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Research Article

On the Dynamics of a Higher-Order Difference Equation

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This paper deals with the investigation of the following more general rational difference equation: $y_{n+1} = \alpha y_n/(\beta + \gamma \sum_{i=0}^k y_{n-(2i+1)}^p \prod_{i=0}^k y_{n-(2i+1)}), n = 0, 1, 2, \ldots$, where $\alpha, \beta, \gamma, p \in (0, \infty)$ with the initial conditions $x_0, x_{-1}, \ldots, x_{-2k}, x_{-2k-1} \in (0, \infty)$. We investigate the existence of the equilibrium points of the considered equation and then study their local and global stability. Also, some results related to the oscillation and the permanence of the considered equation have been presented.

1. Introduction

In this paper we investigate the global stability character and the oscillatory of the solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha y_n}{\beta + \gamma \sum_{i=0}^k y_{n-(2i+1)}^p \prod_{i=0}^k y_{n-(2i+1)}}, \quad n = 0, 1, 2, \dots,$$
(1.1)

where α , β , γ , $p \in (0, \infty)$ with the initial conditions x_0 , $x_{-1}, \dots, x_{-2k}, x_{-2k-1} \in (0, \infty)$. Also we study the permanence of (1.1). The importance of permanence for biological systems was thoroughly reviewed by Huston and Schmidtt [1].

In general, there are a lot of interest in studying the global attractivity, boundedness character, and periodicity of the solutions of nonlinear difference equations. In particular there are many papers that deal with the rational difference equations and that is because many researchers believe that the results about this type of difference equations are of

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paramount importance in their own right, and furthermore they believe that these results offer prototype towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one.

Kulenović and Ladas [2] presented some known results and derived several new ones on the global behavior of the difference equation $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$ and of its special cases. Elabbasy et al. [3–5] established the solutions form and then investigated the global stability and periodicity character of the obtained solutions of the following difference equations:

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \qquad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^k x_{n-i}}, \qquad x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$
 (1.2)

El-Metwally [6] gave some results about the global behavior of the solutions of the following more general rational difference equations

$$x_{n+1} = \frac{ax_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \dots x_{n-k_i}^{l_i} + bx_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \dots x_{n-r_j}^{s_j}}{cx_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \dots x_{n-k_i}^{l_i} + dx_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \dots x_{n-r_j}^{s_j}}, \qquad y_{n+1} = \frac{\alpha_0 y_n + \alpha_1 y_{n-1} + \dots + \alpha_t y_{n-t}}{\beta_0 y_n + \beta_1 y_{n-1} + \dots + \beta_t y_{n-t}}.$$

$$(1.3)$$

Çinar [7–9] obtained the solutions form of the difference equations $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$, $x_{n+1} = x_{n-1}/(-1 + x_n x_{n-1})$ and $x_{n+1} = a x_{n-1}/(1 + b x_n x_{n-1})$. Also, Cinar et al. [10] studied the existence and the convergence for the solutions of the difference equation $x_{n+1} = x_{n-3}/(-1 + x_n x_{n-1} x_{n-2} x_{n-3})$. Simsek et al. [11] obtained the solution of the difference equation $x_{n+1} = x_{n-3}/(1 + x_{n-1})$. In [12] Yalcinkaya got the solution form of the difference equation $x_{n+1} = x_{n-2}/(1 + x_{n-1})$. In [13] Stević studied the difference equation $x_{n+1} = x_{n-1}/(1 + x_n)$. Other related results on rational difference equations can be found in [14–19].

Let *I* be some interval of real numbers and let

$$f: I^{k+1} \longrightarrow I \tag{1.4}$$

be a continuously differentiable function. Then for every set of initial conditions x_{-k} , $x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots$$
 (1.5)

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1 (permanence). The difference equation (1.5) is said to be permanent if there exist numbers m and M with $0 < m \le M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that $m \le x_n \le M$ for all $n \ge N$.

