

## Research Article

# Global Attractivity of a Periodic Delayed $N$ -Species Model of Facultative Mutualism

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Two classes of periodic  $N$ -species Lotka-Volterra facultative mutualism systems with distributed delays are discussed. Based on the continuation theorem of the coincidence degree theory developed by Gaines and Mawhin and the Lyapunov function method, some new sufficient conditions on the existence and global attractivity of positive periodic solutions are established.

## 1. Introduction

Mutualism is the interaction of two species of organisms that benefits both [1]. In general, mutualism may be either obligate or facultative. Obligate mutualist may survive only by association, and facultative mutualist, while benefiting from the presence of each other, may also survive in the absence of any of them [2]. As it is well known, in recent years the nonautonomous and periodic population dynamical systems are extensively studied. The basic and important studied questions for these systems are the persistence, permanence, and extinction of species, global stability of systems and the existence of positive periodic solutions, positive almost periodic solutions and strictly positive solutions, and so forth. Many important and influential results have been established and can be found in many articles and books. Particularly, the existence of positive periodic solutions for various type population dynamical systems has been extensively studied in [1–16] and the references cited therein.

In [7], the authors studied the following delayed two-species model of facultative mutualism:

$$\begin{aligned} \dot{y}_1(t) = & y_1(t) [r_1(t) - a_1(t) y_1(t) - b_1(t) y_1(t - \tau_1(t)) \\ & + c_1(t) y_2(t - p_1(t))], \end{aligned}$$

$$\begin{aligned} \dot{y}_2(t) = & y_2(t) [r_2(t) - a_2(t) y_2(t) - b_2(t) y_2(t - \tau_1(t)) \\ & + c_2(t) y_1(t - p_1(t))]. \end{aligned} \quad (1)$$

By using the technique of coincidence degree and the Lyapunov functionals method, the sufficient conditions for the existence and globally asymptotic stability of positive periodic solutions are obtained for system (1). In [2], the authors considered the following periodic delayed two-species model of facultative mutualism:

$$\begin{aligned} \dot{y}_1(t) = & y_1(t) [r_1(t) - a_1(t) y_1(t) + b_1(t) y_1(t - \tau_1(t)) \\ & + c_1(t) y_2(t - p_1(t))], \\ \dot{y}_2(t) = & y_2(t) [r_2(t) - a_2(t) y_2(t) + b_2(t) y_2(t - \tau_1(t)) \\ & + c_2(t) y_1(t - p_1(t))]. \end{aligned} \quad (2)$$

By means of the methods of coincidence degree and the Lyapunov functional, the sufficient conditions for the existence and globally asymptotic stability of positive periodic solutions are established for system (2). In [12], the following  $n$ -species periodic Lotka-Volterra type competitive systems

with feedback controls and finite and infinite distributed delays are discussed:

$$\begin{aligned} \dot{y}_i(t) &= y_i(t) \left[ r_i(t) - a_{ii}(t) y_i(t) \right. \\ &\quad - \sum_{j=1, j \neq i}^n a_{ij}(t) \int_0^\omega K_{ij}(s) y_j(t-s) ds \\ &\quad \left. - \alpha_i(t) \int_0^\omega H_i(s) u_i(t-s) ds \right], \\ \dot{u}_i(t) &= -\eta_i(t) + a_i(t) \int_0^\omega K_i(s) y_i(t-s) ds, \end{aligned} \quad (3)$$

where  $i = 1, 2, \dots, n$ . By using the technique of coincidence degree and the Lyapunov functionals method, the sufficient conditions for the existence and global stability of positive periodic solutions are obtained for system (3).

Motivated by the above works, in this paper, we investigate the following two classes of  $n$  species periodic model of facultative mutualism with finite distributed delays:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t) \left[ r_i(t) - \alpha_i(t) x_i(t) \right. \\ &\quad \left. + \sum_{j=1}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) x_j(t+s) ds \right] \\ &\quad i = 1, 2, \dots, n, \end{aligned} \quad (4)$$

$$\begin{aligned} \dot{x}_i(t) &= x_i(t) \left[ r_i(t) - \alpha_i(t) x_i(t) \right. \\ &\quad - \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) x_i(t+s) ds \\ &\quad \left. + \sum_{j \neq i}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) x_j(t+s) ds \right] \\ &\quad i = 1, 2, \dots, n. \end{aligned} \quad (5)$$

By using the technique of coincidence degree developed by Gaines and Mawhin in [17] and the Lyapunov functional method, we will establish some new sufficient conditions which guarantee that the system has at least one positive periodic solution and is globally attractive.

The organization of this paper is as follows. In the next section we will present some basic assumptions and main definitions and lemmas. In Section 3, conditions for the existence and global attractivity of positive periodic solution. In Section 4, two examples are given to illustrate that our main results are applicable. In the final section, we will discuss what we study in this paper and what we had in this paper.

## 2. Preliminaries

In systems (4) and (5), we have that  $x_i(t)$  ( $i = 1, 2, \dots, n$ ) represent the density of  $n$  species  $x_i$  ( $i = 1, 2, \dots, n$ ) at time  $t$ , respectively;  $r_i(t)$  ( $i = 1, 2, \dots, n$ ) represent the intrinsic growth rate of species  $x_i$  ( $i = 1, 2, \dots, n$ ) at time  $t$ , respectively;  $\alpha_i(t)$  ( $i = 1, 2, \dots, n$ ) represent the intrapatch restriction density of species  $x_i$  ( $i = 1, 2, \dots, n$ ) at time  $t$ , respectively;  $a_{ijl}(t)$  ( $l = 1, 2, \dots, m$ ,  $i \neq j$ ,  $i, j = 1, 2, \dots, n$ ) represent the mutualism coefficients between  $n$  species  $x_i$  ( $i = 1, 2, \dots, n$ ) at time  $t$ , respectively, while  $\tau \geq 0$  is a constant and  $\tau$  may be  $+\infty$ . System (4) involves positive feedback terms

$$\sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) x_i(t+s) ds, \quad i = 1, 2, \dots, n \quad (6)$$

which can be regarded as the passive effect on the growth rate of a species, and system (5) involves negative feedback terms

$$-\sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) x_i(t+s) ds, \quad i = 1, 2, \dots, n \quad (7)$$

which are due to gestation. In this paper, we always assume the following:

(H1)  $r_i(t)$  are  $\omega$ -periodic continuous functions with  $\int_0^\omega r_i(t) dt > 0$  ( $i = 1, 2, \dots, n$ );  $\alpha_i(t)$  ( $i = 1, 2, \dots, n$ ) and  $a_{ijl}(t)$  ( $i, j = 1, 2, \dots, n$ ;  $l = 1, 2, \dots, m$ ) are positive  $\omega$ -periodic continuous functions;  $k_{ijl}(s)$  ( $i, j = 1, 2, \dots, n$ ;  $l = 1, 2, \dots, m$ ) are non-negative integrable functions on  $[-\tau, 0]$  satisfying  $\int_{-\tau}^0 k_{ijl}(s) ds = 1$ .

From the viewpoint of mathematical biology, in this paper for systems (4) and (5) we only consider the solution with the following initial condition:

$$x_i(t) = \phi_i(s), \quad \forall s \in [-\tau, 0], \quad i = 1, 2, \dots, n, \quad (8)$$

where  $\phi_i(s)$  ( $i = 1, 2, \dots, n$ ) are nonnegative continuous functions defined on  $[-\tau, 0]$  satisfying  $\phi_i(0) > 0$  ( $i = 1, 2, \dots, n$ ).

In this paper, for any  $\omega$ -periodic continuous function  $f(t)$  we denote the following:

$$f^L = \min_{t \in [0, \omega]} f(t), \quad f^M = \max_{t \in [0, \omega]} f(t), \quad (9)$$

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt.$$

In order to obtain the existence of positive  $\omega$ -periodic solutions of systems (4) and (5), we will use the continuation theorem developed by Gaines and Mawhin in [17]. For the reader's convenience, we will introduce the continuation theorem in the following.

Let  $X$  and  $Z$  be two normed vector spaces. Let  $L : \text{Dom } L \subset X \rightarrow Z$  be a linear operator and  $N : X \rightarrow Z$  be a continuous operator. The operator  $L$  is called a Fredholm operator of index zero, if  $\dim \text{Ker } L = \text{codim } \text{Im } L < \infty$  and  $\text{Im } L$  is a closed set in  $Z$ . If  $L$  is a Fredholm operator of index

zero, then there exist continuous projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  such that  $\text{Im } P = \text{Ker } L$  and  $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$ . It follows that  $L \upharpoonright \text{Dom } L \cap \text{Ker } P : \text{Dom } L \cap \text{Ker } P \rightarrow \text{Im } L$  is invertible and its inverse is denoted by  $K_p$  and denote by  $J : \text{Im } Q \rightarrow \text{Ker } L$  an isomorphism of  $\text{Im } Q$  onto  $\text{Ker } L$ . Let  $\Omega$  be a bounded open subset of  $X$ , we say that the operator  $N$  is  $L$ -compact on  $\overline{\Omega}$ , where  $\overline{\Omega}$  denotes the closure of  $\Omega$  in  $X$ , if  $QN(\overline{\Omega})$  is bounded and  $K_p(I - Q)N : \overline{\Omega} \rightarrow X$  is compact.

**Lemma 1** (see [17]). *Let  $L$  be a Fredholm operator of index zero and let  $N$  be  $L$ -compact on  $\overline{\Omega}$ . If*

- (a) for each  $\lambda \in (0, 1)$  and  $x \in \partial\Omega \cap \text{Dom } L$ ,  $Lx \neq \lambda Nx$ ;
- (b) for each  $x \in \partial\Omega \cap \text{Ker } L$ ,  $QNx \neq 0$ ;
- (c)  $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$ ,

then the operator equation  $Lx = Nx$  has at least one solution lying in  $\text{Dom } L \cap \overline{\Omega}$ .

### 3. Main Results

Now, for the convenience of statements, we denote the function

$$a_{ij}(t) = \sum_{l=1}^m a_{ijl}(t), \quad i, j = 1, 2, \dots, n. \quad (10)$$

The following theorem is about the existence and global attractivity of positive periodic solutions of system (4).

**Theorem 2.** *Suppose that assumption (H1) holds and there exists a constant  $\mu_i > 0$  ( $i = 1, 2, \dots, n$ ) such that*

$$\min_{t \in [0, \omega]} \left\{ \mu_i \alpha_i(t) - \sum_{j=1}^n \sum_{l=1}^m \mu_j \int_{-\tau}^0 a_{jil}(t-s) k_{jil}(s) ds \right\} =: \delta_i > 0, \quad i = 1, 2, \dots, n, \quad (11)$$

and the algebraic equation

$$\bar{r}_i - \bar{\alpha}_i v_i + \sum_{j=1}^n \bar{a}_{ij} v_j = 0, \quad i = 1, 2, \dots, n \quad (12)$$

has a unique positive solution. Then, system (4) has a positive  $\omega$ -periodic solution which is globally attractive.

*Proof.* We firstly consider the existence of positive periodic solutions of system (4). For system (4), we introduce new variables  $u_i(t)$  ( $i = 1, 2, \dots, n$ ) such that

$$x_i(t) = \exp \{u_i(t)\}, \quad i = 1, 2, \dots, n. \quad (13)$$

Then, system (4) is rewritten in the following form:

$$\begin{aligned} \dot{u}_i(t) &= r_i(t) - \alpha_i(t) \exp \{u_i(t)\} \\ &+ \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp \{u_i(t+s)\} ds \\ &+ \sum_{j \neq i} \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp \{u_j(t+s)\} ds, \end{aligned} \quad (14)$$

$i = 1, 2, \dots, n.$

In order to apply Lemma 1 to system (14), we introduce the normed vector spaces  $X$  and  $Z$  as follows. Let  $C(R, R^n)$  denote the space of all continuous function  $u(t) = (u_1(t), u_2(t), \dots, u_n(t)) : R \rightarrow R^n$ . We take

$$X = Z = \{u(t) \in C(R, R^n) : u(t) \text{ is an } \omega\text{-periodic function}\} \quad (15)$$

with norm

$$\|u\| = \sum_{i=1}^n \max_{t \in [0, \omega]} |u_i(t)|. \quad (16)$$

It is obvious that  $X$  and  $Z$  are the Banach spaces.

