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Research Article

On the Higher-Order q-Euler Numbers and Polynomials with Weight α

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The main purpose of this paper is to present a systemic study of some families of higher-order q-Euler numbers and polynomials with weight α. In particular, by using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give a new concept of q-Euler numbers and polynomials with weight α.

1. Introduction

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$, will, respectively, denote the ring of p-adic rational integers, the field, of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$ (see [1–14]). When one speaks of q-extension, q can be regarded as an indeterminate, complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$; it is always clear from context. If $q \in \mathbb{C}$, we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1 - q|_p < 1$ (see [1–14]).

In this paper, we use the notation of q-number as follows:

$$[x]_q = \frac{1 - q^x}{1 - q} \tag{1.1}$$

(see [1–14]). Note that $\lim_{q \to 1} [x]_q = x$ for any x with $|x|_p \leq 1$ in the p-adic case.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x,$$
$$= \lim_{N\to\infty} \frac{[2]_q}{2} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.2}$$

(see [4–7]).

From (1.2), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.3}$$

where $f_1(x) = f(x+1)$.

It is well known that the ordinary Euler polynomials are defined by

$$\frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.4}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$ and $E_n(0) = E_n$ are called the nth Euler numbers (see [1–14]).

By (1.5), we get the following recurrence relation as follows:

$$E_0 = 1, \qquad (E+1)^n + E = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{1.5}$$

Recently, (h,q)-Euler numbers are defined by

$$E_{0,q}^{(h)} = \frac{2}{1+q^h}, \qquad q^h \left(qE_q^{(h)} + 1\right)^n + E_q^{(h)} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{1.6}$$

with the usual convention about replacing $(E_q^{(h)})^n$ by $E_{n,q}^{(h)}$ (see [1–12]).

Note that $\lim_{q\to 1} E_{n,q}^{(h)} = E_n$.

For $\alpha \in \mathbb{N}$, the weight q-Euler numbers are also defined by

$$\widetilde{E}_{0,q}^{(\alpha)} = 1, \qquad q\left(q^\alpha \widetilde{E}_q^{(\alpha)} + 1\right)^n + \widetilde{E}_{n,q}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{1.7}$$

with the usual convention about replacing $(\widetilde{E}_q^{(\alpha)})^n$ by $\widetilde{E}_{n,q}^{(\alpha)}$ (see [4]).

The purpose of this paper is to present a systemic study of some families of higher-order q-Euler numbers and polynomials with weight α. In particular, by using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give a new concept of q-Euler numbers and polynomials with weight α.

2. Higher-Order q-Euler Numbers and Polynomials with Weight α

For $h \in \mathbb{Z}$, $\alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, let us consider the expansion of higher-order q-Euler polynomials with weight α as follows:

$$\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \quad (2.1)$$

From (1.2) and (2.1), we note that:

$$\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x) = \frac{[2]_q^k}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+q^{\alpha l+h}) \cdots (1+q^{\alpha l+h-k+1})}. \quad (2.2)$$

In the special case, $x=0$, $\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid 0) = \widetilde{E}_{n,q}^{(\alpha)}(h,k)$ are called the higher-order q-Euler numbers with weight α.

By (2.1), we get

$$\widetilde{E}_{n,q}^{(\alpha)}(h,k) = (q^\alpha - 1)\widetilde{E}_{n+1,q}^{(\alpha)}(h-\alpha,k) + \widetilde{E}_{n,q}^{(\alpha)}(h-\alpha,k). \quad (2.3)$$

From (2.1) and (2.2), we have

$$\begin{aligned}
\widetilde{E}_{0,q}^{(\alpha)}(m\alpha, k+1) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k+1}(m\alpha-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) \\
&= \sum_{l=0}^{m} \binom{m}{l} (q^\alpha-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1+\cdots+x_{k+1}]_{q^\alpha}^l q^{-\sum_{j=1}^{k+1} j x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) \\
&= \sum_{l=0}^{m} \binom{m}{l} (q^\alpha-1)^l \widetilde{E}_{l,q}^{(\alpha)}(0,k+1) \\
&= \frac{[2]_q^{k+1}}{(1+q^{\alpha m})(1+q^{\alpha m-1}) \cdots (1+q^{\alpha m-k})}.
\end{aligned} \quad (2.4)$$