Definition 1.2 (periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \ge -k$.

Definition 1.3 (semicycles). A positive semicycle of a sequence $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to the equilibrium point \overline{x} , with $l \geq -k$ and $m \leq \infty$ such that either l = -k or l > -k and $x_{l-1} < \overline{x}$; and, either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$. A negative semicycle of a sequence $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than the equilibrium point \overline{x} , with $l \geq -k$ and $m \leq \infty$ such that: either l = -k or l > -k and $x_{l-1} \geq \overline{x}$; and, either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

Definition 1.4 (oscillation). A sequence $\{x_n\}_{n=-k}^{\infty}$ is called nonoscillatory about the point \overline{x} if there is exists $N \ge -k$ such that either $x_n > \overline{x}$ for all $n \ge N$ or $x_n < \overline{x}$ for all $n \ge N$. Otherwise $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about \overline{x} .

Recall that the linearized equation of (1.5) about the equilibrium \overline{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}.$$
 (1.6)

2. Dynamics of (1.1)

The change of variables $y_n = (\beta/\gamma)^{1/(p+k+1)} x_n$ reduces (1.1) to the following difference equation

$$x_{n+1} = \frac{rx_n}{1 + \sum_{i=0}^k x_{n-(2i+1)}^p \prod_{i=0}^k x_{n-(2i+1)}}, \quad n = 0, 1, 2, \dots,$$
(2.1)

where $r = \alpha/\beta$.

In this section we study the local stability character and the global stability of the equilibrium points of the solutions of (2.1). Also we give some results about the oscillation and the permanence of (2.1).

Recall that the equilibrium point of (2.1) are given by

$$\overline{x} = \frac{r\overline{x}}{1 + (k+1)\overline{x}^{p+k+1}}. (2.2)$$

Then (2.1) has the equilibrium points $\overline{x} = 0$ and whenever r > 1, (2.1) possesses the unique equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.

The following theorem deals with the local stability of the equilibrium point $\bar{x} = 0$ of (2.1).

Theorem 2.1. *The following statements are true:*

- (i) if r < 1, then the equilibrium point $\overline{x} = 0$ of (2.1) is locally asymptotically stable,
- (ii) if r > 1, then the equilibrium point $\overline{x} = 0$ of (2.1) is a saddle point.

Proof. The linearized equation of (2.1) about $\overline{x} = 0$ is $u_{n+1} - ru_n = 0$. Then the associated eigenvalues are $\lambda = 0$ and $\lambda = r$. Then the proof is complete.

Theorem 2.2. Assume that r < 1, then the equilibrium point $\overline{x} = 0$ of (2.1) is globally asymptotically stable.

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). It was shown by Theorem 2.1 that the equilibrium point $\overline{x} = 0$ of (2.1) is locally asymptotically stable. So, it is suffices to show that

$$\lim_{n \to \infty} x_n = 0. \tag{2.3}$$

Now it follows from (2.1) that

$$x_{n+1} = \frac{rx_n}{1 + x_{n-1}^{p+1} x_{n-3} \cdots x_{n-2k+1} + x_{n-1} x_{n-3}^{p+1} \cdots x_{n-2k+1} + \cdots + x_{n-1} x_{n-3} \cdots x_{n-2k+1}^{p+1}}$$

$$\leq rx_n$$

$$< x_n.$$
(2.4)

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and this completes the proof.

