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Research Article

Nonlinear Dynamical Integral Inequalities in Two Independent Variables and Their Applications

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In this paper, we investigate some nonlinear dynamical integral inequalities involving the forward jump operator in two independent variables. These inequalities provide explicit bounds on unknown functions, which can be used as handy tools to study the qualitative properties of solutions of certain partial dynamical systems on time scales pairs.

1. Introduction

Theory of dynamical equations on time scales, which goes back to Hilger's landmark paper [1], has received considerable attention in recent years. For example, see the monographs [2, 3] and the references cited therein. Since dynamical integral inequalities usually can be used as handy tools to study the qualitative theory of dynamical equations on time scales, many researchers devoted to the study of different types of integral inequalities on time scales. We refer the readers to [4–19].

To the best of our knowledge, the theory of partial dynamic equations on time scales has received less attention [20–24]. The main purpose of this paper is to investigate several nonlinear integral inequalities in two independent variables on time scale pairs, which can be used to estimate explicit bounds of solutions of certain partial dynamical equations on time scales. Unlike some existing results in the literature (e.g., [12]), the integral inequalities considered in this paper involve the forward jump operator $\sigma(t)$ and $\sigma(s)$ on a pair of time scales $\mathbb{T}$ and $\widetilde{\mathbb{T}}$, which results in difficulties in the estimation on the explicit bounds of unknown functions $u(t,s)$ for $t \in \mathbb{T}$ and $s \in \widetilde{\mathbb{T}}$. As an application, we study the qualitative property of certain partial dynamical equations on time scales.

Throughout this paper, a knowledge and understanding of time scales and time scale notations is assumed. In what follows, $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ are two unbounded time scales, $t_0 \in \mathbb{T}$ and

$s_0 \in \widetilde{\mathbb{T}}$. $C_{\mathrm{rd}}(\mathbb{T}, \widetilde{\mathbb{T}})$ is the set of right-dense continuous functions on $\mathbb{T} \times \widetilde{\mathbb{T}}$. For an excellent introduction to the calculus on time scales, we refer the reader to monographs [2, 3].

2. Problem Statements

Before establishing the main results of this paper, we first present two useful lemmas as follows.

Lemma 2.1. *Let $c \geq 0$, $x \geq 0$, and $0 < \lambda < 1$. Then, for any $k > 0$,*

$$cx^{\lambda} \leq kx + \theta(c, k, \lambda) \tag{2.1}$$

holds, where $\theta(c, k, \lambda) = (1-\lambda)\lambda^{\lambda/(1-\lambda)} c^{1/(1-\lambda)} k^{\lambda/(\lambda-1)}$.

Proof. Set $F(x) = cx^{\lambda} - kx$. It is not difficult to see that $F(x)$ obtains its maximum at $x = (\lambda c/k)^{1/(1-\lambda)}$ and

$$F_{\max} = (1-\lambda)\lambda^{\lambda/(1-\lambda)} c^{1/(1-\lambda)} k^{\lambda/(1-\lambda)}. \tag{2.2}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $y, p, q, r \in C_{rd}(\mathbb{T})$ with $p(t), q(t) \geq 0$ for $t \in \mathbb{T}$. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + \frac{q(t)}{1+\mu(t)q(t)} y(\sigma(t)) + r(t), \quad t \in \mathbb{T}, \tag{2.3}$$

implies

$$y(t) \leq y(t_0)e_{p\oplus q}(t, t_0) + \int_{t_0}^{t} e_{p\oplus q}(t, \sigma(s))\left[1+\mu(s)q(s)\right] r(s)\Delta s, \quad t \in \mathbb{T}, \tag{2.4}$$

where $p \oplus q = p + q + \mu pq$ and $\mu(t) = \sigma(t) - t$.

Proof. Note that $y(\sigma(t)) = y(t) + \mu(t)y^{\Delta}(t)$, we have

$$y^{\Delta}(t) \leq p(t)y(t) + \frac{q(t)}{1+\mu(t)q(t)} \left[y(t) + \mu(t)y^{\Delta}(t)\right] + r(t) \tag{2.5}$$

that is,

$$y^{\Delta}(t) \leq (p \oplus q)(t)y(t) + \left[1 + \mu(t)q(t)\right] r(t). \tag{2.6}$$