From (2.1), we can derive the following equation:

$$\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i+j}q^{(h-\alpha-1)x_1+\cdots+(h-\alpha-k)x_k}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)$$

$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i}q^{(h-1)x_1+\cdots+(h-k)x_k}q^{\alpha(x_1+\cdots+x_k)(i-1)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \quad (2.5)$$

$$=\sum_{j=0}^{i-1}(q^{\alpha}-1)^{j}\binom{i-1}{j}\widetilde{E}_{n-i+j,q}^{(\alpha)}(h,k).$$

By (2.1), (2.2), (2.3), and (2.4), we see that

$$\sum_{j=0}^{i}(q^{\alpha}-1)^{j}\binom{i}{j}\widetilde{E}_{n-i+j,q}^{(\alpha)}(h-\alpha,k)=\sum_{j=0}^{i-1}(q^{\alpha}-1)^{j}\binom{i-1}{j}\widetilde{E}_{n-i+j,q}^{(\alpha)}(h,k). \quad (2.6)$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $\alpha,k\in\mathbb{N}$ and $n,i\in\mathbb{Z}_+$, one has*

$$\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\widetilde{E}_{n-i+j,q}^{(\alpha)}(h-\alpha,k)=\sum_{j=0}^{i-1}(q^{\alpha}-1)^{j}\binom{i-1}{j}\widetilde{E}_{n-i+j,q}^{(\alpha)}(h,k). \quad (2.7)$$

By simple calculation, we easily see that

$$\sum_{j=0}^{m}\binom{m}{j}(q^{\alpha}-1)^{j}\widetilde{E}_{j,q}^{(\alpha)}(0,k)=\frac{[2]_q^k}{(1+q^{\alpha m})(1+q^{\alpha m-1})\cdots(1+q^{\alpha m-k+1})}. \quad (2.8)$$

3. Polynomials $\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)$

We now consider the polynomials $\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)$ (in q^x) by

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{-times}}[x+x_1+\cdots+x_k]_{q^{\alpha}}^{n}q^{-\sum_{j=1}^{k}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \quad (3.1)$$

By (3.1), we get

$$(q^{\alpha}-1)^{n}\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)=[2]_q^k\sum_{l=0}^{n}\binom{n}{l}q^{\alpha l x}(-1)^{n-l}\frac{1}{(1+q^{\alpha l})\cdots(1+q^{\alpha l-k+1})}. \quad (3.2)$$

From (3.1) and (3.2), we can derive the following equation:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k}(\alpha n-j)x_j+\alpha nx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=\sum_{j=0}^{n}\binom{n}{j}[\alpha]_q^j(q-1)^j\widetilde{E}_{j,q}^{(\alpha)}(0,k\mid x),$$
$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k}(\alpha n-j)x_j+\alpha nx}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=\frac{[2]_q^k q^{\alpha nx}}{(1+q^{\alpha n})\cdots(1+q^{\alpha n-k+1})}. \tag{3.3}$$

Therefore, by (3.2) and (3.3), we obtain the following theorem.

Theorem 3.1. *For $\alpha\in\mathbb{N}$ and $n,k\in\mathbb{Z}_+$, one has*

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)=\frac{[2]_q^k}{[\alpha]_q^n(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha lx}\frac{1}{(-q^{\alpha l-k+1}:q)_k},$$
$$\sum_{l=0}^{n}\binom{n}{l}[\alpha]_q^l(q-1)^l\widetilde{E}_{l,q}^{(\alpha)}(0,k\mid x)=\frac{q^{\alpha nx}[2]_q^k}{(-q^{\alpha n-k+1}:q)_k}, \tag{3.4}$$

where $(a:q)_0=1$ and $(a:q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})$.

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. Then we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[x+\sum_{j=1}^{k}x_j\right]_{q^\alpha}^n q^{-\sum_{j=1}^{k}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)$$
$$=\frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\ldots,a_k=0}^{d-1}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j} \tag{3.5}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+\sum_{j=1}^{k}a_j}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^n q^{-d\sum_{j=1}^{k}jx_j}d\mu_{-q^d}(x_1)\cdots d\mu_{-q^d}(x_k).$$

Thus, by (3.5), we obtain the following theorem.

