

Research Article

Existence of Positive Periodic Solutions for a Nonautonomous Generalized Predator-Prey System with Time Delay in Two-Patch Environment

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By using continuation theorem of coincidence degree theory, sufficient conditions of the existence of positive periodic solutions are obtained for a generalized predator-prey system with diffusion and delays. In this paper, we construct a V-function to make the prior estimation for periodic solutions, which makes the discussion more concise. Moreover, to compute the mapping's topological degree, a polynomial function matrix is constructed straightforwardly as a homotopic mapping for the generalized one, which improves the methods of computation on topological degree for a generalized mapping.

1. Introduction

In the study of dynamic population models, which are represented by differential equations, to elaborate the realistic factors into models, sometimes one needs to consider the effects caused by various of factors such as delays, diffusion, and others. The global characteristics (including persistent, stable or attractive, oscillatory, and chaotic behavior) of such differential system have been an intensive subject in biomathematics (see e.g., [1–6]). One important aspect is the existence of positive periodic solutions ([7–17]), which imply the system is periodic.

In the literature available, the methods of studying the existence of periodic solutions of a periodic system often fall into one of the following three categories: (1) combining the results of persistence of system with Horn fixed point theorem (usually for a system with delay) or Brower fixed point theorem (usually for a system without delay), the existence

results of periodic solutions are obtained ([7, 8]); (2) employing other kinds of fixed point theorems (usually for one dimension system), the differential model is investigated by transforming it into equivalent integral one ([9–11]); (3) by virtue of the theory of topological degree, especially by using Mawhin's Continuation theorem of coincidence degree (usually for high dimension system), much significant work has been done ([12–17]). However, for a generalized high-dimension system, no matter by fixed point theorems or by Mawhin's continuation theorem of coincidence degree, it is very difficult to accomplish the work. The relevant literature is seldom to find. Zhang considered a generalized prey-predator system with delay ([16]), where the generalized mapping's degree is computed step by step and a new technique of computation on topological degree is presented.

It is well known that the key and the difficult points are the prior estimation for solutions, and the topological degree's computation of generalized mapping as Mawhin's continuation theorem is applied. In this paper, we investigate a generalized model of one predator competing for two preys, which takes the nonautonomous form:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)f_1(t, x_1(t), x_3(t)) + D_1(t)(x_2(t) - x_1(t)), \\ \dot{x}_2(t) &= x_2(t)f_2(t, x_2(t), x_3(t)) + D_2(t)(x_1(t) - x_2(t)), \\ \dot{x}_3(t) &= x_3(t)[-g(t, x_3(t)) + c_1(t)p_1(x_1(t - \tau_1)) + c_2(t)p_2(x_2(t - \tau_2))],\end{aligned}\tag{1.1}$$

where x_i denotes the density of the prey in the i th patch; x_3 represents the total predator population for both patches; $\tau_i > 0$ is a constant and $D_i(t)$ is a positive continuous function and denotes the dispersal rate; p_i is the functional response of the predator population on the prey in the i th patch, and c_i is the conversion ratio of prey into predator; $D_i(t)$, $c_i(t)$, $f_i(t, x_i, x_3)$ and $g(t, x_3)$ are continuous functions in $t \in [0, +\infty)$ with a common period $\omega > 0$; $f_i(t, x_i, x_3)$ and $g(t, x_3)$ are differentiable with the other variables; $i = 1, 2$. The model is transformed from an autonomous one (see [6]). In [6], sufficient conditions for the permanence and the existence of positive attractive equilibrium are derived.

Our purpose is to obtain sufficient conditions for the existence of positive periodic solutions associated with system (1.1) and to consider whether delays and diffusion have effect on the results. What is more important is to present some new techniques in prior estimation of solutions and computation on topological degree. In this paper, we construct a V-function to make the prior estimation, which makes the proof more concise. Furthermore, we select a polynomial function matrix as a homotopic mapping to the generalized one, which plays a key role in the computation of topological degree.

In a biological sense, we take the initial conditions:

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad \varphi_i(0) > 0, \quad i = 1, 2, 3,\tag{1.2}$$

where $\tau = \max\{\tau_1, \tau_2\}$.