By Theorem 6.1 [2, page 255], we get that Lemma 2.2 holds. □

Consider the following nonlinear integral inequalities in two independent variables on time scales $\mathbb{T} \times \widetilde{\mathbb{T}}$:

$$u(t,s) \leq a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \left[g(\tau,\eta)u(\tau,\eta) + h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta) \right] \Delta\eta \Delta\tau, \tag{2.7}$$

$$\begin{aligned} u(t,s) \leq a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} &\Big[g(\tau,\eta)u(\tau,\eta) + h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta) \\ &+ h_2(\tau,\eta)u^{\lambda_2}(\tau,\sigma(\eta)) \Big] \Delta\eta \Delta\tau, \end{aligned} \tag{2.8}$$

$$\begin{aligned} u(t,s) \leq a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} &\Big[g(\tau,\eta)u(\tau,\eta) + h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta) \\ &+ h_2(\tau,\eta)u^{\lambda_2}(\tau,\sigma(\eta)) + h_3(\tau,\eta)u^{\lambda_3}(\sigma(\tau),\sigma(\eta)) \Big] \Delta\eta \Delta\tau, \end{aligned} \tag{2.9}$$

where $u(t,s), a(t,s), b(t,s), g(t,s)$, and $h_i(t,s)$ $(i = 1,2,3)$ are nonnegative right-dense continuous functions on $\mathbb{T} \times \widetilde{\mathbb{T}}$, $0 < \lambda_i < 1$ $(i = 1,2,3)$ are constants.

The reason for studying inequalities of type (2.7)–(2.9) is that sometimes we may need to estimate the solutions of the following partial dynamical equation in the form

$$u^{\Delta_t \Delta_s}(t,s) = f(t,s,u(t,s),u(\sigma(t),s),u(t,\sigma(s)),u(\sigma(t),\sigma(s))) \tag{2.10}$$

with boundary conditions $u(t,s_0) = \alpha(t)$, $u(t_0,s) = \beta(s)$, and $u(t_0,s_0) = u_0$, where $f : \mathbb{T} \times \widetilde{\mathbb{T}} \times \mathbb{R}^3 \to \mathbb{R}$ is right-dense continuous, $\mathbb{R} = (-\infty,\infty)$, and u_0 is a constant. Integrating (2.10) yields

$$\begin{aligned} u(t,s) &= \alpha(t) + \beta(s) - u_0 \\ &\quad + \int_{t_0}^{t} \int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta),u(\sigma(\tau),\eta),u(\tau,\sigma(\eta)),u(\sigma(\tau),\sigma(\eta))) \Delta\eta \Delta\tau. \end{aligned} \tag{2.11}$$

Therefore, the study on the integral inequalities of type (2.7)–(2.9) can provide explicit bounds of solutions of system (2.10) in some cases.

3. Main Results

Now, let us present the main results of this paper.

Theorem 3.1. *If there exists a positive function $k_1(t,s) \in C_{\mathrm{rd}}(\mathbb{T},\widetilde{\mathbb{T}})$, such that*

$$\mu(t)b(\sigma(t),s)k_1(t,s) < 1, \quad (t,s) \in \mathbb{T} \times \widetilde{\mathbb{T}}, \tag{3.1}$$

then inequality (2.7) *implies*

$$u(t,s) \le a(t,s) + b(t,s)\int_{t_0}^{t} e_{(p_1 \oplus q_1)(\cdot,s)}(t,\sigma(\tau))\left[1+\mu(\tau)q_1(\tau,s)\right]r_1(\tau,s)\Delta\tau, \tag{3.2}$$

where

$$\begin{gathered} p_1(t,s) = b(t,s)\int_{s_0}^{s} g(t,\eta)\Delta\eta, \\ q_1(t,s) = \frac{b(\sigma(t),s)k_1(t,s)}{1-\mu(t)b(\sigma(t),s)k_1(t,s)}, \\ r_1(t,s) = a(\sigma(t),s)k_1(t,s) + a(t,s)\int_{s_0}^{s} g(t,\eta)\Delta\eta + \theta\left(\int_{s_0}^{s} h_1(t,\eta)\Delta\eta, k_1(t,s), \lambda_1\right). \end{gathered} \tag{3.3}$$

Proof. Define a function $v(t,s)$ by

$$v(t,s) = \int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)u(\tau,\eta) + h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta)\right]\Delta\eta\Delta\tau. \tag{3.4}$$

Then, $v(t,s) \ge 0$ for $(t,s) \in \mathbb{T}\times\widetilde{\mathbb{T}}$, $v(t,s)$ is nondecreasing with respect to t and s, and

$$u(t,s) \le a(t,s) + b(t,s)v(t,s), \quad (t,s) \in \mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.5}$$