We define a linear operator  $L : \text{Dom } L \subset X \rightarrow Z$  and a continuous operator  $N : X \rightarrow Z$  as follows:

$$\begin{aligned} Lu(t) &= \dot{u}(t), \\ Nu(t) &= (Nu_1(t), Nu_2(t), \dots, Nu_n(t)), \end{aligned} \quad (17)$$

where

$$\begin{aligned} Nu_i(t) &= r_i(t) - \alpha_i(t) \exp \{u_i(t)\} \\ &+ \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp \{u_i(t+s)\} ds \\ &+ \sum_{j \neq i} \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp \{u_j(t+s)\} ds, \end{aligned} \quad (18)$$

$i = 1, 2, \dots, n.$

Further, we define continuous projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  as follows:

$$Pu(t) = \frac{1}{\omega} \int_0^\omega u(t) dt, \quad Qv(t) = \frac{1}{\omega} \int_0^\omega v(t) dt. \quad (19)$$

We easily see that  $\text{Im } L = \{v \in Z : \int_0^\omega v(t) dt = 0\}$  and  $\text{Ker } L = R^n$ . It is obvious that  $\text{Im } L$  is closed in  $Z$  and  $\dim \text{Ker } L = n$ . Since for any  $v \in Z$  there are unique  $v_1 \in R^n$  and  $v_2 \in \text{Im } L$  with

$$v_1 = \frac{1}{\omega} \int_0^\omega v(t) dt, \quad v_2(t) = v(t) - v_1, \quad (20)$$

such that  $v(t) = v_1 + v_2(t)$ , we have  $\text{co dim Im } L = n$ . Therefore,  $L$  is a Fredholm mapping of index zero. Furthermore, the

generalized inverse (to  $L$ )  $K_p : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$  is given in the following form:

$$K_p v(t) = \int_0^t v(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t v(s) ds dt. \quad (21)$$

For convenience, we denote  $F(t) = (F_1(t), F_2(t), \dots, F_n(t))$  as follows:

$$\begin{aligned} F_i(t) &= r_i(t) - \alpha_i(t) \exp\{u_i(t)\} \\ &+ \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp\{u_i(t+s)\} ds \\ &+ \sum_{j \neq i, l=1}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{u_j(t+s)\} ds, \end{aligned} \quad (22)$$

$$i = 1, 2, \dots, n.$$

Thus, we have

$$\begin{aligned} QNu(t) &= \frac{1}{\omega} \int_0^\omega F(t) dt, \\ K_p(I-Q)Nu(t) &= K_p INu(t) - K_p QNu(t) \\ &= \int_0^t F(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t F(s) ds dt \\ &+ \left( \frac{1}{2} - \frac{t}{\omega} \right) \int_0^\omega F(s) ds. \end{aligned} \quad (23)$$

From formulas (23), we easily see that  $QN$  and  $K_p(I-Q)N$  are continuous operators. Furthermore, it can be verified that  $K_p(I-Q)N(\overline{\Omega})$  is compact for any open bounded set  $\Omega \subset X$  by using Arzela-Ascoli theorem and  $QN(\overline{\Omega})$  is bounded. Therefore,  $N$  is  $L$ -compact on  $\overline{\Omega}$  for any open bounded subset  $\Omega \subset X$ .

Now, we reach the position to search for an appropriate open bounded subset  $\Omega$  for the application of the continuation theorem (Lemma 1) to system (4).

Corresponding to the operator equation  $Lu(t) = \lambda Nu(t)$  with parameter  $\lambda \in (0, 1)$ , we have

$$\dot{u}_i(t) = \lambda F_i(t), \quad i = 1, 2, \dots, n, \quad (24)$$

where  $F_i(t)$  ( $i = 1, 2, \dots, n$ ) is given in (22).

Assume that  $u(t) = (u_1(t), u_2(t), \dots, u_n(t)) \in X$  is a solution of system (24) for some parameter  $\lambda \in (0, 1)$ . By

integrating system (24) with the interval  $[0, \omega]$ , we obtain the following:

$$\begin{aligned} &\int_0^\omega \left[ r_i(t) - \alpha_i(t) \exp\{u_i(t)\} \right. \\ &+ \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp\{u_i(t+s)\} ds \\ &+ \left. \sum_{j \neq i, l=1}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{u_j(t+s)\} ds \right] dt = 0, \end{aligned} \quad (25)$$

$$i = 1, 2, \dots, n.$$

Consequently,

$$\begin{aligned} &\int_0^\omega \left[ \alpha_i(t) \exp\{u_i(t)\} \right. \\ &- \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp\{u_i(t+s)\} ds \\ &- \left. \sum_{j \neq i, l=1}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{u_j(t+s)\} ds \right] dt \\ &= \bar{r}_i \omega, \quad i = 1, 2, \dots, n. \end{aligned} \quad (26)$$

From the continuity of  $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$ , there exist constants  $\xi_i, \eta_i \in [0, \omega]$  ( $i = 1, 2, \dots, n$ ) such that

$$\begin{aligned} u_i(\xi_i) &= \max_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \min_{t \in [0, \omega]} u_i(t), \\ &i = 1, 2, \dots, n. \end{aligned} \quad (27)$$

From (26) and (27), we obtain

$$\int_0^\omega \alpha_i(t) \exp\{u_i(\xi_i)\} dt \geq \bar{r}_i \omega, \quad i = 1, 2, \dots, n. \quad (28)$$

Therefore, we further have

$$u_i(\xi_i) \geq \ln \left( \frac{\bar{r}_i}{\alpha_i} \right), \quad i = 1, 2, \dots, n. \quad (29)$$

