is a contradiction, so L and H do not S intersect.

(5) The case $nl+1<0, n+l>0$ is similar to (4); it is easy to prove that L and H do not S intersect. This completes the proof of Lemma 3.3. □

Lemma 3.4. *If* $W_0 = 0, W_1 = 0, b+1 = 0, O(0,0)$ *is a center of system* (1.4).

Proof. If when $W_0 = 0, W_1 = 0, b+1 = 0, O$ is a weak focal. We let $0 < |b+1| \ll 1$, and O change its stability; then there is a limit cycle surrounding O; this means when $W_0 = 0, b+1 = 0, n+l \neq 0$, system (1.4) or (3.1) has a limit cycle surrounding O; it is a contradiction to Lemma 3.3. It is follows that $O(0,0)$ is a center of system (1.4). By Lemma 3.4, Theorem 2.1 can be recension in the following. □

Theorem 3.5. *For system* (1.4), *the focal quantities of* $O(0,0)$ *are* $W_0 = \delta$; *if* $W_0 = 0$, *then* $W_1 = m(n+l)$; *if* $W_0 = W_1 = 0$, *then* $W_2 = -nm(b+1)$; *if* $W_0 = W_1 = W_2 = 0, O$ *is a center. If* $W_0 > 0 (W_0 < 0)$, $W_0 = 0, W_1 > 0 (W_1 < 0)$, *or* $W_0 = W_1 = 0, W_2 > 0 (W_2 < 0), O$ *is an unstable (stable) critical point. If* $0 < |W_0| \ll |W_1| \ll |W_2| \ll 1, W_0W_1 < 0, W_1W_2 < 0$, *then system* (1.4) *has at least two limit cycles surrounding* O.

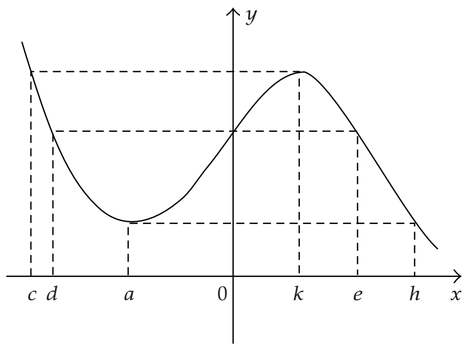

Figure 7: The picture of curve $y = f(x)/g(x)$.

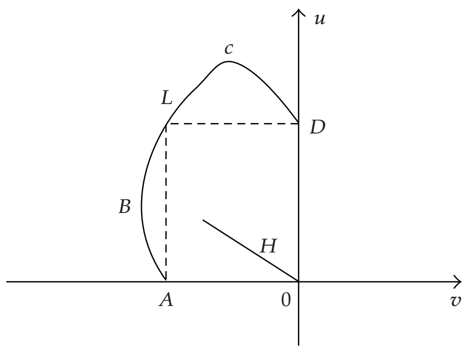

Figure 8: Relative position of curves L and H.

4. Uniqueness of Limit Cycle Surrounding $O(0,0)$

In this section, we study the uniqueness of limit cycle surrounding the weak focal $O(0,0)$; O is weak focal if and only if $\delta = 0$, so we let $\delta = 0$ in system (1.4), we also suppose $m < 0$.

Lemma 4.1. *If $W_0 = 0, W_1W_2 < 0$, H and L have at most one S intersecting point.*

Proof. Since $W_1W_2 = -m^2n(b+1)(n+l)$, so we consider the two care of $n(b+1) > 0, n+l > 0$ and $n(b+1) < 0, n+l < 0$.