Theorem 2.3. Assume that r > 1. Then every solution of (2.1) is either oscillatory or tends to the equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). Without loss of generality assume that $\{x_n\}_{n=-2k+1}^{\infty}$ is a nonoscillatory solution of (2.1), then it suffices to show that $\lim_{n\to\infty}x_n=\overline{x}$. Assume that $x_n\geq \overline{x}$ for $n\geq n_0$ (the case where $x_n\leq \overline{x}$ for $n\geq n_0$ is similar and will be omitted). It follows from (2.1) that

$$x_{n+1} = \frac{rx_n}{1 + x_{n-1}^{p+1} x_{n-3} \cdots x_{n-2k-1} + x_{n-1} x_{n-3}^{p+1} \cdots x_{n-2k-1} + \cdots + x_{n-1} x_{n-3} \cdots x_{n-2k-1}^{p+1}}$$

$$\leq x_n \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}}\right) = x_n.$$
(2.5)

Hence $\{x_n\}$ is monotonic for $n \ge n_0 + 2k + 1$, therefore it has a limit. Let $\lim_{n\to\infty} x_n = \mu$, and for the sake of contradiction, assume that $\mu > \overline{x}$. Then by taking the limit of both side of (2.1), we obtain $\mu = r\mu/(1 + (k+1)\mu^{p+k+1})$, which contradicts the hypothesis that $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$ is the only positive solution of (2.2).

Theorem 2.4. Assume that $\{x_n\}_{n=-2k+1}^{\infty}$ is a solution of (2.1) which is strictly oscillatory about the positive equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$ of (2.1). Then the extreme point in any semicycle occurs in one of the first 2(k+1) terms of the semicycle.

Proof. Assume that $\{x_n\}_{n=-2k-1}^{\infty}$ is a strictly oscillatory solution of (2.1). Let $N \geq 2k+2$ and let $\{x_N, x_{N+1}, \dots, x_M\}$ be a positive semicycle followed by the negative semicycle $\{x_M, x_{M+1}, \dots, x_M\}$. Now it follows from (2.1) that

$$x_{N+2k+2} - x_{N} = \frac{rx_{N+2k+1}}{1 + x_{N+2k}^{p+1} x_{N+2k-2} \cdots x_{N} + x_{N+2k} x_{N+2k-2}^{p+1} \cdots x_{N} + \cdots + x_{N+2k} \cdots x_{N}^{p+1}} - x_{N}$$

$$\leq x_{N+2k+1} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_{N}$$

$$= x_{N+2k+1} - x_{N}$$

$$= \frac{rx_{N+2k}}{1 + x_{N+2k-1}^{p+1} x_{N+2k-3} \cdots x_{N-1} + \cdots + x_{N+2k-1} \cdots x_{N-1}^{p+1}} - x_{N}$$

$$\leq x_{N+2k} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_{N}$$

$$= x_{N+2k} - x_{N} \leq x_{N+2k-1} - x_{N} \leq \cdots \leq x_{N+1} - x_{N} \leq x_{N} - x_{N} = 0.$$

$$(2.6)$$

Then $x_N \ge x_{N+2(k+1)}$ for all $N \ge 2(k+1)$. Similarly, we see from (2.1) that

$$x_{M+2k+2} - x_{M} = \frac{rx_{M+2k+1}}{1 + x_{M+2k}^{p+1} x_{M+2k-2} \cdots x_{M} + x_{M+2k} x_{M+2k-2}^{p+1} \cdots x_{M} + \cdots + x_{M+2k} \cdots x_{M}^{p+1}} - x_{M}$$

$$\geq x_{M+2k+1} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_{M}$$

$$= x_{M+2k+1} - x_{M}$$

$$= \frac{rx_{M+2k}}{1 + x_{M+2k-1}^{p+1} x_{M+2k-3} \cdots x_{M-1} + \cdots + x_{M+2k-1} \cdots x_{M-1}^{p+1}} - x_{M}$$

$$\geq x_{M+2k} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_{M}$$

$$= x_{M+2k} - x_{M} \leq x_{M+2k-1} - x_{M} \geq \cdots \geq x_{M+1} - x_{M} \geq x_{M} - x_{M} = 0.$$

$$(2.7)$$

Therefore $x_{M+2(k+1)} \ge x_M$ for all $M \ge 2(k+1)$. The proof is so complete.