In system (1.1), we assume that

(H₁) for $t \geq 0$, $\partial f_i(t, x_i, x_3)/\partial x_i < 0$, $\partial f_i(t, x_i, x_3)/\partial x_3 < 0$, $i = 1, 2$, and $(\partial g(t, x_3))/\partial x_3 > 0$;

(H₂) $p_i(x)$ is continuous and $p_i(0) = 0$, $p'_i(x) > 0$, $i = 1, 2$.

For a positive continuous ω -periodic function $f(t)$, we define

$$f^u = \max_{t \in [0, \omega]} f(t), \quad f^l = \min_{t \in [0, \omega]} f(t), \quad \bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt. \quad (1.3)$$

In the following sections, we derived the sufficient conditions for the existence of positive periodic solutions and show that the delays and diffusion have no effect on the result.

2. Main Result and Proof

We first introduce some notion of the continuation theorem of coincidence degree theory and the lemma formulated in [18].

Let X, Z be Banach spaces, let $L : \text{Dom } L \subset X \rightarrow Z$ be a linear mapping, and let $N : X \rightarrow Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim } L < +\infty$ and $\text{Im } L$ is closed in Z . Let $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ be two projectors such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$. It follows that $L / (\text{Dom } L \cap \text{Ker } P) : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by K_P . For an open bounded subset Ω of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - P)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 2.1. *Let L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Suppose*

- (a) for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$;
- (b) for each $x \in \text{Ker } L \cap \partial\Omega$, $QNx \neq 0$;
- (c) Brouwer degree $\deg_B(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\text{Dom } L \cap \bar{\Omega}$.

Theorem 2.2. *Assume that there exist four positive constants M, N, m, n such that*

- (1) for $t \in R$, $f_i(t, M, 0) \leq 0$ and $-g(t, N) + c_1(t)p_1(M) + c_2(t)p_2(M) < 0$, $i = 1, 2$;
- (2) for $t \in R$, $f_i(t, m, N) \geq 0$ and $-g(t, n) + c_1(t)p_1(m) + c_2(t)p_2(m) > 0$, $i = 1, 2$.

Then system (1.1) has at least one positive ω -periodic solution.

Proof. Denote $x_i(t) = \exp[u_i(t)]$ ($i = 1, 2, 3$); then system (1.1) can be rewritten as

$$\begin{aligned} \dot{u}_1(t) &= f_1\left(t, e^{u_1(t)}, e^{u_3(t)}\right) + D_1(t)\left(e^{u_2(t)-u_1(t)} - 1\right), \\ \dot{u}_2(t) &= f_2\left(t, e^{u_2(t)}, e^{u_3(t)}\right) + D_2(t)\left(e^{u_1(t)-u_2(t)} - 1\right), \\ \dot{u}_3(t) &= -g\left(t, e^{u_3(t)}\right) + c_1(t)p_1\left(e^{u_1(t-\tau_1)}\right) + c_2(t)p_2\left(e^{u_2(t-\tau_2)}\right). \end{aligned} \quad (2.1)$$

Clearly, if system (2.1) has an ω -periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$, then system (1.1) has a positive ω -periodic solution $(\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T$. We define

$$\begin{aligned} X = Z &= \left\{ u(t) = (u_1(t), u_2(t), u_3(t))^T \in C(\mathbb{R}, \mathbb{R}^3), u(t + \omega) = u(t) \right\}, \\ \|u\| &= \left\| (u_1(t), u_2(t), u_3(t))^T \right\| = \sum_{i=1}^3 \max_{t \in [0, \omega]} |u_i(t)|, \quad u \in X(\text{or } Z). \end{aligned} \quad (2.2)$$

Then X and Z are Banach spaces with the norm $\|\cdot\|$.