A delta derivative of $v(t,s)$ with respect to t yields

$$\begin{aligned} v^{\Delta_t}(t,s) &= \int_{s_0}^{s} g(t,\eta)u(t,\eta) + h_1(t,\eta)u^{\lambda_1}(\sigma(t),\eta)\Delta\eta \\ &\le \left[u(t,s) + u^{\lambda_1}(\sigma(t),s)\right]\int_{s_0}^{s} h_1(t,\eta)\Delta\eta. \end{aligned} \tag{3.6}$$

By Lemma 2.1, we have

$$u^{\lambda_1}(\sigma(t),s)\int_{s_0}^{s} h_1(t,\eta)\Delta\eta \le k_1(t,s)u(\sigma(t),s) + \theta\left(\int_{s_0}^{s} h_1(t,\eta)\Delta\eta, k_1(t,s), \lambda_1\right). \tag{3.7}$$

It follows from (3.5), (3.6), and (3.7) that

$$v^{\Delta_t}(t,s) \le [a(t,s)+b(t,s)v(t,s)]\int_{s_0}^{s} g(t,\eta)\Delta\eta + [a(\sigma(t),s)+b(\sigma(t),s)v(\sigma(t),s)]k_1(t,s) + \theta\left(\int_{s_0}^{s} h_1(t,\eta)\Delta\eta, k_1(t,s), \lambda_1\right). \tag{3.8}$$

Notice the definitions of $p_1(t,s)$, $q_1(t,s)$, and $r_1(t,s)$, we have

$$v^{\Delta_t}(t,s) \le p_1(t,s)v(t,s) + \frac{q_1(t,s)}{1+\mu(t)q_1(t,s)} + r_1(t,s), \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.9}$$

Since $v(t_0,s)=0$, by Lemma 2.2 we get

$$v(t,s) \le \int_{t_0}^{t} e_{(p_1\oplus q_1)(\cdot,s)}(t,\sigma(\tau))\left[1+\mu(\tau)q_1(\tau,s)\right]r_1(\tau,s)\Delta\tau, \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.10}$$

Then, (3.5) and (3.10) imply (3.2). □

Theorem 3.2. *If there exist positive functions $k_1(t,s), k_2(t,s) \in C_{\mathrm{rd}}(\mathbb{T},\widetilde{\mathbb{T}})$, such that $k_2(t,s)$ is Δ-differentiable with respect to s, $k_2^{\Delta_s}(t,s)\in C_{\mathrm{rd}}(\mathbb{T},\widetilde{\mathbb{T}})$, and*

$$\mu(t)b(\sigma(t),s)k_1(t,s) < 1, \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}, \tag{3.11}$$

then inequality (2.8) *implies*

$$u(t,s) \le a(t,s) + b(t,s)\int_{t_0}^{t} e_{(p_2\oplus q_2)(\cdot,s)}(t,\sigma(\tau))\left[1+\mu(\tau)q_2(\tau,s)\right]r_2(\tau,s)\Delta\tau, \tag{3.12}$$

where

$$p_2(t,s) = p_1(t,s) + k_2(t,s) + \int_{s_0}^{s}\left[\overline{k}_2^{\Delta_\eta}(t,\eta) + \frac{k_2(t,\sigma(\eta))b(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\right]\Delta\eta,$$
$$\overline{k}_2^{\Delta_s}(t,s) = \max\left\{0, -k_2^{\Delta_s}(t,s)\right\}, \qquad q_2(t,s) = q_1(t,s),$$
$$r_2(t,s) = r_1(t,s) + \int_{s_0}^{s}\left[\theta\left(h_2(t,\eta), \frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}, \lambda_2\right) + \frac{k_2(t,\sigma(\eta))a(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\right]\Delta\eta. \tag{3.13}$$

Proof. Set

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\Big[g(\tau,\eta)u(\tau,\eta)+h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta)+h_2(\tau,\eta)u^{\lambda_2}(\tau,\sigma(\eta))\Big]\Delta\eta\Delta\tau. \tag{3.14}$$

Then, $z(t,s)$ is nonnegative and nondecreasing with respect to t and s on $\mathbb{T}\times\widetilde{\mathbb{T}}$, and

$$u(t,s)\le a(t,s)+b(t,s)z(t,s),\quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.15}$$