For each  $i, j = 1, 2, \dots, n$  and  $l = 1, 2, \dots, m$ , we have

$$\begin{aligned}
 & \int_0^\omega a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{u_j(t+s)\} ds dt \\
 &= \int_{-\tau}^0 \int_0^\omega a_{ijl}(t) k_{ijl}(s) \exp\{u_j(t+s)\} dt ds \\
 &= \int_{-\tau}^0 \int_s^{s+\omega} a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} dv ds \\
 &= \int_{-\tau}^0 \int_0^\omega a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} dv ds \\
 &= \int_0^\omega \int_{-\tau}^0 a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} ds dv \\
 &= \int_0^\omega \left( \int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds \right) \exp\{u_j(t)\} dt.
 \end{aligned} \tag{30}$$

Hence, from (26) we further obtain

$$\begin{aligned}
 & \int_0^\omega \left[ \left( \alpha_i(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{iil}(t-s) k_{iil}(s) ds \right) \right) \exp\{u_i(t)\} \right. \\
 & \quad \left. - \sum_{j \neq i}^n \sum_{l=1}^m \left( \int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds \right) \exp\{u_j(t)\} \right] dt \\
 &= \bar{r}_i \omega, \quad i = 1, 2, \dots, n.
 \end{aligned} \tag{31}$$

Consequently,

$$\begin{aligned}
 & \int_0^\omega \left[ \left( \alpha_1(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{11l}(t-s) k_{11l}(s) ds \right) \right) \exp\{u_1(t)\} \right. \\
 & \quad \left. - \sum_{j \neq 1}^n \sum_{l=1}^m \left( \int_{-\tau}^0 a_{1jl}(t-s) k_{1jl}(s) ds \right) \exp\{u_j(t)\} \right] dt \\
 &+ \int_0^\omega \left[ \left( \alpha_2(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{22l}(t-s) k_{22l}(s) ds \right) \right) \right. \\
 & \quad \left. \times \exp\{u_2(t)\} - \sum_{j \neq 2}^n \sum_{l=1}^m \left( \int_{-\tau}^0 a_{2jl}(t-s) k_{2jl}(s) ds \right) \right. \\
 & \quad \left. \times \exp\{u_j(t)\} \right] dt
 \end{aligned}$$

$$\begin{aligned}
 & + \dots + \int_0^\omega \left[ \left( \alpha_n(t) \right. \right. \\
 & \quad \left. \left. - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{nml}(t-s) k_{nml}(s) ds \right) \right) \right. \\
 & \quad \times \exp\{u_n(t)\} \\
 & \quad \left. - \sum_{j \neq n}^n \sum_{l=1}^m \left( \int_{-\tau}^0 a_{njl}(t-s) k_{njl}(s) ds \right) \right. \\
 & \quad \left. \times \exp\{u_j(t)\} \right] dt \\
 &= \int_0^\omega \left[ \alpha_1(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{11l}(t-s) k_{11l}(s) ds \right) \right. \\
 & \quad \left. + \sum_{j \neq 1}^n \int_{-\tau}^0 a_{j1l}(t-s) k_{j1l}(s) ds \right) \\
 & \quad \times \exp\{u_1(t)\} dt \\
 & + \int_0^\omega \left[ \alpha_2(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{22l}(t-s) k_{22l}(s) ds \right) \right. \\
 & \quad \left. + \sum_{j \neq 2}^n \int_{-\tau}^0 a_{j2l}(t-s) \right. \\
 & \quad \left. \times k_{j2l}(s) ds \right) \\
 & \quad \times \exp\{u_2(t)\} dt \\
 & + \dots + \int_0^\omega \left[ \alpha_n(t) \right. \\
 & \quad \left. - \sum_{l=1}^m \left( \int_{-\tau}^0 a_{nml}(t-s) k_{nml}(s) ds \right) \right. \\
 & \quad \left. + \sum_{j \neq n}^n \int_{-\tau}^0 a_{jnl}(t-s) \right. \\
 & \quad \left. \times k_{jnl}(s) ds \right) \\
 & \quad \times \exp\{u_n(t)\} dt \\
 &= \int_0^\omega \left[ \alpha_1(t) \right. \\
 & \quad \left. - \sum_{l=1}^m \left( \int_{-\tau}^0 \left[ a_{11l}(t-s) k_{11l}(s) + \sum_{j \neq 1}^n a_{j1l}(t-s) \right. \right. \right. \\
 & \quad \left. \left. \times k_{j1l}(s) \right] ds \right) \right]
 \end{aligned}$$

$$\begin{aligned}
& \times \exp \{u_1(t)\} dt \\
& + \int_0^\omega \left[ \alpha_2(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 \left[ a_{22l}(t-s) k_{22l}(s) \right. \right. \right. \\
& \quad \left. \left. \left. + \sum_{j \neq 2}^n a_{j2l}(t-s) \right. \right. \right. \\
& \quad \left. \left. \left. \times k_{j2l}(s) \right] ds \right) \right] \\
& \times \exp \{u_2(t)\} dt \\
& + \cdots + \int_0^\omega \left[ \alpha_n(t) \right. \\
& \quad \left. - \sum_{l=1}^m \left( \int_{-\tau}^0 \left[ a_{nml}(t-s) k_{nml}(s) \right. \right. \right. \\
& \quad \left. \left. \left. + \sum_{j \neq n}^n a_{jnl}(t-s) \right. \right. \right. \\
& \quad \left. \left. \left. \times k_{jnl}(s) \right] ds \right) \right] \\
& \times \exp \{u_n(t)\} dt = \sum_{i=1}^n \bar{r}_i \omega.
\end{aligned} \tag{32}$$