Theorem 3.2. *For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, one has*

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)=\frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\ldots,a_k=0}^{d-1}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\widetilde{E}_{n,q^d}^{(\alpha)}\left(0,k\mid\frac{x+a_1+\cdots+a_k}{d}\right). \tag{3.6}$$

Moreover,

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid dx)=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\dots,a_k=0}^{d-1}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\widetilde{E}_{n,q^d}^{(\alpha)}\left(0,k\mid x+\frac{a_1+\cdots+a_k}{d}\right). \tag{3.7}$$

By (3.1), we get

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x)=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-l}q^{\alpha l x}\widetilde{E}_{l,q}^{(\alpha)}(0,k), \tag{3.8}$$

where $\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid 0)=\widetilde{E}_{n,q}^{(\alpha)}(0,k)$.

Thus, we note that

$$\widetilde{E}_{n,q}^{(\alpha)}(0,k\mid x+y)=\sum_{l=0}^{n}\binom{n}{l}[y]_{q^{\alpha}}^{n-l}q^{\alpha l y}\widetilde{E}_{l,q}^{(\alpha)}(0,k\mid x). \tag{3.9}$$

4. Polynomials $\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)$

Let us define polynomials $\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)$ as follows:

$$\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)=\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}q^{x_1(h-1)}d\mu_{-q}(x_1). \tag{4.1}$$

From (4.1), we have

$$\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)=\frac{[2]_q}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{\alpha l x}\frac{1}{(1+q^{\alpha l+h})}. \tag{4.2}$$

By the calculation of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned}
&q^{\alpha x}\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}q^{x_1(h-1)}d\mu_{-q}(x_1)\\
&\quad=(q^{\alpha}-1)\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n+1}q^{x_1(h-\alpha-1)}d\mu_{-q}(x_1)+\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}q^{x_1(h-\alpha-1)}d\mu_{-q}(x_1).
\end{aligned} \tag{4.3}$$

Thus, by (4.3), we obtain the following theorem.

Theorem 4.1. *For $\alpha\in\mathbb{N}$ and $h\in\mathbb{Z}$, one has*

$$q^{\alpha x}\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)=(q^{\alpha}-1)\widetilde{E}_{n+1,q}^{(\alpha)}(h-\alpha-1,1\mid x)+\widetilde{E}_{n,q}^{(\alpha)}(h-\alpha-1,1\mid x). \tag{4.4}$$

It is easy to show that

$$\begin{aligned}\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)&=\int_{\mathbb{Z}_p}[x+x_1]_{q^\alpha}^n q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha l x}\int_{\mathbb{Z}_p}[x_1]_{q^\alpha}^{l}q^{x_1(h-1)}d\mu_{-q}(x_1)\\&=\sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha l x}\widetilde{E}_{l,q}^{(\alpha)}(h,1)\\&=\left(q^{\alpha x}\widetilde{E}_{q}^{(\alpha)}(h,1)+[x]_{q^\alpha}\right)^n,\quad\text{for } n\geq 1,\end{aligned}\tag{4.5}$$

with the usual convention about replacing $(\widetilde{E}_{q}^{(\alpha)}(h,1))^n$ by $\widetilde{E}_{n,q}^{(\alpha)}(h,1)$.

From $qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0)$, we have

$$q^h\int_{\mathbb{Z}_p}[x+x_1+1]_{q^\alpha}^n q^{x_1(h-1)}d\mu_{-q}(x_1)+\int_{\mathbb{Z}_p}[x+x_1]_{q^\alpha}^n q^{x_1(h-1)}d\mu_{-q}(x_1)=[2]_q[x]_{q^\alpha}^n.\tag{4.6}$$

By (4.3) and (4.6), we get

$$q^h\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x+1)+\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid x)=[2]_q[x]_{q^\alpha}^n.\tag{4.7}$$

For $x=0$ in (4.7), we have

$$q^h\widetilde{E}_{n,q}^{(\alpha)}(h,1\mid 1)+\widetilde{E}_{n,q}^{(\alpha)}(h,1)=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases}\tag{4.8}$$

Therefore, by (4.8), we obtain the following theorem.

