Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 925629, 6 pages
http://dx.doi.org/10.1155/2013/925629

Research Article

Global Regularity Criterion for the Magneto-Micropolar Fluid Equations

Fanhui Meng[1] and Gang Wang[2]

[1] *Changchun Finance College, Changchun, Jilin 130028, China*
[2] *Shandong Transport Vocational College, Weifang, Shandong 261206, China*

Correspondence should be addressed to Fanhui Meng; fanhuimeng_02@126.com

Received 24 December 2012; Accepted 22 February 2013

Academic Editor: Hua Su

We are concerned with the magneto-micropolar fluid equations in $\mathbb{R}^3$. Using Littlewood-Paley decomposition, we obtain an Osgood type global regularity criterion for the system.

1. Introduction

In this paper, we consider the following magneto-micropolar fluid equations in $\mathbb{R}^3$:

$$\begin{gathered}
\partial_t u - (\mu + \chi)\Delta u + u\cdot\nabla u - b\cdot\nabla b + \nabla\left(p + b^2\right) \\
- \chi\nabla\times\omega = 0, \\
\partial_t\omega - \gamma\Delta\omega - \kappa\nabla \operatorname{div}\omega + 2\chi\omega + u\cdot\nabla\omega - \chi\nabla\times u = 0, \\
\partial_t b - \nu\Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, \\
\operatorname{div} u = \operatorname{div} b = 0, \\
u(0,x) = u_0(x), \omega(0,x) = \omega_0(x), b(0,x) = b_0(x),
\end{gathered} \tag{1}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x)) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3$, $t \in [0,T)$, $\omega(t,x) \in \mathbb{R}^3$and $b(t,x) \in \mathbb{R}^3$, and $p(t,x) \in \mathbb{R}$ denote, respectively, the microrotational velocity, the magnetic field, and the hydrostatic pressure. μ, χ, κ, γ, and ν are positive numbers associated with properties of the material: μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities, and $1/\nu$ is the magnetic Reynold. u_0, ω_0, and b_0 are initial data for the velocity, the angular velocity, and the magnetic field with properties $\operatorname{div} u_0 = 0$ and $\operatorname{div} b_0 = 0$.

It is well known that the question of global existence or finite time blowup of smooth solutions for the 3D incompressible Euler or Navier-Stokes equations has been one of the most outstanding open problems in applied analysis, as well as that for the 3D incompressible magneto-micropolar fluid equations. This challenging problem has attracted significant attention. Therefore, it is interesting to study the global regularity criterion of the solutions for system (1). But there are few theories about regularity and blow-up criteria of magneto-micropolar fluid equations. Some blow-up criterion are obtained by Yuan [1] in 2010. His paper implies that most classical blow-up criteria of smooth solutions to Navier-Stokes or magneto-hydrodynamic equations also hold for magneto-micropolar fluid equations. In particular, using Fourier frequency localization, Yuan proved the Beale-Kato-Majda criterion only relying on ∇u; that is, if

$$\int_0^T \|\nabla u(t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \tag{2}$$

then the solution (u, ω, b) can be extended past time $t = T$. In 2008, Yuan [2] obtain the following blow-up criteria: if

$$\lim_{\varepsilon\to 0}\sup_{j\in Z}\int_{T-\varepsilon}^T \left\|\Delta_j(\nabla\times u)(t)\right\|_\infty dt < \infty, \tag{3}$$

then the solution (u, ω, b) can be extended $t = T$. Recently, Xu [3] also studied the regularity of weak solutions to magneto-micropolar fluid equations in Besov spaces.

In this paper, we establish a refined global regularity criterion by means of Osgood norm which improves the results (2) and (3). As we know, Osgood condition plays an important role in solving uniqueness of solutions to the incompressible fluids equations. This induces us to apply it to global regularity criterion problems of smooth solution. To achieve this goal, taking full advantage of Fourier frequency localization method and using the low-high decomposition technique, we show the following main results.

Theorem 1. *Suppose that for* $s > 1/2$, $(u, \omega, b) \in C([0,T); H^s) \cap C^1((0,T); H^s) \cap C((0,T); H^{s+2})$ *is the smooth solution to* (1). *If the following Osgood type condition:*

$$\sup_{2\leq q<\infty} \int_0^T \frac{\left\|\overline{S}_q \nabla u(t)\right\|_\infty}{q\log q} dt < \infty, \tag{4}$$

then the solution (u, ω, b) *can be extended past time* $t = T$. *Here one denotes that* $\overline{S}_q := \sum_{k=-q}^{q} \dot{\Delta}_k$.

