Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 360583, 6 pages
doi:10.1155/2011/360583

Research Article

A Note on Ergodicity of Systems with the Asymptotic Average Shadowing Property

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Received 23 July 2011; Revised 22 September 2011; Accepted 5 October 2011

Academic Editor: Victor S. Kozyakin

We prove that if a continuous, Lyapunov stable map f from a compact metric space X into itself is topologically transitive and has the asymptotic average shadowing property, then X is consisting of one point. As an application, we prove that the identity map $i_X : X \to X$ does not have the asymptotic average shadowing property, where X is a compact metric space with at least two points.

1. Introduction

It is well known that the pseudoorbit tracing property is one of the most important notions in dynamical systems (see [1–4]). Blank [5] introduced the notion of the average shadowing property to characterize Anosov diffeomorphisms (see [6]). Yang [7] proved that if a continuous map $f : X \to X$ of a compact metric space X has the pseudoorbit tracing property and is chain transitive, then it is topologically ergodic. Gu and Guo [8] discussed the relation between the average shadowing property and topological ergodicity for flows and showed that a Lyapunov stable flow with the average shadowing property is topologically ergodic. In a recent work, the author [9] introduced the notion of the asymptotic average shadowing property and found that the asymptotic average shadowing property is closely related with transitivity. More recently, Gu [10] showed that every Lyapunov stable map from a compact metric space onto itself is topologically ergodic, provided it has the asymptotic average shadowing property. However, in this paper, we will show that a continuous, Lyapunov stable map f from a compact metric space X into itself is topologically transitive and has the asymptotic average shadowing property, then X is consisting of one point. As an application, it is shown that the identity map $i_X : X \to X$ does not have the asymptotic average shadowing property, where X is a compact metric space with at least two points. Moreover, we point out that the proof of [10, Theorem 3.1] cannot be true.

The organization of this paper is as follows. In Section 2, we recall some concepts and useful lemmas. Main results are established in Section 3.

2. Preliminaries

Firstly, we complete some notations and recall some concepts.

In this paper, by a dynamical system we mean a pair (X, f), where $f : X \to X$ is a continuous map and X is a compact metric space with metric d. For $x \in X$, we write $O(x, f) = \{f^n(x)\}_{n=0}^{\infty}$. Let $|A|$ denote the cardinality of a set A and $Z^+ = \{0, 1, \ldots\}$.

A subset $S \subset Z^+$ is called the positive upper density if

$$\limsup_{k \to \infty} \frac{1}{k+1} \left|\{0 \le j \le k : j \in S\}\right| > 0. \tag{2.1}$$

For any two nonempty sets $U, V \subset X$, we write $N_f(U, V) = \{n \in Z^+ : U \bigcap f^{-n}(V) \ne \emptyset\}$. Obviously, we have $N_f(U, V) = \{n \in Z^+ : f^n(U) \cap V \ne \emptyset\}$.

A map $f : X \to X$ is topologically transitive if $N_f(U, V)$ is nonempty, for any nonempty open sets $U, V \subset X$.

A map $f : X \to X$ is topologically ergodic if $N_f(U, V)$ has positive upper density, for any nonempty open sets $U, V \subset X$.

A map $f : X \to X$ is topologically weak mixing if $f \times f$ is topologically transitive.

Let (X, f) be a dynamical system. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a δ pseudoorbit of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \ge 0$. A sequence $\{x_i\}_{i=0}^{\infty}$ is said to be ε traced by some point $z \in X$ if $d(f^i(z), x_i) < \varepsilon$ for every $i \ge 0$. A point $x \in X$ is said to be a stable point of f if, for any $\varepsilon > 0$, there exists $\delta = \delta(x) > 0$ satisfying that $d(f^n(x), f^n(y)) < \varepsilon$ for any $y \in X$ with $d(x, y) < \delta$ and any $n \ge 0$.

A map $f : X \to X$ is called Lyapunov stable, if every point of X is a stable point of f.

A map $f : X \to X$ is said to have sensitive dependence on initial conditions if every point of X is not a stable point of f.

A map $f : X \to X$ is said to have the pseudoorbit tracing property, if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ pseudoorbit of f can be ε traced by some point in X.

A sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a δ average pseudoorbit of f if there exists an integer $N = N(\delta) > 0$ such that for any integer $n \ge N$ and every integer $k \ge 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d\left(f(x_{i+k}), x_{i+k+1}\right) < \delta. \tag{2.2}$$

A map $f : X \to X$ is said to have the average shadowing property, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ average pseudoorbit $\{x_i\}_{i=0}^{\infty}$ of f is ε shadowed in average by some point z in X, that is,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i(z), x_i\right) < \varepsilon. \tag{2.3}$$

A sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called an asymptotic average pseudoorbit of f if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f(x_i), x_{i+1}\right) = 0. \tag{2.4}$$

For a given dynamical system (X, f), a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is said to be asymptotically shadowed in average by some point $z \in X$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i(z), x_i\right) = 0. \tag{2.5}$$

A map $f : X \to X$ is said to have the asymptotic average shadowing property, if every asymptotic average pseudoorbit of f is asymptotically shadowed in average by some point in X.

We know from [9] that the pseudoorbit tracing property does not imply the asymptotic average shadowing property and the asymptotic average shadowing property does not imply the pseudoorbit tracing property.

3. Main Results

The following lemmas will be used in the proof of Theorem 3.5.

Lemma 3.1 (see [10]). *Let X be a compact metric space containing at least two points and $f : X \to X$ a continuous map. If f is topologically weakly mixing, then f has sensitive dependence on initial conditions.*

Let (X, f) and (X', g) be dynamical systems and $X \times X'$ the product space with metric $d''((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}$, where d and d' are, respectively, metrics for X and X'. Let the map $f \times g$ be defined by $(f \times g)(x, x') = (f(x), g(x'))(x \in X, x' \in X')$.

Lemma 3.2. *Let (X, f) be a dynamical system. Then, f has the asymptotic average shadowing property if and only if $f \times f$ has the asymptotic average shadowing property.*

Proof. Suppose that f has the asymptotic average shadowing property. Let $\{(x_i, y_i)\}_{i=0}^{\infty}$ be an asymptotic average pseudoorbit of $f \times f$, that is,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d''\left((f \times f)(x_i, y_i), (x_{i+1}, y_{i+1})\right) = 0. \tag{3.1}$$

This implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f(x_i), x_{i+1}\right) = 0,$$
$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f(y_i), y_{i+1}\right) = 0. \tag{3.2}$$

So $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are asymptotic average pseudoorbits of f. Hence, there are two points $z_1, z_2 \in X$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d\left(f^i(z_1), x_i\right) = 0, \tag{3.3}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d\left(f^i(z_2), y_i\right) = 0. \tag{3.4}$$

By Lemma 2.3 in [9] and (3.3), there is a set $J_0 \subset Z^+$ of zero density such that

$$\lim_{j\to\infty} d\left(f^j(z_1), x_j\right) = 0, \tag{3.5}$$

where $j \notin J_0$. Similarly, there is a set $J_1 \subset Z^+$ of zero density such that

$$\lim_{j\to\infty} d\left(f^j(z_2), y_j\right) = 0, \tag{3.6}$$

where $j \notin J_1$. Let $J = J_0 \bigcup J_1$. Then, J is a subset of Z^+ of zero density and

$$\lim_{j\to\infty} d''\left((f\times f)^j(z_1, z_2), (x_j, y_j)\right) = 0, \tag{3.7}$$

where $j \notin J$. So, by Lemma 2.3 in [9], we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d''\left((f\times f)^j((z_1, z_2)), (x_j, y_j)\right) = 0. \tag{3.8}$$

Consequently, $f \times f$ has the asymptotic average shadowing property.

Similarly, one can easily prove that if $f \times f$ has the asymptotic average shadowing property, then so does f. Thus, the proof is ended. □

Lemma 3.3. *A topologically transitive map from a compact metric space into itself is a surjective map.*

Proof. By the definition, the proof is easy and is omitted here. □

We do not know whether a continuous self-map of a compact metric space with the asymptotic average shadowing property is topologically transitive. However, we have the following lemma which is from [11].

Lemma 3.4. *Let (X, d) be a compact metric space. If $f : X \to X$ is a equicontinuous surjection and has the asymptotic average shadowing property, then f is topologically transitive.*

In [10], Gu proved that every Lyapunov stable map from a compact metric space onto itself is topologically ergodic, provided it has the asymptotic average shadowing property. However, we obtain the following theorem.