Theorem 4.2. *For $h\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$, one has*

$$q^h\left(q^{\alpha}\widetilde{E}_{q}^{(\alpha)}(h,1)+1\right)^n+\widetilde{E}_{n,q}^{(\alpha)}(h,1)=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}\tag{4.9}$$

with the usual convention about replacing $(\widetilde{E}_{q}^{(\alpha)}(h,1))^n$ by $\widetilde{E}_{n,q}^{(\alpha)}(h,1)$.

From the fermionic p-adic q-integral on $\mathbb{Z}_p$, we easily get

$$\widetilde{E}_{0,q}^{(\alpha)}(h,1)=\int_{\mathbb{Z}_p}q^{x_1(h-1)}d\mu_{-q}(x_1)=\frac{[2]_q}{[2]_{q^h}}.\tag{4.10}$$

By (4.1), we see that

$$
\begin{aligned}
\widetilde{E}_{n,q^{-1}}^{(\alpha)}(h,1 \mid 1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^{n} q^{-x_1(h-1)} d\mu_{-q^{-1}}(x_1) \\
&= (-1)^n q^{\alpha n+h-1} \frac{[2]_q}{(1-q^{\alpha})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l+h}} \\
&= (-1)^n q^{\alpha n+h-1} \widetilde{E}_{n,q^{-1}}^{(\alpha)}(h,1 \mid x).
\end{aligned} \tag{4.11}
$$

Therefore, by (4.11), we obtain the following theorem.

Theorem 4.3. *For $\alpha \in \mathbb{N}$, $h \in \mathbb{Z}$, and $n \in \mathbb{Z}_+$, one has*

$$
\widetilde{E}_{n,q^{-1}}^{(\alpha)}(h,1 \mid 1-x) = (-1)^n q^{\alpha n+h-1} \widetilde{E}_{n,q}^{(\alpha)}(h,1 \mid x). \tag{4.12}
$$

In particular, for $x = 1$, one gets

$$
\begin{aligned}
\widetilde{E}_{n,q}^{(\alpha)}(h,1) &= (-1)^n q^{\alpha n+h-1} \widetilde{E}_{n,q}^{(\alpha)}(h,1 \mid 1) \\
&= (-1)^{n-1} q^{\alpha n-1} \widetilde{E}_{n,q}^{(\alpha)}(h,1) \quad \textit{if } n \geq 1.
\end{aligned} \tag{4.13}
$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then one has

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x_1(h-1)} [x+x_1]_{q^{\alpha}}^{n} d\mu_{-q}(x_1) \\
&\quad = \frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \int_{\mathbb{Z}_p} \left[\frac{x+a}{d} + x_1\right]_{q^{\alpha d}}^{n} q^{x_1(h-1)d} d\mu_{-q^d}(x_1).
\end{aligned} \tag{4.14}
$$

Therefore, by (4.14), we obtain the following theorem.

Theorem 4.4 (Multiplication formula). *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$
\widetilde{E}_{n,q}^{(\alpha)}(h,1 \mid x) = \frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \widetilde{E}_{n,q^d}^{(\alpha)}\left(h,1 \mid \frac{x+a}{d}\right). \tag{4.15}
$$

5. Polynomials $\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x)$ and $k=h$

In (2.1), we know that

$$\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k+x]_{q^\alpha}^{n} q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \tag{5.1}$$

Thus, we get

$$\begin{aligned}
(q^\alpha-1)^n\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x) &= [2]_q^k\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}\frac{q^{\alpha l x}}{(1+q^{\alpha l+h})\cdots(1+q^{\alpha l+h-k+1})},\\
q^h\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_k]_{q^\alpha}^{n} q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&= -\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n} q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&\quad + [2]_q\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_2+\cdots+x_k]_{q^\alpha}^{n} q^{(h-2)x_2+\cdots+(h-k)x_k} d\mu_{-q}(x_2)\cdots d\mu_{-q}(x_k).
\end{aligned} \tag{5.2}$$

Therefore, by (2.1) and (5.2), we obtain the following theorem.