Care A. Let $n(b+1) > 0, n+l > 0$, so $W(x) = n(b+1)x^2 + (nl-b)x - (n+l) = 0$ have two real roots : $a, k, a < 0 < k$. Let $c < d < a < 0 < k < e < h$, such that (see Figure 7)

$$\frac{f(c)}{g(c)} = \frac{f(k)}{g(k)}, \qquad \frac{f(d)}{g(d)} = \frac{f(e)}{g(e)} = \frac{f(0)}{g(0)}, \qquad \frac{f(h)}{g(h)} = \frac{f(a)}{g(a)}. \tag{4.1}$$

Without loss of generality, we suppose that $h < 1, -1 < c$ and $nh < 1, nc < 1$(otherwise, only part of L exist; it does not influence the proof); then the graph of $y = f/g$ is shown in Figure 7, and the relative position of H and L is shown in Figure 8, where $A(u=0,\ v=d)$, $B(u=k, v=c)$, $C(u=h, v=a)$, $D(u=e, v=0)$. Now we suppose H and L intersect in $P_1(u_1, v_1)$ (the first S intersecting point from O, the same as below, and if P_1 does not exist,

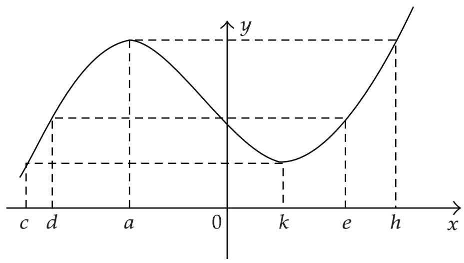

Figure 9: The picture of curve $y = f(x)/g(x)$.

then H and L do not intersect, so the system has no limit cycle surrounding O). We denoted the curve of L from A to B by $L(A,B)$, and $L[A,B]$ $(A, B \in L[A,B], A, B \overline{\in} L(A,B))$ (also define $L(B,C), L(C,D), L[B,C], L[C,D]$ to see the Figure 8), then the following occurs.

(1) If $P_1 \in L(A,B)$, and If H and L have a second S intersecting point $P_2(u_2, v_2)$, then from Figure 8

$$K_1 = K_1(u_1, v_1) \leq 0, \qquad K_2 = K_2(u_2, v_2) \geq 0. \tag{4.2}$$

Since $P_1 \in L(A,B)$, so $0 < u_1 < k$, and $W(u_1) < 0$, hence from (3.8) $n(b+1)u_1v_1 + n + l < 0$. Since $(du/dv)|_H < 0$, so $P_2 \in L(A,B)$ and $0 < u_1 < u_2 < k$, hence $W(u_2) < 0$. From $n(b+1)u_2v_2 + n + l < n(b+1)u_1v_1 + n + l < 0$, it is follows that $K_2 = K(u_2, v_2) < 0$, this is a contradiction, so H and L have no second S intersecting point $P_2(u_2, v_2)$.

(2) If $P_1 \in L[B,C]$, since in $L[B,C]$, $(du/dv)|_{L[B,C]} > 0, (du/dv)|_H < 0$, so L and H have only one S intersecting point.

(3) If $P_1 \in L(C,D)$, and If H and L have a second S intersecting point $P_2(u_2, v_2)$, then from Figure 8

$$K_1 = K_1(u_1, v_1) \geq 0, \qquad K_2 = K_2(u_2, v_2) \leq 0. \tag{4.3}$$

Since $P_1 \in L(C,D)$, so $k < u_1$, and $W(u_1) > 0$; hence from (3.8), $n(b+1)u_1v_1 + n + l < 0$. Since $(du/dv)|_H < 0$, so $P_2 \in L(C,D)$ and $k < u_1 < u_2$, hence $W(u_2) > 0$. From $n(b+1)u_2v_2 + n + l < n(b+1)u_1v_1 + n + l < 0$, it is follows that $K_2 = K(u_2, v_2) > 0$; this is a contradiction, so H and L have no second S interesting point $P_2(u_2, v_2)$.