By Lemma 2.1, we have

$$\begin{aligned}
z^{\Delta_t}(t,s)&\le u(t,s)\int_{s_0}^{s}g(t,\eta)\Delta\eta+u^{\lambda_1}(\sigma(t),s)\int_{s_0}^{s}h_1(t,\eta)\Delta\eta\\
&\quad+\int_{s_0}^{s}h_2(t,\eta)u^{\lambda_2}(t,\sigma(\eta))\Delta\eta\\
&\le u(t,s)\int_{s_0}^{s}g(t,\eta)\Delta\eta+k_1(t,s)u(\sigma(t),s)\\
&\quad+\theta\left(\int_{s_0}^{s}h_1(t,\eta)\Delta\eta,k_1(t,s),\lambda_1\right)\\
&\quad+\int_{s_0}^{s}\frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}u(t,\sigma(\eta))\Delta\eta\\
&\quad+\int_{s_0}^{s}\theta\left(h_2(t,\eta),\frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))},\lambda_2\right)\Delta\eta.
\end{aligned} \tag{3.16}$$

Substituting (3.15) into (3.16), we get

$$\begin{aligned}
z^{\Delta_t}(t,s)&\le[a(t,s)+b(t,s)z(t,s)]\int_{s_0}^{s}g(t,\eta)\Delta\eta\\
&\quad+[a(\sigma(t),s)+b(\sigma(t),s)z(\sigma(t),s)]k_1(t,s)\\
&\quad+\theta\left(\int_{s_0}^{s}h_1(t,\eta)\Delta\eta,k_1(t,s),\lambda_1\right)\\
&\quad+\int_{s_0}^{s}\frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}[a(t,\sigma(\eta))+b(t,\sigma(\eta))z(t,\sigma(\eta))]\Delta\eta\\
&\quad+\int_{s_0}^{s}\theta\left(h_2(t,\eta),\frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))},\lambda_2\right)\Delta\eta.
\end{aligned} \tag{3.17}$$

Note that

$$z(t,\sigma(\eta)) = z(t,\eta) + \mu(\eta) z^{\Delta_\eta}(t,\eta). \tag{3.18}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_{s_0}^{s} \frac{k_2(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\left[a(t,\sigma(\eta)) + b(t,\sigma(\eta))z(t,\sigma(\eta))\right]\Delta\eta \\
&\quad\le \int_{s_0}^{s} \frac{k_2(t,\sigma(\eta))a(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\Delta\eta + \int_{s_0}^{s} \frac{k_2(t,\sigma(\eta))b(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))} z(t,\eta)\Delta\eta \\
&\qquad + \int_{s_0}^{s} k_2(t,\sigma(\eta)) z^{\Delta_\eta}(t,\eta)\Delta\eta \\
&\quad\le \int_{s_0}^{s} \frac{k_2(t,\sigma(\eta))a(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\Delta\eta + z(t,s)\int_{s_0}^{s} \frac{k_2(t,\sigma(\eta))b(t,\sigma(\eta))}{1+\mu(\eta)b(t,\sigma(\eta))}\Delta\eta \\
&\qquad + z(t,s)\left[k_2(t,s) + \int_{s_0}^{s} \overline{k}_2^{\Delta_\eta}(t,\eta)\Delta\eta\right].
\end{aligned} \tag{3.19}$$

Therefore, it follows from (3.17) and (3.19) that

$$z^{\Delta_t}(t,s) \le p_2(t,s)z(t,s) + \frac{q_2(t,s)}{1+\mu(t)q_2(t,s)} z(\sigma(t),s) + r_2(t,s), \quad (t,s)\in \mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.20}$$

This together with Lemma 2.2 and (3.15) yields (3.12). □

Theorem 3.3. *If there exist positive functions $k_1(t,s), k_2(t,s), k_3(t,s) \in C_{\mathrm{rd}}(\mathbb{T},\widetilde{\mathbb{T}})$, such that $k_2(t,s), k_3(t,s)$ are Δ-differentiable with respect to s, $k_2^{\Delta_s}(t,s), k_3^{\Delta_s}(t,s) \in C_{\mathrm{rd}}(\mathbb{T},\widetilde{\mathbb{T}})$, and*

$$\mu(t)\Lambda(t,s) < 1, \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}, \tag{3.21}$$

then inequality (2.9) *implies*

$$u(t,s) \le a(t,s) + b(t,s)\int_{t_0}^{t} e_{(p_3\oplus q_3)(\cdot,s)}(t,\sigma(\tau))\left[1+\mu(\tau)q_3(\tau,s)\right] r_3(\tau,s)\Delta\tau, \tag{3.22}$$