From the assumptions of Theorem 2, we can obtain

$$\begin{aligned}
& \int_0^\omega \left[ \alpha_i(t) - \sum_{l=1}^m \left( \int_{-\tau}^0 \left[ a_{iil}(t-s) k_{iil}(s) \right. \right. \right. \\
& \quad \left. \left. \left. + \sum_{j \neq i}^n a_{jil}(t-s) \right. \right. \right. \\
& \quad \left. \left. \left. \times k_{jil}(s) \right] ds \right) \right] \\
& \times \exp \{u_i(t)\} dt \\
& \leq \sum_{i=1}^n \bar{r}_i \omega, \quad i = 1, 2, \dots, n.
\end{aligned} \tag{33}$$

Hence,

$$\delta_i \int_0^\omega \exp \{u_i(t)\} dt \leq \sum_{i=1}^n \bar{r}_i \omega, \quad i = 1, 2, \dots, n. \tag{34}$$

Consequently,

$$\int_0^\omega \exp \{u_i(t)\} dt \leq \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i}, \quad i = 1, 2, \dots, n. \tag{35}$$

From (35), we further obtain

$$u_i(r_i) \leq \ln \left( \frac{\sum_{i=1}^n \bar{r}_i}{\delta_i} \right), \quad i = 1, 2, \dots, n. \tag{36}$$

On the other hand, directly from system (14) we have

$$\begin{aligned}
& \int_0^\omega |\dot{u}_i(t)| dt \\
& \leq \int_0^\omega \left[ |r_i(t)| + \alpha_i(t) \exp \{u_i(t)\} \right. \\
& \quad \left. + \sum_{l=1}^m a_{iil}(t) \int_{-\tau}^0 k_{iil}(s) \exp \{u_i(t+s)\} ds \right. \\
& \quad \left. + \sum_{j \neq i}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \right. \\
& \quad \left. \times \exp \{u_j(t+s)\} ds \right] dt \\
& = \int_0^\omega |r_i(t)| dt \\
& \quad + \int_0^\omega \left[ \alpha_i(t) + \sum_{l=1}^m \int_{-\tau}^0 a_{iil}(t-s) k_{iil}(s) ds \right] \\
& \quad \times \exp \{u_i(t)\} dt \\
& \quad + \int_0^\omega \left( \sum_{j \neq i}^n \sum_{l=1}^m \int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds \right) \\
& \quad \times \exp \{u_j(t)\} dt \\
& \leq \int_0^\omega |r_i(t)| dt + \int_0^\omega \alpha_i(t) \exp \{u_i(t)\} dt \\
& \quad + \sum_{l=1}^m a_{iil}^M \int_0^\omega \exp \{u_i(t)\} dt \\
& \quad + \sum_{j \neq i}^n \sum_{l=1}^m a_{ijl}^M \int_0^\omega \exp \{u_j(t)\} dt \\
& \leq \overline{|r_i|} \omega + \sum_{j=1}^n \sum_{l=1}^m a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i} \\
& \quad + \alpha_i^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i}, \quad i = 1, 2, \dots, n.
\end{aligned} \tag{37}$$

From (36) and (37), we have, for any  $t \in [0, \omega]$ ,

$$u_i(t) \leq u_i(\eta_i) + \int_0^\omega |\dot{u}_i(t)| dt \leq \ln \left( \frac{\sum_{i=1}^n \bar{r}_i}{\delta_i} \right) + |\bar{r}_i| \omega + \sum_{j=1}^n \sum_{l=1}^m a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i} + \alpha_i^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i} =: M_i, \quad (38)$$

$$i = 1, 2, \dots, n.$$

Further, from (29) and (37), we have, for any  $t \in [0, \omega]$ ,

$$u_i(t) \geq u_i(\xi_i) - \int_0^\omega |\dot{u}_i(t)| dt \geq \ln \left( \frac{\bar{r}_i}{\bar{\alpha}_i} \right) - |\bar{r}_i| \omega - \sum_{j=1}^n \sum_{l=1}^m a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i} - \alpha_i^M \frac{\sum_{i=1}^n \bar{r}_i \omega}{\delta_i} =: N_i, \quad i = 1, 2, \dots, n. \quad (39)$$

Therefore, from (38) and (39) we have

$$\max_{t \in [0, \omega]} |u_i(t)| \leq \max \{ |M_i|, |N_i| \} =: B_i, \quad (40)$$

$$i = 1, 2, \dots, n.$$

It can be seen that the constants  $B_i$  ( $i = 1, 2, \dots, n$ ) are independent of parameter  $\lambda \in (0, 1)$ .

For any  $u = (u_1, u_2, \dots, u_n) \in R^n$ , from (18) we can obtain

$$QNu = (QNu_1, QNu_2, \dots, QNu_n), \quad (41)$$

where

$$QNu = \bar{r}_i - (\bar{\alpha}_i - \bar{a}_{ii}) \exp \{u_i\} + \sum_{j \neq i}^n \bar{a}_{ij} \exp \{u_j\}, \quad (42)$$

$$i = 1, 2, \dots, n.$$

We consider the following algebraic equation:

$$\bar{r}_i - (\bar{\alpha}_i - \bar{a}_{ii}) v_i + \sum_{j \neq i}^n \bar{a}_{ij} v_j = 0, \quad i = 1, 2, \dots, n. \quad (43)$$

From the assumption of Theorem 2, the equation has a unique positive solution  $v^* = (v_1^*, v_2^*, \dots, v_n^*)$ . Hence, the equation  $QNu = 0$  has a unique solution  $u^* = (u_1^*, u_2^*, \dots, u_n^*) = (\ln v_1^*, \ln v_2^*, \dots, \ln v_n^*) \in R^n$ .