Let

$$\begin{aligned} N(u, \lambda) &= \begin{bmatrix} f_1(t, e^{u_1(t)}, e^{u_3(t)}) + \lambda D_1(t)(e^{u_2(t)-u_1(t)} - 1) \\ f_2(t, e^{u_2(t)}, e^{u_3(t)}) + \lambda D_2(t)(e^{u_1(t)-u_2(t)} - 1) \\ -g(t, e^{u_3(t)}) + c_1(t)p_1(e^{u_1(t-\tau_1)}) + c_2(t)p_2(e^{u_2(t-\tau_2)}) \end{bmatrix}, \quad u \in X, \lambda \in (0, 1), \\ Lu = u' &= \frac{du(t)}{dt}, \end{aligned} \quad (2.3)$$

$$Pu = \frac{1}{\omega} \int_0^\omega u(t) dt, \quad u \in X, \quad Qz = \frac{1}{\omega} \int_0^\omega z(t) dt, \quad z \in Z.$$

Then, L is a Fredholm mapping of index zero and the generalized inverse of L is $K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$, which is given by

$$K_P(z) = \int_0^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) ds dt. \quad (2.4)$$

Hence

$$QN(u, \lambda) = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega F_1(s) ds \\ \frac{1}{\omega} \int_0^\omega F_2(s) ds \\ \frac{1}{\omega} \int_0^\omega F_3(s) ds \end{bmatrix}, \quad (2.5)$$

where

$$\begin{aligned} F_1(s) &= f_1(s, e^{u_1(s)}, e^{u_3(s)}) + \lambda D_1(s)(e^{u_2(s)-u_1(s)} - 1), \\ F_2(s) &= f_2(s, e^{u_2(s)}, e^{u_3(s)}) + \lambda D_2(s)(e^{u_1(s)-u_2(s)} - 1), \\ F_3(s) &= -g(s, e^{u_3(s)}) + c_1(s)p_1(e^{u_1(s-\tau_1)}) + c_2(s)p_2(e^{u_2(s-\tau_2)}). \end{aligned} \quad (2.6)$$

Consequently,

$$K_p(I - Q)N(u, \lambda) = \begin{bmatrix} \int_0^t F_1(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_1(s) ds dt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_1(s) ds \\ \int_0^t F_2(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_2(s) ds dt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_2(s) ds \\ \int_0^t F_3(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_3(s) ds dt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_3(s) ds \end{bmatrix}. \quad (2.7)$$

By Arzela-Ascoli theorem, we can verify that the map $K_p(I - Q)N(\bar{\Omega})$ is compact. Thus, N is L -compact on $\bar{\Omega}$ for any open bounded $\Omega \subset X$. Since $\text{Im } Q = \text{Ker } L$, the isomorphism J is the identical mapping. Corresponding to the operator equation $Lx = \lambda N(x, \lambda)$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} \dot{u}_1(t) &= \lambda [f_1(t, e^{u_1(t)}, e^{u_3(t)}) + \lambda D_1(t)(e^{u_2(t)-u_1(t)} - 1)], \\ \dot{u}_2(t) &= \lambda [f_2(t, e^{u_2(t)}, e^{u_3(t)}) + \lambda D_2(t)(e^{u_1(t)-u_2(t)} - 1)], \\ \dot{u}_3(t) &= \lambda [-g(t, e^{u_3(t)}) + c_1(t)p_1(e^{u_1(t-\tau_1)}) + c_2(t)p_2(e^{u_2(t-\tau_2)})]. \end{aligned} \quad (2.8)$$

Assume that $u = u(t) \in X$ is a solution of (2.8) for a certain $\lambda \in (0, 1)$. Then there exist $\xi_i, \eta_i \in [0, \omega]$ ($i = 1, 2, 3$) such that

$$\begin{aligned} u_i(\xi_i) &= \max_{t \in [0, \omega]} u_i(t), & u_i(\eta_i) &= \min_{t \in [0, \omega]} u_i(t), \\ u'_i(\xi_i) &= 0, & u'_i(\eta_i) &= 0. \end{aligned} \quad (2.9)$$