Theorem 5.1. *For $h \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has*

$$q^h\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x+1) + \widetilde{E}_{n,q}^{(\alpha)}(h,k \mid x) = [2]_q\widetilde{E}_{n,q}^{(\alpha)}(h-1,k-1 \mid x). \tag{5.3}$$

Note that

$$\begin{aligned}
q^{\alpha x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n} q^{hx_1+(h-1)x_2+\cdots+(h+1-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&= (q^\alpha-1)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n+1} q^{(h-\alpha)x_1+\cdots+(h+1-\alpha-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&\quad + \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^{n} q^{(h-\alpha)x_1+\cdots+(h+1-\alpha-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&= (q^\alpha-1)\widetilde{E}_{n+1,q}^{(\alpha)}(h+1-\alpha,k \mid x) + \widetilde{E}_{n,q}^{(\alpha)}(h+1-\alpha,k \mid x).
\end{aligned} \tag{5.4}$$

Therefore, by (5.4), we obtain the following theorem.

Theorem 5.2. *For $n \in \mathbb{Z}_+$, one has*

$$q^{\alpha x}\widetilde{E}_{n,q}^{(\alpha)}(h+1,k \mid x) = \left(q^{\alpha}-1\right)\widetilde{E}_{n+1,q}^{(\alpha)}(h+1-\alpha,k \mid x) + \widetilde{E}_{n,q}^{(\alpha)}(h+1-\alpha,k \mid x). \tag{5.5}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[x+\sum_{j=1}^{k}x_j\right]_{q^{\alpha}}^{n} q^{\sum_{j=1}^{k}(h-j)x_j}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
&\quad=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\ldots,a_k=0}^{d-1} q^{h\sum_{j=1}^{k}a_j-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\\
&\qquad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+\sum_{j=1}^{k}a_j}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^{n} q^{d\sum_{j=1}^{k}(h-j)x_j}\,d\mu_{-q^{d}}(x_1)\cdots d\mu_{-q^{d}}(x_k).
\end{aligned} \tag{5.6}$$

Therefore, by (5.6), we obtain the following theorem.

Theorem 5.3. *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, one has*

$$\begin{aligned}
&\widetilde{E}_{n,q}^{(\alpha)}(h,k \mid dx)\\
&\quad=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\ldots,a_k=0}^{d-1} q^{h\sum_{j=1}^{k}a_j-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\widetilde{E}_{n,q^{d}}^{(\alpha)}\left(h,k \mid x+\frac{a_1+\cdots+a_k}{d}\right).
\end{aligned} \tag{5.7}$$

Let $\widetilde{E}_{n,q}^{(\alpha)}(k,k \mid x) = \widetilde{E}_{n,q}^{(\alpha)}(k \mid x)$. Then we get

$$\begin{aligned}
&\left(q^{\alpha}-1\right)^{n}\widetilde{E}_{n,q}^{(\alpha)}(k \mid x),\\
&\quad=\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}q^{\alpha l x}\frac{[2]_q^{k}}{\left(1+q^{\alpha l+k}\right)\cdots\left(1+q^{\alpha l+1}\right)}\\
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[k-x+x_1+\cdots+x_k]_{q^{-\alpha}}^{n}q^{-(k-1)x_1-\cdots-(k-k)x_k}\,d\mu_{-q^{-1}}(x_1)\cdots d\mu_{-q^{-1}}(x_k)\\
&\quad=\frac{q^{\alpha\binom{k+1}{2}-k}}{\left(1-q^{-\alpha}\right)^{n}}[2]_q^{k}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{\alpha l x}\frac{1}{\left(1+q^{\alpha l+1}\right)\cdots\left(1+q^{\alpha l+k}\right)}\\
&\quad=(-1)^{n}q^{n\alpha}q^{\alpha\binom{k+1}{2}-k}\frac{[2]_q^{k}}{\left(1-q^{\alpha}\right)^{n}}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^{l}q^{\alpha l x}}{\left(1+q^{\alpha l+1}\right)\cdots\left(1+q^{\alpha l+k}\right)}\\
&\quad=(-1)^{n}q^{\alpha\left(n+\binom{k+1}{2}\right)-k}\widetilde{E}_{n,q}^{(\alpha)}(k \mid x).
\end{aligned} \tag{5.8}$$

Therefore, by (5.8), we obtain the following theorem.