Choosing constant  $B > 0$  large enough such that  $|u_1^*| + |u_2^*| + \dots + |u_n^*| < B$  and  $B > B_1 + B_2 + \dots + B_n$ , we define a bounded open set  $\Omega \subset X$  as follows:

$$\Omega = \{u \in X : \|u\| < B\}. \quad (44)$$

It is clear that  $\Omega$  satisfies conditions (a) and (b) of Lemma 1. On the other hand, by direct calculating we can obtain

$$\deg \{JQN, \Omega \cap \text{Ker } L, (0, 0, \dots, 0)\} = \text{sgn} \begin{vmatrix} f_{u_1}^1 & f_{u_2}^1 & \dots & f_{u_n}^1 \\ f_{u_1}^2 & f_{u_2}^2 & \dots & f_{u_n}^2 \\ \dots & \dots & \dots & \dots \\ f_{u_1}^n & f_{u_2}^n & \dots & f_{u_n}^n \end{vmatrix}, \quad (45)$$

where

$$f_{u_j}^i = -(\bar{\alpha}_i - \bar{a}_{ij}) \exp \{u_j^*\}, \quad i = j, \quad (46)$$

$$f_{u_j}^i = \bar{a}_{ij} \exp \{u_j^*\}, \quad i \neq j, \quad i, j = 1, 2, \dots, n.$$

From the assumption of Theorem 2, we have

$$\begin{vmatrix} f_{u_1}^1 & f_{u_2}^1 & \dots & f_{u_n}^1 \\ f_{u_1}^2 & f_{u_2}^2 & \dots & f_{u_n}^2 \\ \dots & \dots & \dots & \dots \\ f_{u_1}^n & f_{u_2}^n & \dots & f_{u_n}^n \end{vmatrix} \neq 0. \quad (47)$$

From this, we finally have

$$\deg \{JQN, \Omega \cap \text{Ker } L, (0, 0, \dots, 0)\} \neq 0. \quad (48)$$

This shows that  $\Omega$  satisfies condition (c) of Lemma 1. Therefore, system (14) has a  $\omega$ -periodic solution  $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t)) \in \bar{\Omega}$ . Further, from (13), system (4) has a positive  $\omega$ -periodic solution  $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ .

Next, we will consider the global attractivity of positive periodic solutions  $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$  of system (4). Choose positive constants  $m_i > 0, M_i > 0$  such that

$$m_i \leq x_i^*(t) \leq M_i, \quad i = 1, 2, \dots, n. \quad (49)$$

From the assumption of Theorem 2, there exists constant  $\beta > 0$  such that for all  $t \geq 0$  we have

$$\delta_i \geq \beta > 0, \quad i = 1, 2, \dots, n. \quad (50)$$

Let  $(x_1(t), x_2(t), \dots, x_n(t))$  be any solution of system (4), we define Lyapunov function as follows:

$$V_i(t) = \mu_i |\ln x_i^*(t) - \ln x_i(t)| + \sum_{j=1}^n \sum_{l=1}^m \mu_j \int_{-\tau}^0 k_{ijl}(s) \int_{t+s}^t a_{ijl}(\theta - s) \times |x_j^*(\theta) - x_j(\theta)| d\theta ds. \quad (51)$$



Calculating the upper right derivation of  $V_i(t)$  along system (4) for  $i = 1, 2, \dots, n$ , we have

$$\begin{aligned}
D^+V_i(t) &= \text{sign}(x_i^*(t) - x_i(t)) \\
&\times \left[ -\mu_i \alpha_i(t) (x_i^*(t) - x_i(t)) \right. \\
&\quad + \sum_{j=1}^n \sum_{l=1}^m \mu_j a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \\
&\quad \times (x_j^*(t+\theta) - x_j(t+\theta)) ds \\
&\quad + \sum_{j=1}^n \sum_{l=1}^m \mu_j \int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds |x_j^*(t) - x_j(t)| \\
&\quad - \sum_{j=1}^n \sum_{l=1}^m \mu_j a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \\
&\quad \times |x_j^*(t+\theta) - x_j(t+\theta)| ds \\
&\leq -\mu_i \alpha_i(t) |x_i^*(t) - x_i(t)| \\
&\quad + \sum_{j=1}^n \sum_{l=1}^m \mu_j \int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds |x_j^*(t) - x_j(t)|.
\end{aligned} \tag{52}$$

Further, we define a Lyapunov function as follows:

$$V(t) = \sum_{i=1}^n V_i(t). \tag{53}$$

Calculating the upper right derivation of  $V(t)$ , from (52) we finally can obtain, for all  $t \geq 0$ ,

$$D^+V(t) \leq -\sum_{i=1}^n \delta_i |x_i^*(t) - x_i(t)|. \tag{54}$$

Integrating from 0 to  $t$  on both sides of (54) and by (50) produces

$$V(t) + \beta \int_0^t \left( \sum_{i=1}^n |x_i^*(s) - x_i(s)| \right) ds \leq V(0), \quad t \geq 0, \tag{55}$$

then

$$\int_0^t \left( \sum_{i=1}^n |x_i^*(s) - x_i(s)| \right) ds \leq \frac{V(0)}{\beta}, \quad t \geq 0. \tag{56}$$

By the definition of  $V(t)$  and (53), we have

$$\sum_{i=1}^n \mu_i |\ln x_i^*(t) - \ln x_i(t)| \leq V(t) \leq V(0), \quad t \geq 0. \tag{57}$$

Therefore, for  $i = 1, 2, \dots, n$  we have

$$\mu_i |\ln x_i^*(t) - \ln x_i(t)| \leq V(0), \quad t \geq 0, \tag{58}$$

which, together with (49), leads to

$$\begin{aligned}
m_i \exp \left\{ -\frac{V(0)}{\mu_i} \right\} &\leq x_i(t) \\
&\leq M_i \exp \left\{ \frac{V(0)}{\mu_i} \right\}, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{59}$$

and, hence,  $\sum_{i=1}^n |x_i^*(t) - x_i(t)| \in L^1[0, +\infty)$ . From the boundedness of  $x_i^*(t)$  and (58), it follows that  $x_i(t)$  ( $i = 1, 2, \dots, n$ ) are bounded for  $t \geq 0$ . It is obvious that both  $x_i(t)$  and  $x_i^*(t)$  satisfy the equations of system (4), then by system (4) and the boundedness of  $x_i(t)$  and  $x_i^*(t)$ , we know that the derivatives  $\dot{x}_i(t)$  and  $\dot{x}_i^*(t)$  are bounded. Furthermore, we can obtain that  $\dot{x}_i^*(t) - \dot{x}_i(t)$  ( $i = 1, 2, \dots, n$ ) and their derivatives remain bounded on  $[0, +\infty)$ . Therefore  $\sum_{i=1}^n |x_i^*(t) - x_i(t)|$  is uniformly continuous on  $[0, +\infty)$ . Thus, from (56), we have

$$\lim_{t \rightarrow +\infty} \sum_{i=1}^n |x_i^*(t) - x_i(t)| = 0. \tag{60}$$

Therefore,

$$\lim_{t \rightarrow +\infty} (x_i^*(t) - x_i(t)) = 0, \quad i = 1, 2, \dots, n. \tag{61}$$

This completes the proof of Theorem 2.  $\square$

From the proof of Theorem 2, on the existence and global attractivity of positive periodic solutions of system (5), we have the following result.