Care B. Let $n(b+1) < 0, n + l < 0$, so $W(x) \equiv n(b+1)x^2 + (nl - b)x - (n + l) = 0$ have two real roots : $a, k, a < 0 < k$. Let $c < d < a < 0 < k < e < h$, such that (see Figure 9)

$$\frac{f(c)}{g(c)} = \frac{f(k)}{g(k)}, \qquad \frac{f(d)}{g(d)} = \frac{f(e)}{g(e)} = \frac{f(0)}{g(0)}, \qquad \frac{f(h)}{g(h)} = \frac{f(a)}{g(a)}. \tag{4.4}$$

Without loss of generality, we suppose that $h < 1, -1 < c$ and $nh < 1, nc < 1$, then the graph of $y = f/g$ is shown in Figure 9, and the relative position of H and L is shown in Figure 10, where $A(u = 0, v = d), B(u = k, v = c), C(u = h, v = a),$ and $D(u = e, v = 0)$. Now we suppose that H and L interest in $P_1(u_1, v_1)$ (the first S interesting point from O, and if P does not exist, then H and L have no intersecting point, so the system has no limit cycle surrounding O), then the following occurs.

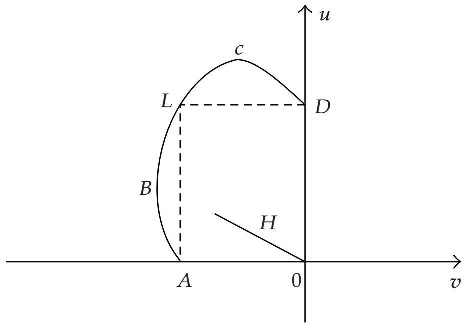

Figure 10: Relative position of curves L and H.

$(1)'$ If $P_1 \in L(A, B)$, and If H and L have a second S intersecting point $P_2(u_2, v_2)$, then from Figure 10,

$$K_1 = K_1(u_1, v_1) \leq 0, \qquad K_2 = K_2(u_2, v_2) \geq 0. \tag{4.5}$$

Since $P_1 \in L(A, B)$, so $0 < u_1 < k$, and $W(u_1) > 0$, hence from (3.8), $n(b+1)u_1v_1+n+l > 0$. Since $(du/dv)|_H < 0$, so $P_2 \in L(A, B)$ and $0 < u_1 < u_2 < k$, so $W(u_2) > 0$. From $n(b+1)u_2v_2 + n + l > n(b+1)u_1v_1 + n + l > 0$, it follows that $K_2 = K(u_2, v_2) < 0$; this is a contradiction, so H and L have no second S intersecting point $P_2(u_2, v_2)$.

$(2)'$ If $P_1 \in L[B, C]$, since in $L[B, C]$, $(du/dv)|_{L[B,C]} > 0, (du/dv)|_H < 0$, so L and H have only one S intersecting point.

$(3)'$ If $P_1 \in L(C, D)$, and If H and L have a second S intersecting point $P_2(u_2, v_2)$, then from Figure 10,

$$K_1 = K_1(u_1, v_1) \geq 0, \qquad K_2 = K_2(u_2, v_2) \leq 0. \tag{4.6}$$

Since $P_1 \in L(C, D)$, so $k < u_1$, and $W(u_1) < 0$; hence $n(b+1)u_1v_1+n+l > 0$. Since $(du/dv)|_H < 0$, so $P_2 \in L(C, D)$ and $k < u_1 < u_2$, so $W(u_2) < 0$. From $n(b+1)u_2v_2+n+l > n(b+1)u_1v_1+n+l > 0$, it follows that $K_2 = K(u_2, v_2) > 0$; this is a contradiction, so H and L have no second S intersecting point $P_2(u_2, v_2)$.

From (1)–(3) and $(1)'$–$(3)'$, we have proved that L and H have at most one S intersecting point under the conditions $W_0 = 0, W_1W_2 < 0$. □

Lemma 4.2. *Let $W_0 = 0; W_1W_2 < 0$, if $n(b + 1) > 0$, then $(Ff/g)' \leq 0, -1 < x < 0, nx < 1$; if $n(b + 1) < 0$, then $(Ff/g)' \geq 0, 0 < x < 1, nx < 1$.*

Proof.