Theorem 5.4. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{E}_{n,q^{-1}}^{(\alpha)}(k \mid k-x) = (-1)^n q^{\alpha\left(n+\binom{k+1}{2}\right)-k} \widetilde{E}_{n,q}^{(\alpha)}(k \mid x). \tag{5.9}$$

Let $x = k$ in Theorem 5.4. Then we see that

$$\widetilde{E}_{n,q^{-1}}^{(\alpha)}(k \mid 0) = (-1)^n q^{\alpha\left(n+\binom{k+1}{2}\right)-k} \widetilde{E}_{n,q}^{(\alpha)}(k \mid k). \tag{5.10}$$

From (4.6) and Theorem 5.1, we note that

$$q^k \widetilde{E}_{n,q}^{(\alpha)}(k \mid x+1) + \widetilde{E}_{n,q}^{(\alpha)}(k \mid x) = [2]_q \widetilde{E}_{n,q}^{(\alpha)}(k-1 \mid x). \tag{5.11}$$

It is easy to show that

$$(q^\alpha - 1)^n \widetilde{E}_{n,q}^{(\alpha)}(k \mid 0) = \sum_{l=0}^{n} \binom{n}{1} (-1)^{l+n} \frac{[2]_q^k}{\left(1+q^{\alpha l+1}\right)\cdots\left(1+q^{\alpha l+k}\right)}. \tag{5.12}$$

By simple calculation, we get

$$\begin{aligned} &\sum_{l=0}^{n} \binom{n}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^l q^{\sum_{l=1}^{k}(k-l)x_l} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ &\quad = \frac{[2]_q^k}{\left(1+q^{\alpha n+k}\right)\left(1+q^{\alpha n+k-1}\right)\cdots\left(1+q^{\alpha n+1}\right)}. \end{aligned} \tag{5.13}$$

From (5.13), we note that

$$\begin{aligned} &\sum_{l=0}^{n} \binom{n}{l} (q^\alpha - 1)^l \widetilde{E}_{l,q}^{(\alpha)}(k \mid 0) = \frac{[2]_q^k}{\left(1+q^{\alpha n+k}\right)\left(1+q^{\alpha n+k-1}\right)\cdots\left(1+q^{\alpha n+1}\right)}, \\ &\widetilde{E}_{n,q}^{(\alpha)}(k \mid x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(k-1)x_1+\cdots+(k-k)x_k} d\mu_{-q}(x_1) \\ &\qquad\qquad \cdots d\mu_{-q}(x_k) \\ &\qquad = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} \widetilde{E}_{l,q}^{(\alpha)}(k \mid 0)[x]_{q^\alpha}^{n-l} \\ &\qquad = \left(q^{x\alpha} \widetilde{E}_q^{(\alpha)}(k \mid 0) + [x]_{q^\alpha}\right)^n \quad \text{for } n \in \mathbb{Z}_+, \end{aligned} \tag{5.14}$$

with the usual convention about replacing $(\widetilde{E}_q^{(\alpha)}(k \mid 0))^n$ by $\widetilde{E}_{n,q}^{(\alpha)}(k \mid 0)$.

Put $x = 0$ in (5.11); we get

$$q^k \widetilde{E}_{n,q}^{(\alpha)}(k \mid 1) + \widetilde{E}_{n,q}^{(\alpha)}(k \mid 0) = [2]_q \widetilde{E}_{n,q}^{(\alpha)}(k-1 \mid 0). \tag{5.15}$$

Thus, we have

$$q^k \left(q^\alpha \widetilde{E}_q^{(\alpha)}(k \mid 0) + 1\right)^n + \widetilde{E}_{n,q}^{(\alpha)}(k \mid 0) = [2]_q \widetilde{E}_{n,q}^{(\alpha)}(k-1 \mid 0), \tag{5.16}$$

with the usual convention about replacing $(\widetilde{E}_q^{(\alpha)}(k \mid 0))^n$ by $\widetilde{E}_{n,q}^{(\alpha)}(k \mid 0)$.