**Corollary 3.** Suppose that assumption (H1) holds and there exists a constant  $\rho_i > 0$  ( $i = 1, 2, \dots, n$ ) such that

$$\begin{aligned}
\min_{t \in [0, \omega]} \left\{ \rho_i \alpha_i(t) - \sum_{j=1}^n \sum_{l=1}^m \rho_j \int_{-\tau}^0 a_{jil}(t-s) k_{jil}(s) ds \right\} \\
=: \lambda_i > 0, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{62}$$

and the algebraic equation

$$\bar{r}_i - (\bar{\alpha}_i + \bar{a}_{ii}) v_i + \sum_{j \neq i} \bar{a}_{ij} v_j = 0, \quad i = 1, 2, \dots, n, \tag{63}$$

has a unique positive solution. Then, system (5) has a positive  $\omega$ -periodic solution which is globally attractive.



### 4. Two Examples

*Example 4.* First, we consider the following delayed system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) \left[ 2 + \cos(t) - (6 + \cos(t)) x_1(t) \right. \\ &\quad + \frac{2 + \cos(t)}{3} \int_{-\tau}^0 k_{111}(s) x_1(t+s) ds \\ &\quad + \frac{2 + \cos(t)}{3} \int_{-\tau}^0 k_{121}(s) x_2(t+s) ds \\ &\quad \left. + \frac{2 + \cos(t)}{3} \int_{-\tau}^0 k_{131}(s) x_3(t+s) ds \right], \\ \dot{x}_2(t) &= x_2(t) \left[ 2 + \cos(t) - (5 + \cos(t)) x_2(t) \right. \\ &\quad + \frac{2 + \cos(t)}{7} \int_{-\tau}^0 k_{221}(s) x_2(t+s) ds \\ &\quad + \frac{3 + \cos(t)}{4} \int_{-\tau}^0 k_{211}(s) x_1(t+s) ds \\ &\quad \left. + \frac{3 + \cos(t)}{4} \int_{-\tau}^0 k_{231}(s) x_3(t+s) ds \right], \\ \dot{x}_3(t) &= x_3(t) \left[ 2 + \cos(t) - (6 + \cos(t)) x_3(t) \right. \\ &\quad + \frac{3 + \cos(t)}{4} \int_{-\tau}^0 k_{331}(s) x_3(t+s) ds \\ &\quad + \frac{4 + \cos(t)}{5} \int_{-\tau}^0 k_{311}(s) x_1(t+s) ds \\ &\quad \left. + \frac{3 + \cos(t)}{5} \int_{-\tau}^0 k_{321}(s) x_2(t+s) ds \right]. \end{aligned} \tag{64}$$

Corresponding to system (4),  $n = 3, m = 1, \omega = 2\pi$ , by direct calculation, we can get

$$\sigma_1 \approx 2, \quad \sigma_2 \approx 1.6, \quad \sigma_3 \approx 2.2, \tag{65}$$

and the following equations have unique positive solutions:

$$\begin{aligned} \bar{a}_{11}v_1 - \bar{a}_{12}v_2 - \bar{a}_{13}v_3 &= \bar{r}_1, \\ \bar{a}_{22}v_1 - \bar{a}_{21}v_2 - \bar{a}_{23}v_3 &= \bar{r}_2, \\ \bar{a}_{33}v_1 - \bar{a}_{31}v_2 - \bar{a}_{32}v_3 &= \bar{r}_3, \end{aligned} \tag{66}$$

where

$$v_1 \approx 0.1944, \quad v_2 \approx 0.5625, \quad v_3 \approx 0.8819. \tag{67}$$

It is clear that all the conditions of Theorem 2 hold. Hence, system (64) has a positive periodic solution which is globally attractive.

*Example 5.* Next, we consider the following delayed system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) \left[ 2 + \cos(t) - \frac{5 + 4 \cos(t)}{20} x_1(t) \right. \\ &\quad + \frac{3 + 2 \cos(t)}{2} \int_{-\tau}^0 k_{111}(s) x_1(t+s) ds \\ &\quad + \frac{3 + 2 \cos(t)}{4} \int_{-\tau}^0 k_{121}(s) x_2(t+s) ds \\ &\quad \left. + \frac{2 + \cos(t)}{2} \int_{-\tau}^0 k_{131}(s) x_3(t+s) ds \right], \\ \dot{x}_2(t) &= x_2(t) \left[ 2 + \cos(t) - \frac{41 + 40 \cos(t)}{100} x_2(t) \right. \\ &\quad + (2 + \cos(t)) \int_{-\tau}^0 k_{221}(s) x_2(t+s) ds \\ &\quad + \frac{9 + 6 \cos(t)}{8} \int_{-\tau}^0 k_{211}(s) x_1(t+s) ds \\ &\quad \left. + \frac{3 + 2 \cos(t)}{4} \int_{-\tau}^0 k_{231}(s) x_3(t+s) ds \right], \\ \dot{x}_3(t) &= x_3(t) \left[ 2 + \cos(t) - \frac{5 + 4 \cos(t)}{20} x_3(t) \right. \\ &\quad + (4 + \cos(t)) \int_{-\tau}^0 k_{331}(s) x_3(t+s) ds \\ &\quad + \frac{5 + 4 \cos(t)}{8} \int_{-\tau}^0 k_{311}(s) x_1(t+s) ds \\ &\quad \left. + \frac{6 + 3 \cos(t)}{8} \int_{-\tau}^0 k_{321}(s) x_2(t+s) ds \right]. \end{aligned} \tag{68}$$