$$
\begin{aligned}
\left(\frac{Ff}{g}\right)' &= \frac{f^2}{g} + F\left(\frac{f}{g}\right)' \\
&= \frac{m^2 x}{(1-nx)(1-x^2)} + F(x)\frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)^2(1-x^2)}\left[n(b+1)x^2 + (nl-b)x - (n+l)\right] \\
&\equiv \frac{m^2 x}{(1-nx)(1-x^2)} + F(x)\frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)^2(1-x^2)}W(x) \\
&\equiv \frac{me^{-\int_0^x ((l+bs)/(s^2-1))ds}}{(1-nx)^2(1-x^2)}\left[mx(1-nx)e^{\int_0^x ((l+bs)/(s^2-1))ds} + F(x)W(x)\right].
\end{aligned}
\tag{4.7}
$$

Now we only have to prove that

$$
V(x) \equiv mx(1-nx)e^{\int_0^x ((l+bs)/(s^2-1))ds} + F(x)W(x) \geq 0, \tag{4.8}
$$

as $n(b+1) > 0, 0 < x < 1, nx < 1$ (or $V(x) \leq 0$, as $-1 < x < 0, n(b+1) < 0, nx < 1$), since

$$
\begin{aligned}
V'(x) &= m(1-nx)e^{\int_0^x ((l+bs)/(s^2-1))ds} + F(x)(2n(b+1)x + (nl-b)), \\
V''(x) &= \frac{m\left(n(b+1)x^2 + n + l\right)}{x^2-1}e^{\int_0^x ((l+bs)/(s^2-1))ds} + F(x)(2n(b+1)).
\end{aligned}
\tag{4.9}
$$

Care A. $n(b+1) > 0$, according to $W_1W_2 < 0$, we have $n+l > 0$. Since $f(x) > 0, x > 0$, and $f(x) < 0, x < 0$, so $F(x) > 0, x \neq 0, F(0) = 0$; hence when $0 < x < 1$, $V''(x) > 0$, that is, $V'(x) > V'(0) = -m > 0$, $x > 0$, so $V(x) > V(0) = 0$, $x > 0$, it follows that $V(x) > 0$, as $0 < x < 1$.

Care B. $n(b+1) < 0$, according to $W_1W_2 < 0$, we have $n+l < 0$, since $F(x) > 0$, $x \neq 0, F(0) = 0$; hence when $-1 < x < 0$, $V''(x) < 0$, that is, $V'(x) > V'(0) = -m > 0$, $-1 < x < 0$, so $V(x) < V(0) = 0$, $-1 < x < 0$, it follows that $V(x) < 0$, as $-1 < x < 0$.

Since $f/g \to -m > 0$, as $x \to 0$, from Lemmas 4.1-4.2 and paper [1], we have the following theorem. □

Theorem 4.3. *If* $W_0 = 0, W_1W_2 < 0$, *then system* (1.4) *has at most one limit cycle surrounding* $O(0,0)$.

5. The Remaining Issues in Paper [8]

In this section we will study the remaining issues in paper [8]. Paper [8] considers a class of cubic system (1.2), where $b \neq 0, 1$, and has proved that the focal quantities of $O(0,0)$ in (1.2) are $W_0 = \delta$, $W_1 = m(n+l), W_2 = -mn(b-1)$, if $W_0 = W_1 = W_2 = 0$, O is a center, and if $W_0 = 0$, then system (1.2) has at most one limit cycle surrounding O. But in paper [8], the case $b = 1$ is not considered.

Lemma 5.1. *If $\delta = 0, b = 1, n + l \neq 0$, system* (1.2) *has no limit cycle surrounding* $O(0,0)$.

Proof. Paper [8] has proved that when $\delta = 0, W_1 W_2 \geq 0, L$ and H in paper [8] do not S intersect. Since $b - 1 = 0, n + l \neq 0$, so $K(u,v) \neq 0$(to see (1.8) in paper [8]), hence $L \neq H$ (if $L \equiv H$, then $K(u,v) \equiv 0$). Now if when $\delta = 0, b = -1, n + l \neq 0$, L and H have an S intersecting point P, then under condition: $\delta = 0, 0 < |b+1| \ll 1, n + l \neq 0$, this S intersecting point P also exists, this is a contradiction to above. Hence under condition of Lemma 5.1, L and H do not S intersect, and system (1.2) has no limit cycle surrounding $O(0,0)$. □