Therefore, we obtain the equations

$$\begin{aligned} f_1(\xi_1, e^{u_1(\xi_1)}, e^{u_3(\xi_1)}) + \lambda D_1(\xi_1)(e^{u_2(\xi_1)-u_1(\xi_1)} - 1) &= 0, \\ f_2(\xi_2, e^{u_2(\xi_2)}, e^{u_3(\xi_2)}) + \lambda D_2(\xi_2)(e^{u_1(\xi_2)-u_2(\xi_2)} - 1) &= 0, \\ -g(\xi_3, e^{u_3(\xi_3)}) + c_1(\xi_3)p_1(e^{u_1(\xi_3-\tau_1)}) + c_2(\xi_3)p_2(e^{u_2(\xi_3-\tau_2)}) &= 0 \end{aligned} \quad (2.10)$$

and equations

$$\begin{aligned} f_1(\eta_1, e^{u_1(\eta_1)}, e^{u_3(\eta_1)}) + \lambda D_1(\eta_1)(e^{u_2(\eta_1)-u_1(\eta_1)} - 1) &= 0, \\ f_2(\eta_2, e^{u_2(\eta_2)}, e^{u_3(\eta_2)}) + \lambda D_2(\eta_2)(e^{u_1(\eta_2)-u_2(\eta_2)} - 1) &= 0, \\ -g(\eta_3, e^{u_3(\eta_3)}) + c_1(\eta_3)p_1(e^{u_1(\eta_3-\tau_1)}) + c_2(\eta_3)p_2(e^{u_2(\eta_3-\tau_2)}) &= 0. \end{aligned} \quad (2.11)$$

To make estimation of $u_i(\xi_i)$ ($i = 1, 2, 3$), we define a V-function as $\bar{V}(t) = \max\{u_1(t), u_2(t)\}$. Obviously, $\bar{V}(\xi_i) = u_i(\xi_i)$ and $\dot{\bar{V}}(\xi_i) = 0$, $i = 1, 2$. According to the hypothesis and equations (2.10), we have

$$\begin{aligned} f_i(\xi_i, e^{\bar{V}(\xi_i)}, 0) &> f_i(\xi_i, e^{\bar{V}(\xi_i)}, e^{u_3(\xi_i)}) \\ &= f_i(\xi_i, e^{u_i(\xi_i)}, e^{u_3(\xi_i)}) \\ &\geq 0 \\ &\geq f_i(\xi_i, M, 0), \quad i = 1, 2, \end{aligned} \tag{2.12}$$

which implies that $e^{\bar{V}(\xi_i)} < M$, that is,

$$u_i(\xi_i) < \ln M, \quad i = 1, 2. \tag{2.13}$$

The third equation of (2.10) gives

$$\begin{aligned} g(\xi_3, e^{u_3(\xi_3)}) &= c_1(\xi_3)p_1(e^{u_1(\xi_3-\tau_1)}) + c_2(\xi_3)p_2(e^{u_2(\xi_3-\tau_2)}) \\ &\leq c_1(\xi_3)p_1(e^{u_1(\xi_1)}) + c_2(\xi_3)p_2(e^{u_2(\xi_2)}) \\ &< c_1(\xi_3)p_1(M) + c_2(\xi_3)p_2(M) \\ &< g(\xi_3, N). \end{aligned} \tag{2.14}$$

Thus

$$e^{u_3(\xi_3)} < N, \tag{2.15}$$

or

$$u_3(\xi_3) < \ln N. \tag{2.16}$$

Similarly, we define $\underline{V}(t) = \min\{u_1(t), u_2(t)\}$. Then $\underline{V}(\eta_i) = u_i(\eta_i)$, and $\dot{\underline{V}}(\eta_i) = 0$, $i = 1, 2$. From (2.11), we have

$$\begin{aligned} f_i(\eta_i, e^{\underline{V}(\eta_i)}, N) &= f_i(\eta_i, e^{u_i(\eta_i)}, N) \\ &< f_i(\eta_i, e^{u_i(\eta_i)}, e^{u_3(\eta_i)}) \\ &\leq 0 \\ &< f_i(\eta_i, m, N), \end{aligned} \tag{2.17}$$