Corresponding to system (4),  $n = 3, m = 1, \omega = 2\pi$ , by direct calculation we can get

$$\sigma_1 \approx -5.2, \quad \sigma_2 \approx -6.1, \quad \sigma_3 \approx -7.2, \tag{69}$$

and the following equations have a unique positive solutions:

$$\begin{aligned} \bar{a}_{11}v_1 - \bar{a}_{12}v_2 - \bar{a}_{13}v_3 &= \bar{r}_1, \\ \bar{a}_{22}v_1 - \bar{a}_{21}v_2 - \bar{a}_{23}v_3 &= \bar{r}_2, \\ \bar{a}_{33}v_1 - \bar{a}_{31}v_2 - \bar{a}_{32}v_3 &= \bar{r}_3, \end{aligned} \tag{70}$$

where

$$v_1 = -0.1622, \quad v_2 = 0.7007, \quad v_3 = 1.2717. \tag{71}$$

Clearly, the conditions of Theorem 2 do not hold.

From Figure 1 we can see that system (68) has no globally attractive positive periodic solution.

*Remark 6.* From these two examples, we can see that if the conditions of Theorem 2 hold, then the system has a globally attractive positive periodic solution. If the conditions of Theorem 2 do not hold, then the system has no globally attractive positive periodic solution.

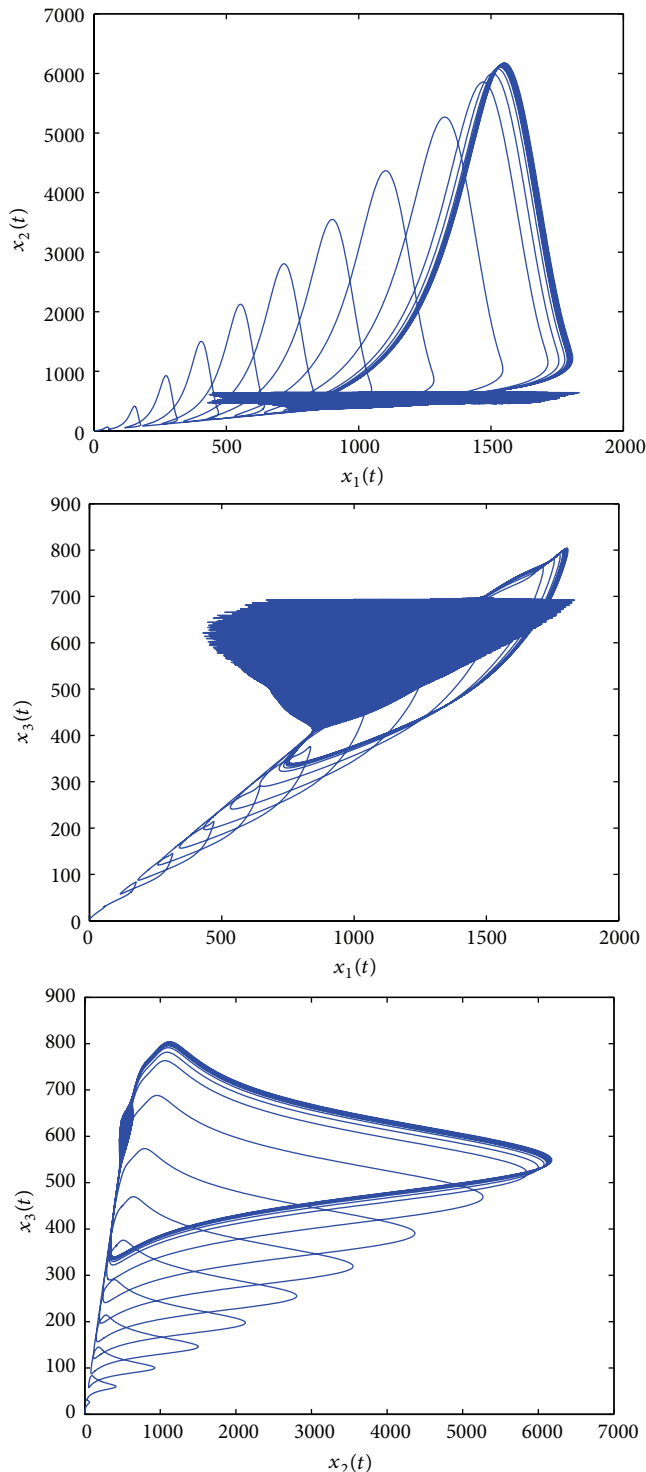


FIGURE 1: Numerical simulation for system (68). Here, we take the initial value  $x_0 = (x_{10}, x_{20}, x_{30}) = (1, 1.5, 2)$ .

## 5. Conclusions

Mawhin's continuation theorem is a powerful tool for studying the existence of periodic solutions of periodic high-dimensional time-delayed problems. When dealing with time-delayed problem, it is very convenient and the result is

relatively simple. The most critical thing in the using of the theorem is the calculation of topological degree, that is, the condition (c) of the theorem.

In this paper, motivated by [2, 7] of Liu et al. we propose two classes of periodic  $N$ -species Lotka-Volterra facultative mutualism systems with distributed delays. By applying the continuation theorem of the coincidence degree theory developed by Gaines and Mawhin and the Lyapunov function method, we easily obtain sufficient conditions for the existence and global attractivity of positive periodic solutions of the system. From Theorem 2 and Corollary 3, we can see that the distributed time delays have effect on the existence and global attractivity of positive periodic solutions, and conditions (11) and (62) are very crucial to find the criteria for globally attractive positive periodic solutions. Further, the conditions (11) and (62) indicate that the undelayed intraspecific competition dominates the delayed intraspecific reproduction, and the intraspecific competition is more significant than the interspecific cooperation.

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