Theorem 5.2. *For system* (1.2)*,the focus quantities of* $O(0,0)$ *are* $W_0 = \delta$; *if* $W_0 = 0$, *then* $W_1 = m(n+l)$; *if* $W_0 = W_1 = 0$, *then* $W_2 = -nm(b-1)$; *if* $W_0 = W_1 = W_2 = 0$, *then* O *is a center. If* $W_0 > 0(W_0 < 0)$, $W_0 = 0, W_1 > 0(W_1 < 0)$, *or* $W_0 = W_1 = 0, W_2 > 0(W_2 < 0)$, *then* O *is unstable (stable) critical point.*

Proof. By the theorem 1 of paper [8], we only need to prove that under conditions: $\delta = 0, b = 1, n + l = 0$, O is a center of system (1.2).

If when $\delta = 0, n+l = 0, b-1 = 0$, O is a weak focus (not a center). We let $0 < |b-1| \ll 1$, and O change its stability, then there is a limit cycle surrounding O; this means that when $W_0 = 0, b-1 = 0, n+l \neq 0$, system (1.2) has a limit cycle surrounding O; this is a contradiction to Lemma 5.1, it follows that $O(0,0)$ is a center of system (1.2). □

Acknowledgments

This work is supported by the Natural Science Foundation of Fujian Province (2010J01010), the Technology Innovation Platform Project of Fujian Province (2009J1007).

References

[1] W. A. Coppel, "A new class of quadratic systems," *Journal of Differential Equations*, vol. 92, no. 2, pp. 360–372, 1991.
[2] X. A. Yang, "A survey of cubic systems," *Annals of Differential Equations*, vol. 7, no. 3, pp. 323–363, 1991.
[3] M. S. Wang and D. J. Luo, "Global bifurcation of some cubic planar systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 8, no. 7, pp. 711–722, 1984.
[4] W. Mingshu and L. Dingjun, "Global bifurcation of some cubic planar systems," *Nonlinear Analysis*, vol. 8, no. 7, pp. 711–722, 1984.
[5] J. M. Hill, N. G. Lloyd, and J. M. Pearson, "Centres and limit cycles for an extended Kukles system," *Electronic Journal of Differential Equations*, vol. 2007, no. 119, pp. 1–23, 2007.
[6] X. D. Xie, "Uniqueness of limit cycles of a class of cubic systems," *Applied Mathematics Series A*, vol. 19, no. 1, pp. 23–30, 2004.
[7] Y. Yujun, "A class of cubit system with one order's fine focus," *Journal of Minjiang University*, vol. 25, no. 5, pp. 14–17, 2004.
[8] Y. H. Zheng and X. D. Xie, "Limit cycle for a cubic system with two imaginary invariant lines," *Applied Mathematics. A Journal of Chinese Universities. Series A*, vol. 24, no. 1, pp. 47–52, 2009.
[9] X. Xie and Q. Zhan, "Uniqueness of limit cycles for a class of cubic system with an invariant straight line," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4217–4225, 2009.
[10] X. D. Xie and J. F. Zhang, "A plane polynomial system and its accompanying system," *Journal of Mathematical Study*, vol. 37, no. 2, pp. 161–166, 2004.
[11] C. Suilin and M. Hui, "The centre-focus decision problem for the equations $\dot{x} = \varphi(y) - F(x), \dot{y} = -g(x)$," *Journal of Zhejiang University*, vol. 25, no. 5, pp. 562–569, 1991.
[12] Z. Zhifen and D. Tongren, *Qualitative Theory of Differential Equation*, Scientific Publishers, Beijing, China, 1985.

[13] H. Maoan, "A note on the nonexistence of limit cycles," *Chinese Science Bulletin*, vol. 16, pp. 1445–1447, 1992.
[14] Z. Xianwu, "The problem of uniqueness of limit cycles of Lienard equation," *China Science A*, vol. 1, pp. 14–20, 1982.
[15] Y. Yanqian, *Qualitative Theory of Polynomial Differential System*, Shanghai Scientific and Technical Publishers, Shanghai, China, 1995.