Remark 2. The Osgood type condition (4) is weaker than (2) and (3). Note that, for $q \in [2, \infty)$, we have

$$\frac{\left\|\overline{S}_q \nabla u\right\|_\infty}{q\log q} \leq \frac{\sum_{k=-q}^{q} \left\|\dot{\Delta}_k(\nabla u)\right\|_\infty}{q\log q} \leq C\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}. \tag{5}$$

2. Preliminaries

The proof of the results presented in this paper is based on a dyadic partition of unity in Fourier variables, the so-called homogeneous Littlewood-Paley decomposition. So, we first introduce the Littlewood-Paley decomposition and review the so-called Beinstein estimate and commutator estimate, which are to be used in the proof of our theorem.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, the Fourier transform of f is defined by

$$\widehat{f}(\xi) = (2\pi)^{-(3/2)} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx. \tag{6}$$

We consider that $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$, respectively, support in $B = \{\xi \in \mathbb{R}^3,\ |\xi| \leq 4/3\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3,\ 3/4 \leq |\xi| \leq 8/3\}$, such that

$$\begin{gathered} \chi(\xi) + \sum_{j\geq 0} \varphi\left(2^{-j}\xi\right) = 1, \quad \forall \xi \in \mathbb{R}^3, \\ \sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}. \end{gathered} \tag{7}$$

Setting $\varphi_j = \varphi(2^{-j}\xi)$, then $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_j' = \emptyset$ if $|j - j'| \geq 2$ and $\operatorname{supp}\chi \cap \operatorname{supp}\varphi_j' = \emptyset$ if $|j - j'| \geq 1$. Let $h = F^{-1}\varphi$ and $\widetilde{h} = F^{-1}\chi$; the dyadic blocks are defined as follows:

$$\begin{gathered} \Delta_j f = \varphi\left(2^{-j}D\right) f = 2^{3j} \int_{\mathbb{R}^3} h\left(2^j y\right) f(x-y)\, dy, \\ S_j f = \sum_{k\leq j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}\left(2^j y\right) f(x-y)\, dy, \quad j \in \mathbb{Z}. \end{gathered} \tag{8}$$

Informally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection to the annulus $|\xi| \approx 2^j$, while S_j is frequency projection to the ball $|\xi| \lesssim 2^j$. The details of Littlewood-Paley decomposition can be found in Triebel [4] and Chemin [5]. Now Besov spaces in $\mathbb{R}^3$ can be defined as follows:

$$\begin{gathered} \dot{B}^s_{p,q} = \left\{ f \in \mathcal{Z}'\left(\mathbb{R}^3\right) \mid \|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\|\Delta_j f\right\|_p^q\right)^{1/q} < \infty \right\}, \\ q \neq \infty, \\ \dot{B}^s_{p,\infty} = \left\{ f \in \mathcal{Z}'\left(\mathbb{R}^3\right) \mid \|f\|_{\dot{B}^s_{p,q}} = \sup_{j\in\mathbb{Z}} 2^{js} \left\|\Delta_j f\right\|_p < \infty \right\}, \end{gathered} \tag{9}$$

where $\mathcal{Z}'$ denotes the dual space of $\mathcal{Z} = \{f \in \mathcal{S};\ D^\alpha \widehat{f}(0) = 0;\ \forall \alpha \in \mathbb{N}^n \text{ multi-index}\}$.

Now we introduce well-known Bernstetin's Lemma and commutator estimate, the proof is omitted here, and we can find the details in Chemin [5], Chemin and Lerner [6], and Kato and Ponce [7].

Lemma 3 (Bernstein's lemma). *Let* $1 \leq p \leq q \leq \infty$. *Assume that* $f \in L^p$, *then there exist constants* C, C_1, *and* C_2 *independent of* f, j, *such that*

$$\begin{gathered} \sup_{|\alpha|=k} \left\|\partial^\alpha f\right\|_q \leq C 2^{jk+3j(1/p-1/q)} \|f\|_p \\ \operatorname{supp} \widehat{f} \subset \left\{|\xi| \lesssim 2^j\right\}, \\ C_1 2^{jk} \|f\|_p \leq \sup_{|\alpha|=k} \left\|\partial^\alpha f\right\|_p \leq C_2 2^{jk} \|f\|_p \\ \operatorname{supp} \widehat{f} \subset \left\{|\xi| \approx 2^j\right\}. \end{gathered} \tag{10}$$