where

$$\Lambda(t,s) = b(\sigma(t),s)k_1(t,s) + k_3(t,s) + \int_{s_0}^{s}\left(\overline{k}_3^{\Delta_\eta}(t,\eta) + \frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}\right)\Delta\eta,$$

$$p_3(t,s) = p_2(t,s),$$

$$q_3(t,s) = \frac{\Lambda(t,s)}{1-\mu(t)\Lambda(t,s)},$$

$$r_3(t,s) = r_2(t,s) + \int_{s_0}^{s}\frac{k_3(t,\sigma(\eta))a(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}\Delta\eta + \int_{s_0}^{s}\theta_3\left(h_3(t,\eta), \frac{k_3(t,\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}, \lambda_3\right)\Delta\eta, \tag{3.23}$$

and $\overline{k}_3^{\Delta_s}(t,s) = \max\{0, -k_3^{\Delta_s}(t,s)\}$.

Proof. Let the nonnegative and nondecreasing function $w(t,s)$ be defined by

$$w(t,s) = \int_{t_0}^{t}\int_{s_0}^{s}\Big[g(\tau,\eta)u(\tau,\eta) + h_1(\tau,\eta)u^{\lambda_1}(\sigma(\tau),\eta) + h_2(\tau,\eta)u^{\lambda_2}(\tau,\sigma(\eta)) + h_3(\tau,\eta)u^{\lambda_3}(\sigma(\tau),\sigma(\eta))\Big]\Delta\eta\Delta\tau. \tag{3.24}$$

Then,

$$u(t,s) \le a(t,s) + b(t,s)w(t,s), \quad (t,s) \in \mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.25}$$

Based on the same arguments as in Theorem 3.2, we have

$$\begin{aligned} w^{\Delta_t}(t,s) \le{}& p_2(t,s)z(t,s) + b(\sigma(t),s)k_1(t,s)w(\sigma(t),s) + r_2(t,s) \\ &+ \int_{s_0}^{s}\frac{k_3(t,\sigma(\eta))a(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}\Delta\eta \\ &+ \int_{s_0}^{s}\theta_3\left(h_3(t,\eta), \frac{k_3(t,\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}, \lambda_3\right)\Delta\eta \\ &+ \int_{s_0}^{s}\frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}z(\sigma(t),\sigma(\eta))\Delta\eta. \end{aligned} \tag{3.26}$$

Notice that

$$\begin{aligned}
&\int_{s_0}^{s} \frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))} z(\sigma(t),\sigma(\eta))\Delta\eta \\
&\quad = \int_{s_0}^{s} \frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))} \left[z(\sigma(t),\eta)+\mu(\eta)z^{\Delta_\eta}(\sigma(t),\eta)\right]\Delta\eta \\
&\quad \le z(\sigma(t),s)\int_{s_0}^{s} \frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}\Delta\eta + \int_{s_0}^{s} k_3(t,\sigma(\eta))z^{\Delta_\eta}(\sigma(t),\eta)\Delta\eta \\
&\quad \le z(\sigma(t),s)\left[k_3(t,s)+\int_{s_0}^{s}\left(\overline{k}_3^{\Delta_\eta}(t,\eta)+\frac{k_3(t,\sigma(\eta))b(\sigma(t),\sigma(\eta))}{1+\mu(\eta)b(\sigma(t),\sigma(\eta))}\right)\Delta\eta\right].
\end{aligned} \tag{3.27}$$

By (3.26) and (3.27), we have

$$w^{\Delta_t}(t,s) \le p_3(t,s)w(t,s)+\Lambda(t,s)w(\sigma(t),s)+r_3(t,s), \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}. \tag{3.28}$$

Using the fact $\Lambda(t,s) = q_3(t,s)/(1+\mu(t)q_3(t,s))$, Lemma 2.2 and (3.25), we get that (3.22) holds.

It is worthy to mention that although some additional assumptions such as $\mu(t)b(\sigma(t),s)k_1(t,s) < 1$ and $\mu(t)\Lambda(t,s) < 1$ are imposed in Theorems 3.1–3.3, they are easy to be satisfied by choosing appropriate adjusting functions $k_1(t,s)$ and $k_3(t,s)$. □

4. Applications

We now consider some applications of the main results in the partial dynamical system (2.10) under the boundary condition

$$u(t,s_0)=\alpha(t), \qquad u(t_0,s)=\beta(s), \qquad u(t_0,s_0)=u_0. \tag{4.1}$$

Denote $a(t,s)=|\alpha(t)|+|\beta(s)|+|u_0|$. We have the following corollaries.