which implies that

$$e^{u_i(\eta_i)} > m, \quad (2.18)$$

that is,

$$u_i(\eta_i) > \ln m. \quad (2.19)$$

From (2.19), it follows that

$$\begin{aligned} g(\eta_3, e^{u_3(\eta_3)}) &= c_1(\eta_3)p_1(e^{u_1(\eta_3-\tau_1)}) + c_2(\eta_3)p_2(e^{u_2(\eta_3-\tau_2)}) \\ &\geq c_1(\eta_3)p_1(e^{u_1(\eta_1)}) + c_2(\eta_3)p_2(e^{u_2(\eta_2)}) \\ &\geq c_1(\eta_3)p_1(m) + c_2(\xi_3)p_2(m) \\ &> g(\eta_3, n). \end{aligned} \quad (2.20)$$

Hence

$$e^{u_3(\eta_3)} > n, \quad (2.21)$$

that is,

$$u_3(\eta_3) > \ln n. \quad (2.22)$$

Denote

$$R_1 = \max\{|\ln m|, |\ln M|\}, \quad R_2 = \max\{|\ln n|, |\ln N|\}. \quad (2.23)$$

In view of (2.13)–(2.22), we have

$$|u_1(t)| < R_1, \quad |u_2(t)| < R_1, \quad |u_3(t)| < R_2, \quad \forall t \in R. \quad (2.24)$$

Clearly, R_i ($i = 1, 2$) are independent of λ . Denote $\widetilde{M} = 2R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that the solution $(\alpha^*, \beta^*, \gamma^*)^T$ of the following system:

$$\begin{aligned} \frac{1}{\omega} \int_0^\omega f_1(s, e^\alpha, e^\gamma) ds &= 0, \\ \frac{1}{\omega} \int_0^\omega f_2(s, e^\beta, e^\gamma) ds &= 0, \\ \frac{1}{\omega} \int_0^\omega [-g(s, e^\gamma) + c_1(s)p_1(e^\alpha) + c_2(s)p_2(e^\beta)] ds &= 0 \end{aligned} \quad (2.25)$$

satisfies $\|(\alpha^*, \beta^*, \gamma^*)^T\| < \widetilde{M}$, provided that system (2.25) has one or a number of solutions. Let $\Omega = \{u = (u_1(x), u_2(x), u_3(x))^T \in X; \|u\| < \widetilde{M}\}$. It is easy to see that the condition (a) of Lemma 2.1 is satisfied.

When $u \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap R^3$, u is a constant vector in R^3 with $\sum_{i=1}^3 |u_i| = \widetilde{M}$. If system (2.25) has one or a number of solutions, then

$$QN(u, 0) = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega f_1(s, e^{u_1}, e^{u_3}) ds \\ \frac{1}{\omega} \int_0^\omega f_2(s, e^{u_2}, e^{u_3}) ds \\ \frac{1}{\omega} \int_0^\omega [-g(s, e^{u_3}) + c_1(s)p_1(e^{u_1}) + c_2(s)p_2(e^{u_2})] ds \end{bmatrix} \neq (0, 0, 0)^T. \quad (2.26)$$

If system (2.25) has no solution, then naturally $QN(u, 0) \neq (0, 0, 0)^T$. Hence, the condition (b) of Lemma 2.1 is satisfied.

Finally, we will prove that condition (c) of Lemma 2.1 is satisfied. To this end, we define a mapping $\psi : \text{Dom } L \times [0, 1] \rightarrow X$ by

$$\psi(u, \nu) = \nu \begin{bmatrix} \frac{1}{\omega} \int_0^\omega f_1(s, e^{u_1}, e^{u_3}) ds \\ \frac{1}{\omega} \int_0^\omega f_2(s, e^{u_2}, e^{u_3}) ds \\ \frac{1}{\omega} \int_0^\omega [-g(s, e^{u_3}) + c_1(s)p_1(e^{u_1}) + c_2(s)p_2(e^{u_2})] ds \end{bmatrix} + (1 - \nu) \begin{bmatrix} a - e^{u_1} \\ a - e^{u_2} \\ b - e^{u_3} \end{bmatrix}, \quad (2.27)$$

where $\nu \in [0, 1]$ is a parameter and a, b are two chosen numbers as follows:

$$m < a < M, \quad n < b < N. \quad (2.28)$$

In the following, we will show that when $u \in \partial\Omega \cap \text{Ker } L$, $\psi(u, \nu) \neq (0, 0, 0)^T$.