Remark 4. From the above Beinstein estimate, we easily know that in $\mathbb{R}^3$, for the Reisz transform R_k $(k = 1, 2, 3)$, it has for $\forall 1 \leq p \leq q \leq \infty$

$$\left\|R_k \Delta_j u\right\|_q \leq C 2^{3j(1/p-1/q)} \|u\|_p. \tag{11}$$

If suppose vector valued funtion u be divergence free, by Biot Savard law $\nabla u = (-\Delta)^{-1} \nabla\nabla \times v$ with $v = \nabla \times u$ and the boundedness of Reisz transform on L^p $(1 < p < \infty)$, we have, there exist constants C independent u such that

$$\|\nabla u\|_p \leq C\|v\|_p, \quad \forall 1 < p < \infty. \tag{12}$$

If the frequency of u is restricted to annulus $|\xi| \approx 2^j$, then (11) implies that

$$\|\nabla u\|_p \leq C\|v\|_p, \quad \forall 1 \leq p \leq \infty. \tag{13}$$

Now we denote that $\Lambda = (I - \Delta)^{1/2}$, which satisfies

$$\widehat{\Lambda f}(\xi) = \left(1 + |\xi|^2\right)^{1/2} \widehat{f}(\xi). \tag{14}$$

Λ^s $(s \in \mathbb{R})$ can be defined in the same way as follows:

$$\widehat{\Lambda^s f}(\xi) = \left(1 + |\xi|^2\right)^{s/2} \hat{f}(\xi). \tag{15}$$

Using the perivious notation, we define the norm of Sobolev space $W^{s,p}$

$$\|f\|_{W^{s,p}} \triangleq \|\Lambda^s f\|_{L^p}, \tag{16}$$

especially by Fourier transform, and $H^s \triangleq W^{s,2}$ can be defined as

$$H^s \triangleq \left\{f \in \mathcal{S}'\left(\mathbb{R}^3\right) \mid \|f\|_{H^s} < \infty\right\}, \tag{17}$$

where

$$\|f\|_{H^s} \triangleq \|\Lambda^s f\|_{L^2}\left(\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}. \tag{18}$$

Lemma 5 (Commutator estimate). *Let* $1 < p < \infty$, $s > 0$*; assume that* $f, g \in W^{s,p}$*, then there exists a constant* C *independent of* f, g*, such that*

$$\|[\Lambda^s, f]\, g\|_{L^p} \le C\left(\|\nabla f\|_{L^{p_1}} \|g\|_{W^{s-1,p_2}} + \|f\|_{W^{s,p_3}} \|g\|_{L^{p_4}}\right) \tag{19}$$

with $1 < p_2, p_3 < \infty$*, such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \tag{20}$$

Here $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

3. Proof of the Theorem 1

First we go on with the H^1 estimates of the solution (u, ω, b) under the condition (3). Denote that $H = \nabla \times u$, $I = \nabla \times \omega$, and $J = \nabla \times b$ we take curl on both sides of (1); we get the following equation:

$$\begin{gathered}
\partial_t H - (\mu + \chi)\Delta H + u \cdot \nabla H - H \cdot \nabla u - b \cdot \nabla J + J \cdot \nabla b \\
- \chi \nabla \times I = 0, \\
\partial_t I - \gamma \Delta I + 2\chi I + u \cdot \nabla I - H \cdot \nabla \omega - \chi \nabla \times H = 0, \\
\partial_t J - \nu \Delta J + u \cdot \nabla J - H \cdot \nabla b - b \cdot \nabla H + J \cdot \nabla u = 2T(b, u)
\end{gathered} \tag{21}$$

with

$$T(b, u) = \begin{pmatrix} \partial_2 b \cdot \partial_3 u - \partial_3 b \cdot \partial_2 u \\ \partial_3 b \cdot \partial_1 u - \partial_1 b \cdot \partial_3 u \\ \partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u \end{pmatrix} \tag{22}$$

which uses the fact $\nabla \times \nabla \operatorname{div} \omega = 0$.