Theorem 3.5. *If a Lyapunov stable map f from a compact metric space X into itself is topologically transitive and has the asymptotic average shadowing property, then X is consisting of one point.*

Proof. Let $f : X \to X$ be a Lyapunov stable map of a compact metric space X, and suppose that f is topologically transitive and has the asymptotic average shadowing property. Assuming that X consists of at least two points, we derive a contradiction. First of all, we note that every Lyapunov stable map is continuous by the definition and that f is a surjective map by Lemma 3.3 since f is topologically transitive. Since $f \times f$ also has the asymptotic average shadowing property by Lemma 3.2, $f \times f$ is topologically transitive by Lemma 3.4. Thus, f is topologically weakly mixing by definition. Hence, by Lemma 3.1, f has sensitive dependence on initial conditions, so that f is not Lyapunov stable. This is a contradiction. □

Remark 3.6. Theorem 3.5 shows that [10, Theorem 3.1] cannot be true. Indeed, let X be a compact metric space containing at least two points, and let $f : X \to X$ be a continuous Lyapunov stable map. Suppose that f has the asymptotic average shadowing property. If [10, Theorem 3.1] is correct, then f is topologically ergodic, so that f is topologically transitive by definition. Thus, by Theorem 3.5 proved above, X must be consisting one point. This is a contradiction, and, thus, [10, Theorem 3.1] cannot be true.

To prove Theorem 3.1 in [10], the author first construct a sequence $\{w_i\}_{i=0}^{\infty}$ as follows.

For any two nonempty open subsets $U, V \subset X$ and any $\varepsilon > 0$, choose $x \in U, y \in V$ with $\{z \in X : d(x,z) < \varepsilon\} \subset U$ and $\{z \in X : d(y,z) < \varepsilon\} \subset V$. Let $w_0 = x$, $w_1 = y$, $w_2 = x$, $w_3 = y$, $w_4 = x_{-1}$, $w_5 = x$, $w_6 = y_{-1}$, $w_7 = y$, $w_{2^k+j} = x_{-2^{k-1}+1-j}$ for $j \in \{0, 1, \ldots, 2^{k-1} - 1\}$, and $w_{2^k+j} = y_{-(2^{k-1}-1)+j-2^{k-1}}$ for $j \in \{2^{k-1}, 2^{k-1}+1, \ldots, 2^k - 1\}$, where $f(x_{-t}) = x_{-t+1}$ for every $t > 0$, $x_0 = x$ and $f(y_{-l}) = y_{-l+1}$ for every $l > 0$, $y_0 = y$. And then the author showed that the sequence $\{w_i\}_{i=0}^{\infty}$ is an asymptotic average pseudoorbit of f. Finally, the author defined $J_z = \{i : w_i \in \{z_{-2^{i-1}+1}, z_{-2^{i-1}+2}, \ldots, z_{-1}, z\}$ and $d(f^i(w), w_i) < \delta\}$ for $z \in \{x, y\}$ with the positive constant δ satisfying that for any $u, v \in X$, $d(u,v) < \delta$ implies $d(f^n(u), f^n(v)) < \varepsilon$ for every integer $n \geq 0$ and

$$J_m(y) = \{i \in J_y : w_i = y_{-m}\} \tag{3.9}$$

for every $m \geq 0$ and showed that there is an integer $m_0 \geq 0$ such that $\overline{D}(J_{m_0}(y)) > 0$ which leads to the fact that f is topologically ergodic, where $\overline{D}(J_{m_0}(y))$ is the upper density of the set J_{m_0}.

Remark 3.7. In fact, for a given integer $m \geq 0$ and any $n \geq m + 1$, let $k = [(\ln n)/(\ln 2)]$, where $[(\ln n)/(\ln 2)]$ is the greatest integer less than or equal to $\ln n / \ln 2$. Then, by the definition of the sequence $\{w_i\}_{i=0}^{\infty}$, $\overline{D}(J_m(y)) \leq ((k+1)/2^k)$, which implies $\overline{D}(J_m(y)) = 0$. Consequently, the proof of Theorem 3.1 from [10] is not correct.

As an application, we obtain the following theorem which follows from Theorem 3.1 in [12]. For completeness, we now give a different proof here.

Theorem 3.8. *Let (X, d) be a compact metric space with at least two points. Then the identity map $i_X : X \to X$ does not have the asymptotic average shadowing property.*

Proof. Clearly, the identity map i_X is Lyapunov stable. Assume that the identity map $i_X : X \to X$ has the asymptotic average shadowing property. By Lemma 3.4, the identity map i_X is topologically transitive. Since (X, d) is a compact metric space with at least two points, by

Theorem 3.5, the identity map i_X does not have the asymptotic average shadowing property. Thus, the proof is ended. □

Remark 3.9. Theorem 3.8 extends Example 5.1 from [9], and the proof of Theorem 3.8 is simpler than that of Example 5.1 from [9].

Acknowledgments

The authors thank the (unknown) referee of the paper for many valuable suggestions, in particular for giving a clear proof of Theorem 3.5 and a good description of Remark 3.6. This research was supported by the NSF of Guangdong Province (Grant no. 10452408801004217) and the Key Scientific and Technological Research Project of Science and Technology Department of Zhanjiang City (Grant no. 2010C3112005).

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