Corollary 4.1. *Let* $\mathbb{T}=Z=\{0,1,2,\ldots\}$, $\widetilde{\mathbb{T}}=R_+=[0,\infty)$, *and*

$$\left|f(t,s,u(t,s),u(t+1,s)\right| \le |u(t,s)|+|u(t+1,s)|^{\lambda_1}, \quad (t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}. \tag{4.2}$$

Then, the solution of system (2.10) *under the boundary condition* (4.1) *satisfies*

$$|u(t,s)| \le a(t,s)+2\sum_{\tau=0}^{t-1}(2+2s)^{t-1-\tau}\left[\frac{a(\tau+1,s)}{2}+a(\tau,s)s+\theta\left(s,\frac{1}{2},\lambda_1\right)\right], \tag{4.3}$$

for $(t,s)\in\mathbb{T}\times\widetilde{\mathbb{T}}$.

Proof. For $(t,s) \in \mathbb{T} \times \widetilde{\mathbb{T}}$, it follows from (2.11) and (4.2) that

$$|u(t,s)| \leq a(t,s) + \sum_{\tau=0}^{t-1} \int_0^s \left[|u(\tau,\eta)| + |u(\tau+1,\eta)|^{\lambda_1} \right]. \tag{4.4}$$

Let $k_1(t,s) = 1/2$ be a constant. A straightforward computation yields

$$\begin{aligned} p_1(t,s) &= s, \qquad q_1(t,s) = 1, \\ r_1(t,s) &= \frac{a(t+1,s)}{2} + a(t,s)s + \theta\left(s, \frac{1}{2}, \lambda_1\right). \end{aligned} \tag{4.5}$$

Since $(p_1 \oplus q_1)(t,s) = 2 + 2s$, we get (4.3) by Theorem 3.1. □

Corollary 4.2. *Let* $\mathbb{T} = \widetilde{\mathbb{T}} = Z$, *and*

$$\left| f(t,s,u(t,s),u(t+1,s),u(t,s+1) \right| \leq |u(t,s)| + |u(t+1,s)|^{\lambda_1} + |u(t,s+1)|^{\lambda_2}. \tag{4.6}$$

Then, the solution of system (2.10) *under the boundary condition* (4.1) *satisfies*

$$|u(t,s)| \leq a(t,s) + 2\sum_{\tau=0}^{t-1}(3+3s)^{t-1-\tau} \left[r_1(\tau,s) + s\theta\left(s, \frac{1}{2}, \lambda_2\right) + \frac{\sum_{\eta=0}^{s-1} a(\tau,\eta+1)}{2} \right], \tag{4.7}$$

where $r_1(t,s)$ *is defined as in Corollary 4.1 for* $(t,s) \in \mathbb{T} \times \widetilde{\mathbb{T}}$.

Proof. For $(t,s) \in \mathbb{T} \times \widetilde{\mathbb{T}}$, it follows from (2.11) and (4.6) that

$$|u(t,s)| \leq a(t,s) + \sum_{\tau=0}^{t-1} \sum_{\eta=0}^{s-1} \left[|u(\tau,\eta)| + |u(\tau+1,\eta)|^{\lambda_1} + |u(\tau,\eta+1)|^{\lambda_2} \right] \tag{4.8}$$

holds for $(t,s) \in \mathbb{T} \times \widetilde{\mathbb{T}}$. Let $k_1(t,s) = 1/2$ and $k_2(t,s) = 1$. A straightforward computation yields

$$\begin{aligned} p_2(t,s) &= 1 + \left(\frac{3s}{2}\right), \qquad q_2(t,s) = 1, \\ r_2(t,s) &= r_1(t,s) + s\theta\left(s, \frac{1}{2}, \lambda_2\right) + \frac{\sum_{\eta=0}^{s-1} a(t,\eta+1)}{2}. \end{aligned} \tag{4.9}$$

Hence, $p_2 \oplus q_2 = 3 + 3s$. By Theorem 3.2, we have that (4.7) holds.

For the case when f satisfies

$$\begin{aligned}&\left|f(t,s,u(t,s),u(t+1,s),u(t,s+1),u(t+1,s+1))\right|\\&\quad\le |u(t,s)|+|u(t+1,s)|^{\lambda_1}+|u(t,s+1)|^{\lambda_2}+|u(t+1,s+1)|^{\lambda_3}\end{aligned}\tag{4.10}$$

on $Z\times Z$, the solution of system (2.10) under the boundary condition (4.1) can be similarly estimated by Theorem 3.3. We omit it here. □

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