Theorem 2.5. *Equation* (2.1) *is permanent.*

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). There are two cases to consider: (i) $\{x_n\}_{n=-2k+1}^{\infty}$ is a nonoscillatory solution of (2.1). Then it follows from Theorem 2.3 that

$$\lim_{n \to \infty} x_n = \overline{x},\tag{2.8}$$

that is there is a sufficiently large positive integer N such that $|x_n - \overline{x}| < \varepsilon$ for all $n \ge N$ and for some $\varepsilon > 0$. So, $\overline{x} - \varepsilon < x_n < \overline{x} + \varepsilon$, this means that there are two positive real numbers, say C and D, such that

$$C \le x_n \le D. \tag{2.9}$$

(ii) $\{x_n\}_{n=-2k+1}^{\infty}$ is strictly oscillatory about $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.

Now let $\{x_{s+1}, x_{s+2}, \ldots, x_t\}$ be a positive semicycle followed by the negative semicycle $\{x_{t+1}, x_{t+2}, \ldots, x_u\}$. If x_V and x_W are the extreme values in these positive and negative semicycle, respectively, with the smallest possible indices V and W, then by Theorem 2.4 we see that $V - s \le 2(k+1)$ and $W - u \le 2(k+1)$. Now for any positive indices μ and U with $\mu < L$, it follows from (2.1) for $n = \mu, \mu + 1, \ldots, L - 1$ that

$$x_{L} = x_{L-1} \left(\frac{r}{1 + x_{L-2}^{p+1} x_{L-4} \cdots x_{L-2k-2} + x_{L-2} x_{L-4}^{p+1} \cdots x_{L-2k-2} + \cdots + x_{L-2} x_{L-4} \cdots x_{L-2k-2}^{p+1}} \right)$$

$$= \frac{r^{2} x_{L-2}}{\left(1 + x_{L-3}^{p+1} \cdots x_{L-2k-3} + \cdots + x_{L-3} \cdots x_{L-2k-3}^{p+1} \right) \left(1 + x_{L-2}^{p+1} \cdots x_{L-2k-2} + \cdots + x_{L-2} \cdots x_{L-2k-2}^{p+1} \right)}$$

$$\vdots$$

$$= x_{L-\zeta} r^{\zeta} \prod_{\eta=1}^{\zeta} \left(\frac{1}{1 + \sum_{i=0}^{k} x_{L-(2i+1)-\eta}^{p} \prod_{i=0}^{k} x_{L-(2i+1)-\eta}} \right)$$

$$= x_{\mu} r^{L-\mu} \prod_{\eta=\mu}^{L-1} \left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}} \right). \tag{2.10}$$

Therefor for V = L and $s = \mu$ we obtain

$$x_{V} = x_{s} r^{V-s} \prod_{\eta=s}^{V-1} \left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}} \right)$$

$$\leq \overline{x} r^{2k+1} = H.$$
(2.11)

Again whenever W = L and $\mu = t$, we see that

$$x_{W} = x_{t} r^{W-t} \prod_{\eta=t}^{W-1} \left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}} \right)$$

$$\geq \overline{x} r^{W-t} \prod_{\eta=t}^{W-1} \left(\frac{1}{1 + (k+1)H^{p+k+1}} \right)$$

$$= \overline{x} r^{W-t} \left(\frac{1}{1 + (k+1)H^{p+k+1}} \right)^{W-t-1}$$

$$\geq \overline{x} \left(\frac{1}{1 + (k+1)H^{p+k+1}} \right)^{2k+1} = G.$$
(2.12)

That is, $G \le x_n \le H$. It follows from (i) and (ii) that

$$\min\{C, G\} \le x_n \le \max\{D, H\}. \tag{2.13}$$

Then the proof is complete.