Multiplying the three equations with (H, I, J), respectively, integrating by parts over $\mathbb{R}^3$ about the variable x, then adding the resulting equations yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) \\
&\quad + (\mu + \chi)\|\nabla H\|_2^2 + \gamma\|\nabla I\|_2^2 + \nu\|\nabla J\|_2^2 + 2\chi\|I\|_2^2 \\
&= \int_{\mathbb{R}^3} (H \cdot \nabla) u \cdot H\, dx + \int_{\mathbb{R}^3} (H \cdot \nabla)\omega \cdot I\, dx \\
&\quad + \int_{\mathbb{R}^3} (J \cdot \nabla) u \cdot J\, dx \\
&\quad - \int_{\mathbb{R}^3} (J \cdot \nabla) b \cdot H\, dx + \int_{\mathbb{R}^3} (H \cdot \nabla) b \cdot J\, dx \\
&\quad + 2\chi \int_{\mathbb{R}^3} (\nabla \times H) \cdot I\, dx + 2\int_{\mathbb{R}^3} T(b, u) \cdot J\, dx \\
&= II_1 + II_2 + II_3 + II_4 + II_5 + II_6 + II_7,
\end{aligned} \tag{23}$$

where we use the following fact due to the divergence free condition of u, b:

$$\begin{aligned}
&\int_{\mathbb{R}^3} (u \cdot \nabla) H \cdot H\, dx = \int_{\mathbb{R}^3} (u \cdot \nabla) I \cdot I\, dx \\
&\qquad = \int_{\mathbb{R}^3} (u \cdot \nabla) J \cdot J\, dx = 0, \\
&\int_{\mathbb{R}^3} (b \cdot \nabla) J \cdot H\, dx + \int_{\mathbb{R}^3} (b \cdot \nabla) H \cdot J\, dx = 0, \\
&\int_{\mathbb{R}^3} (\nabla \times H) \cdot I\, dx = \int_{\mathbb{R}^3} (\nabla \times I) \cdot H\, dx.
\end{aligned} \tag{24}$$

Let us begin with estimating II_1 and II_3. Using Littlewood-Paley decomposition to ∇u, we have

$$\begin{aligned}
II_1 &= \sum_{j<-N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H\, dx \\
&\quad + \sum_{-N \le j \le N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H\, dx \\
&\quad + \sum_{j>N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H\, dx, \\
&= II_1^1 + II_1^2 + II_1^3.
\end{aligned} \tag{25}$$

For the terms II_1^1 and II_1^2, using Hölder's inequality, Beinstein's inequality, and (12), (13), we obtain

$$\begin{aligned}
\left|II_1^1\right| &\le \|H\|_2^2 \sum_{j<-N} \left\|\nabla\Delta_j u\right\|_\infty \le C\|H\|_2^2 \sum_{j<-N} 2^{(3/2)j}\left\|\Delta_j H\right\|_2 \\
&\le C 2^{-(3/2)N}\|H\|_2^3, \\
\left|II_1^2\right| &\le \|H\|_2^2 \sum_{-N \le j \le N} \left\|\nabla\Delta_j u\right\|_\infty \le C\|H\|_2^2 \left\|\overline{S}_N \nabla u\right\|_\infty,
\end{aligned} \tag{26}$$

for II_1^3, we get

$$\begin{aligned}\left|II_1^3\right| &\le \|H\|_3^2 \sum_{j>N} \left\|\nabla \Delta_j u\right\|_3 \\ &\le C\|H\|_3^2 \sum_{j>N} 2^{j/2} \left\|\Delta_j H\right\|_2 \\ &\le C\|H\|_3^2 \left(\sum_{j>N} 2^{-(j/2)\cdot 2}\right)^{1/2} \left(\sum_{j>N} 2^{j\cdot 2} \|H\|_2^2\right)^{1/2} \\ &\le C 2^{-(N/2)} \|H\|_2 \|\nabla H\|_2^2,\end{aligned} \tag{27}$$

where we use the interpolation inequality

$$\|H\|_3 \le C\|H\|_2^{1/2} \|\nabla H\|_2^{1/2}. \tag{28}$$

Summing up (26)-(27), we have

$$\begin{aligned}\left|II_1\right| \le C \Big(& 2^{-(3/2)N} \|H\|_2^3 + \|H\|_2^2 \left\|\overline{S}_N \nabla u\right\|_\infty \\ &+ 2^{-(N/2)} \|H\|_2 \|\nabla H\|_2^2\Big).\end{aligned} \tag{29}$$