We consider three possible cases: (1) either of e^{u_i} satisfies $e^{u_i} \geq M$ ($i = 1, 2$), (2) either of e^{u_i} satisfies $e^{u_i} \leq m$ ($i = 1, 2$), and (3) both of e^{u_i} satisfy $m < e^{u_i} < M$ ($i = 1, 2$).

Case 1. Either of e^{u_i} ($i = 1, 2$) satisfies $e^{u_i} \geq M$: since there exists a constant t_1 such that

$$\frac{1}{\omega} \int_0^\omega f_i(s, e^{u_i}, e^{u_3}) ds = f_i(t_1, e^{u_i}, e^{u_3}), \quad (2.29)$$

then

$$\begin{aligned} \nu f_i(t_1, e^{u_i}, e^{u_3}) + (1 - \nu)(a - e^{u_i} - be^{u_3}) &< \nu f_i(t_1, e^{u_i}, 0) + (1 - \nu)(a - e^{u_i}) \\ &\leq \nu f_i(t_1, M, 0) + (1 - \nu)(a - M) < 0. \end{aligned} \quad (2.30)$$

Case 2. Either of e^{u_i} ($i = 1, 2$) satisfies $e^{u_i} \leq m$: we consider two subcases as follows.

Subcase 1. $e^{u_3} < N$. There exists a constant t_2 and u_i ($i = 1$ or 2) such that

$$\frac{1}{\omega} \int_0^\omega f_i(s, e^{u_i}, e^{u_3}) ds = f_i(t_2, e^{u_i}, e^{u_3}), \quad (2.31)$$

then

$$\begin{aligned} \nu f_i(t_2, e^{u_i}, e^{u_3}) + (1 - \nu)(a - e^{u_i}) &> \nu f_i(t_2, m, N) \\ + (1 - \nu)(a - m) &\geq \nu f_i(t_2, m, N) + (1 - \nu)(a - m) > 0 \end{aligned} \quad (2.32)$$

Subcase 2. $e^{u_3} \geq N$. Because the other u_j ($j = 2$ or 1) must satisfy either $e^{u_j} \leq M$ or $e^{u_j} \geq M$. The later one is discussed in Case 1. Then we only consider the subcase when $e^{u_i} \leq m$ ($i = 1$ or 2) and $e^{u_j} \leq M$ ($j = 2$ or 1). Under these conditions, there exists a constant t_3 such that

$$\begin{aligned} \nu [-g(t_3, e^{u_3}) + c_1(t_3)p_1(e^{u_1}) + c_2(t_3)p_2(e^{u_2})] + (1 - \nu)(b - e^{u_3}) \\ < [-g(t_3, N) + c_1(t_3)p_1(M) + c_2(t_3)p_2(M)] + (1 - \nu)(b - N) < 0. \end{aligned} \quad (2.33)$$

Case 3. Both of e^{u_i} ($i = 1, 2$) satisfy $m < e^{u_i} < M$: we also consider the following two subcases.

Subcase 3. $e^{u_3} \geq N$. There exists a constant t_4 such that

$$\begin{aligned} \frac{1}{\omega} \int_0^\omega [-g(s, e^{u_3}) + c_1(s)p_1(e^{u_1(s)}) + c_2(s)p_2(e^{u_2(s)})] ds \\ = -g(t_4, e^{u_3}) + c_1(t_4)p_1(e^{u_1}) + c_2(t_4)p_2(e^{u_2}). \end{aligned} \quad (2.34)$$

Hence

$$\begin{aligned} \nu [-g(t_4, e^{u_3}) + c_1(t_4)p_1(e^{u_1}) + c_2(t_4)p_2(e^{u_2})] + (1 - \nu)(b - e^{u_3}) \\ < \nu [-g(t_4, N) + c_1(t_4)p_1(M) + c_2(t_4)p_2(M)] + (1 - \nu)(b - N) \\ < 0. \end{aligned} \quad (2.35)$$