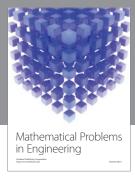
References

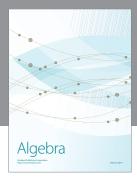
- [1] V. Huston and K. Schmidtt, "Persistence and the dynamics of biological systems," *Mathematical Biosciences*, vol. 111, pp. 1–71, 1992.
- [2] M. R. S. Kulenović and G. Ladas, Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures, Chapman & Hall, Boca Raton, Fla, USA, 2002.
- [3] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, "On the difference equation $x_{n+1} = ax_n bx_n/(cx_n dx_{n-1})$," Advances in Difference Equations, vol. 2006, Article ID 82579, 10 pages, 2006.
- [4] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, "On the difference equation $x_{n+1} = \alpha x_{n-k}/\beta + \gamma \prod_{i=0}^k x_{n-i}$," *Journal of Concrete and Applicable Mathematics*, vol. 5, no. 2, pp. 101–113, 2007.
- [5] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, "Qualitative behavior of higher order difference equation," *Soochow Journal of Mathematics*, vol. 33, no. 4, pp. 861–873, 2007.
- [6] H. El-Metwally, "Qualitative properties of some higher order difference equations," Computers & Mathematics with Applications, vol. 58, no. 4, pp. 686–692, 2009.
- [7] C. Çinar, "On the positive solutions of the difference equation $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$," Applied Mathematics and Computation, vol. 150, no. 1, pp. 21–24, 2004.
- [8] C. Çinar, "On the difference equation $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$," Applied Mathematics and Computation, vol. 158, no. 3, pp. 813–816, 2004.
- [9] C. Çinar, "On the positive solutions of the difference equation $x_{n+1} = ax_{n-1}/(1 + bx_nx_{n-1})$," Applied Mathematics and Computation, vol. 156, no. 2, pp. 587–590, 2004.
- [10] C. Cinar, R. Karatas, and I. Yalçinkaya, "On solutions of the difference equation $x_{n+1} = x_{n-3}/(-1 + x_n x_{n-1} x_{n-2} x_{n-3})$," *Mathematica Bohemica*, vol. 132, no. 3, pp. 257–261, 2007.
- [11] D. Simsek, C. Cinar, and I. Yalcinkaya, "On the recursive sequence $x_{n+1} = x_{n-3}/(1+x_{n-1})$," *International Journal of Contemporary Mathematical Sciences*, vol. 1, no. 9–12, pp. 475–480, 2006.
- [12] I. Yalcinkaya, "On the global attractivity of positive solutions of a rational difference equation," *Selçuk Journal of Applied Mathematics*, vol. 9, no. 2, pp. 3–8, 2008.
- [13] S. Stević, "On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$," Taiwanese Journal of Mathematics, vol. 6, no. 3, pp. 405–414, 2002.

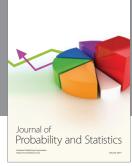
- [14] M. Aloqeili, "Dynamics of a rational difference equation," *Applied Mathematics and Computation*, vol. 176, no. 2, pp. 768–774, 2006.
- [15] E. M. Elabbasy and E. M. Elsayed, "Dynamics of a rational difference equation," *Chinese Annals of Mathematics. Series B*, vol. 30, no. 2, pp. 187–198, 2009.
- [16] E. M. Elsayed, "Qualitative behavior of a rational recursive sequence," *Indagationes Mathematicae*, vol. 19, no. 2, pp. 189–201, 2008.
- [17] R. Karatas and C. Cinar, "On the solutions of the difference equation $x_{n+1} = ax_{n-(2k+2)}/(-a + \prod_{i=0}^{2k+2} x_{n-i})$," International Journal of Contemporary Mathematical Sciences, vol. 2, no. 29–32, pp. 1505–1509, 2007.
- [18] V. L. Kocić and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, vol. 256 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
- [19] C. Wang and S. Wang, "Oscillation of partial population model with diffusion and delay," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1793–1797, 2009.













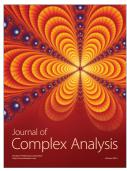




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