II_3 can be treated in the same way, and we decompose it as

$$\begin{aligned}II_3 = &\sum_{j<-N} \int_{\mathbb{R}^3} (J\cdot\nabla)\,\Delta_j u \cdot J\,dx \\ &+ \sum_{-N\le j\le N} \int_{\mathbb{R}^3} (J\cdot\nabla)\,\Delta_j u \cdot J\,dx \\ &+ \sum_{j>N} \int_{\mathbb{R}^3} (J\cdot\nabla)\,\Delta_j u \cdot J\,dx,\end{aligned} \tag{30}$$

then we obtain

$$\begin{aligned}\left|II_3\right| \le C \Big(& 2^{-(3/2)N} \|J\|_2^2 \|H\|_2 + \|J\|_2^2 \left\|\overline{S}_N \nabla u\right\|_\infty \\ &+ 2^{-(N/2)} \|J\|_2 \|\nabla J\|_2 \|\nabla H\|_2\Big).\end{aligned} \tag{31}$$

Now we study II_2, we decompose H by using Littlewood-Paley theory; that is,

$$\begin{aligned}II_2 = &\sum_{j<-N} \int_{\mathbb{R}^3} \left(\Delta_j H \cdot\nabla\right)\omega \cdot I\,dx \\ &+ \sum_{-N\le j\le N} \int_{\mathbb{R}^3} \left(\Delta_j H \cdot\nabla\right)\omega \cdot I\,dx \\ &+ \sum_{j>N} \int_{\mathbb{R}^3} \left(\Delta_j H \cdot\nabla\right)\omega \cdot I\,dx,\end{aligned} \tag{32}$$

then

$$\begin{aligned}\left|II_2\right| \le C \Big(& 2^{-(3/2)N} \|I\|_2^2 \|H\|_2 + \|I\|_2^2 \left\|\overline{S}_N \nabla u\right\|_\infty \\ &+ 2^{-(N/2)} \|I\|_2 \|\nabla I\|_2 \|\nabla H\|_2\Big).\end{aligned} \tag{33}$$

Similarly for II_4, II_5, and II_7, we have

$$\begin{aligned}&\left|II_4\right| + \left|II_5\right| + \left|II_7\right| \\ &\quad\le C \Big(2^{-(3/2)N} \|J\|_2^2 \|H\|_2 + \|J\|_2^2 \left\|\overline{S}_N \nabla u\right\|_\infty \\ &\qquad + 2^{-(N/2)} \|J\|_2 \|\nabla J\|_2 \|\nabla H\|_2\Big).\end{aligned} \tag{34}$$

Using Young's inequality, the term II_6 can be written as

$$\left|II_6\right| \le 2\chi \|\nabla\times H\|_2 \|I\|_2 \le \frac{\chi}{2} \|\nabla H\|_2^2 + 2\chi \|I\|_2^2. \tag{35}$$

Summing up (29)–(35) and taking the sum into (23), by Young's inequality, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) + (2\mu+\chi)\,\|\nabla H\|_2^2 \\ &\quad + 2\gamma \nabla I_2^2 + 2\nu \|\nabla J\|_2^2 \\ &\quad\le C\Big(2^{-(3/2)N}\left(\|H\|_2^3 + \|I\|_2^3 + \|J\|_2^3\right)\Big) \\ &\quad + \left\|\overline{S}_N \nabla u\right\|_\infty \left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) \\ &\quad + 2^{-(N/2)}\left(\|H\|_2 + \|I\|_2 + \|J\|_2\right) \\ &\quad \times \left(\|\nabla H\|_2^2 + \|\nabla I\|_2^2 + \|\nabla J\|_2^2\right).\end{aligned} \tag{36}$$

If we let $2^{-(N/2)}(\|H\|_2 + \|I\|_2 + \|J\|_2) \le \min(\mu,\gamma,\nu)$, that is, if we choose

$$N \ge \left[\frac{2}{\log 2}\log^+\left(\frac{C}{\min(\mu,\gamma,\nu)}\left(\|H\|_2 + \|I\|_2 + \|J\|_2\right)\right)\right] + 1, \tag{37}$$

where $[a]$ stands for the integral parts of $a \in \mathbb{R}$, $\log^+(x) = \log(x+e)$, then we have

$$\begin{aligned}&\frac{d}{dt}\left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) + (\mu+\chi)\,\|\nabla H\|_2^2 \\ &\quad + \gamma\|\nabla I\|_2^2 + \nu\|\nabla J\|_2^2 \\ &\quad\le C\left\|\overline{S}_N \nabla u\right\|_\infty \left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) + C \\ &\quad\le C f_N N \log N \left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) + C \\ &\quad\le C f_N \log^+\left(\|H\|_2 + \|I\|_2 + \|J\|_2\right) \\ &\qquad \times \log^+\log^+\left(\|H\|_2 + \|I\|_2 + \|J\|_2\right) \\ &\qquad \times \left(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2\right) + C,\end{aligned} \tag{38}$$

where $f_N = \|\overline{S}_N \nabla u\|_\infty / N\log N$.