Subcase 4. $e^{u_3} < N$. If $n < e^{u_3} < N$, then $\sum_{i=1}^3 |u_i| < \widetilde{M}$, which is a contradiction to $\sum_{i=1}^3 |u_i| = \widetilde{M}$. Hence, $e^{u_3} \leq n$. When $m < e^{u_i} < M$ and $e^{u_3} \leq n$, there exists a t_5 such that

$$\begin{aligned} & \nu[-g(t_5, e^{u_3}) + c_1(t_5)p_1(e^{u_1}) + c_2(t_5)p_2(e^{u_2})] + (1-\nu)(b - e^{u_3}) \\ & > \nu[-g(t_5, n) + c_1(t_5)p_1(m) + c_2(t_5)p_2(m)] + (1-\nu)(b - n) \\ & > 0. \end{aligned} \quad (2.36)$$

To sum up, $\psi(u, \nu) \neq (0, 0, 0)^T$ when $u \in \partial\Omega \cap \text{Ker } L$.

Since the algebraic equations

$$\begin{aligned} a - x &= 0, \\ a - y &= 0, \\ b - z &= 0 \end{aligned} \quad (2.37)$$

have a unique solution (x^*, y^*, z^*) such that

$$x^* = y^* = a > 0, \quad z^* = b > 0, \quad (2.38)$$

then

$$\begin{aligned} & \deg(JQN(u, 0), \Omega \cap \text{Ker } L, (0, 0, 0)^T) \\ & = \text{sign} \begin{vmatrix} -x^* & 0 & 0 \\ 0 & -y^* & 0 \\ 0 & 0 & -z^* \end{vmatrix} = \text{sign}[-x^*y^*z^*] = -1 \neq 0. \end{aligned} \quad (2.39)$$

This completes the proof of Theorem 2.2. \square

Remark 2.3. Theorem 2.2 implies that the delays and the diffusion have no effect on the result provided (H_1) - (H_2) holds.

3. Application

Example 3.1. Consider the system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)x_3(t)] + D_1(t)(x_2(t) - x_1(t)), \\ \dot{x}_2(t) &= x_2(t)[r_2(t) - a_2(t)x_2(t) - b_2(t)x_3(t)] + D_2(t)(x_1(t) - x_2(t)), \\ \dot{x}_3(t) &= x_3(t)[-r_3(t) - a_3(t)x_3 + c_1(t)x_1(t - \tau_1) + c_2(t)x_2(t - \tau_2)], \end{aligned} \quad (3.1)$$

where τ_i ($i = 1, 2$) are positive constants and all the coefficients are positive continuous functions with period ω .

It is easy to see that

$$f_i(t, x_i, x_3) = r_i(t) - a_i(t)x_i - b_i(t)x_3, \quad i = 1, 2; \quad g(t, x_3) = r_3(t) + a_3(t)x_3; \quad p_i(x) = x. \quad (3.2)$$

Obviously, the assumptions (H₁)-(H₂) are satisfied. Let

$$M = \max \left\{ \frac{r_1^u}{a_1^l}, \frac{r_2^u}{a_2^l} \right\}. \quad (3.3)$$

Then

$$f_i(t, M, 0) \leq 0. \quad (3.4)$$

Let

$$N = \frac{(c_1^u + c_2^u)M - r_3^l}{a_3^l}; \quad (3.5)$$

then $-g(t, N) + c_1(t)M + c_2(t)M < 0$. While

$$m = \min \left\{ \frac{r_1^l - b_1^u N}{a_1^u}, \frac{r_2^l - b_2^u N}{a_2^u} \right\}. \quad (3.6)$$

$f_i(t, m, N) \geq 0$. Let

$$n = \frac{(c_1^l + c_2^l)m - r_3^u}{a_3^u}; \quad (3.7)$$

then $-g(t, n) + c_1(t)m + c_2(t)m > 0$. Therefore, the conditions of Theorem 2.2 are satisfied. We can draw the conclusion that system (3.1) has at least one positive periodic solution.

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