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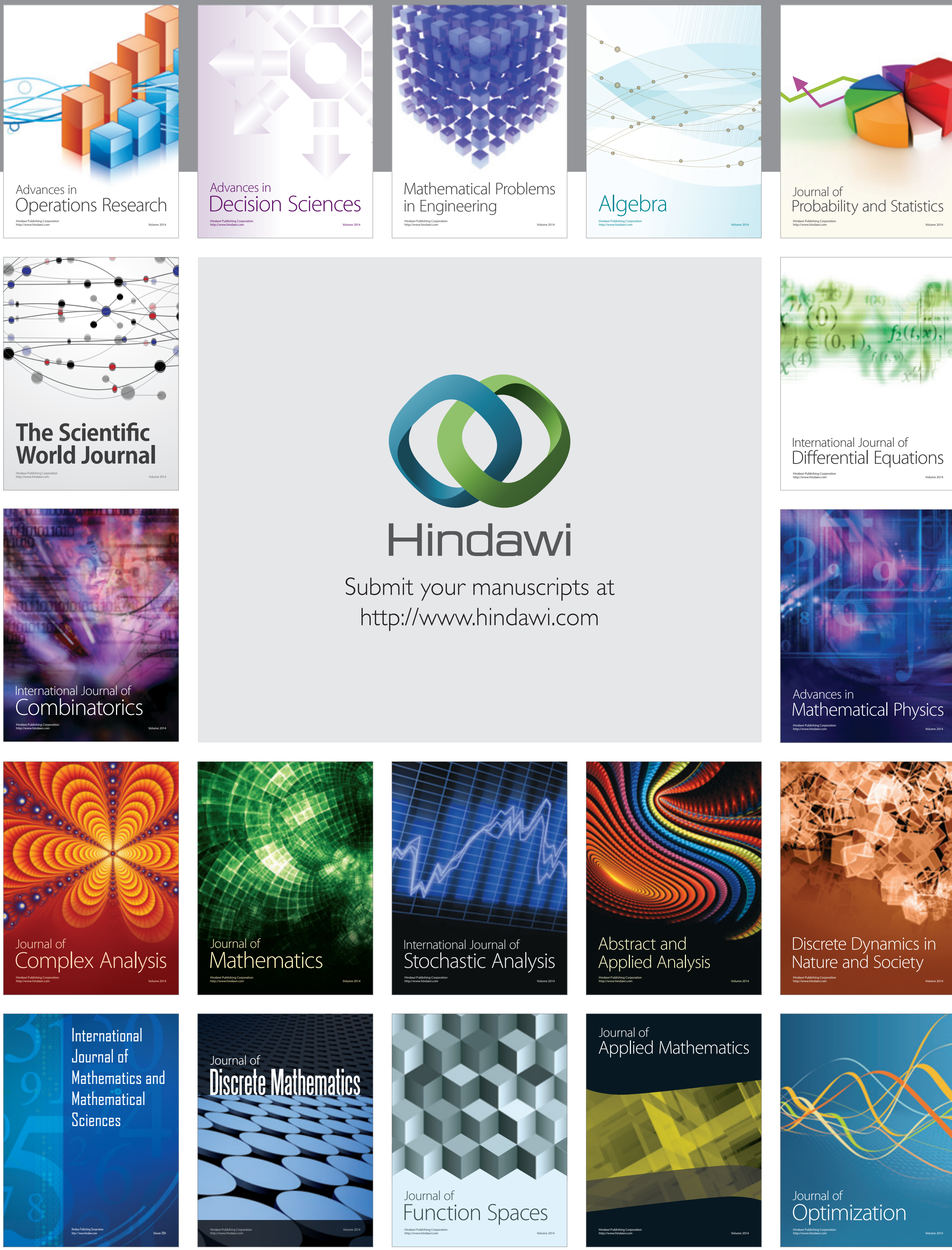

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