Using Gronwall's inequality, we have

$$\left(\|H\|_2^2+\|I\|_2^2+\|J\|_2^2\right) + \int_0^t \left[(\mu+\chi)\|\nabla H\|_2^2+\gamma\|\nabla I\|_2^2+\nu\|\nabla J\|_2^2\right]dt \le \exp\exp\exp\left(C\int_0^t f_N(t')\,dt'\right). \quad (39)$$

On the other hand, by multiplying (u,ω,b), it can be easily derived from magneto-micropolar fluid equation (1) that

$$\|u\|_2^2+\|\omega\|_2^2+\|b\|_2^2+2\mu\int_0^t\|\nabla u\|_2^2 t' + 2\gamma\int_0^t\|\nabla\omega\|_2^2 dt' + 2\nu\int_0^t\|\nabla b\|_2^2 dt' + 2\kappa\int_0^t\|\operatorname{div}\omega\|_2^2 dt' + 2\chi\int_0^t\|\omega\|_2^2 dt' \le \|u_0\|_2^2+\|\omega_0\|_2^2+\|b_0\|_2^2. \quad (40)$$

Equation (39) along with (40) implies that the H^1 estimate of solution (u,ω,b).

Next, we will show how to deduce H^s estimates based on the H^1 estimates. We apply operator Λ^s on the two sides of (1), multiply $(\Lambda^s u,\Lambda^s\omega,\Lambda^s b)$ by the resulting equations and integrate the final form over $\mathbb{R}^3$, and ge

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\Lambda^s u\|_2^2+\|\Lambda^s\omega\|_2^2+\|\Lambda^s b\|_2^2\right) + (\mu+\chi)\|\nabla\Lambda^s u\|_2^2+\gamma\|\nabla\Lambda^s\omega\|_2^2 + \nu\|\nabla\Lambda^s b\|_2^2 + \kappa\|\operatorname{div}\Lambda^s\omega\|_2^2+2\chi\|\Lambda^s b\|_2^2\\ &= -\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\Lambda^s u\,dx - \int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla\omega)\Lambda^s\omega\,dx - \int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla b)\Lambda^s b\,dx + \int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla b)\Lambda^s u\,dx\\ &\quad + \int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla u)\Lambda^s b\,dx - 2\chi\int_{\mathbb{R}^3}\Lambda^s(\nabla\times u)\Lambda^s\omega\,dx,\end{aligned} \quad (41)$$

where we use the fact

$$\int_{\mathbb{R}^3}\Lambda^s(\nabla\times\omega)\Lambda^s u\,dx = \int_{\mathbb{R}^3}\Lambda^s(\nabla\times u)\Lambda^s\omega\,dx. \quad (42)$$

Furthermore, the divergence free conditions of (u,b) imply that

$$\begin{aligned}&\int_{\mathbb{R}^3}(u\cdot\nabla\Lambda^s u)\Lambda^s u\,dx = \int_{\mathbb{R}^3}(u\cdot\nabla\Lambda^s\omega)\Lambda^s\omega\,dx = \int_{\mathbb{R}^3}(u\cdot\nabla\Lambda^s b)\Lambda^s b\,dx = 0,\\ &\int_{\mathbb{R}^3}(b\cdot\nabla\Lambda^s b)\Lambda^s u\,dx + \int_{\mathbb{R}^3}(b\cdot\nabla\Lambda^s u)\Lambda^s b\,dx = 0,\end{aligned} \quad (43)$$

then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\Lambda^s u\|_2^2+\|\Lambda^s\omega\|_2^2+\|\Lambda^s b\|_2^2\right) + (\mu+\chi)\|\nabla\Lambda^s u\|_2^2+\gamma\|\nabla\Lambda^s\omega\|_2^2+\nu\|\nabla\Lambda^s b\|_2^2 + \kappa\|\operatorname{div}\Lambda^s\omega\|_2^2+2\chi\|\Lambda^s\omega\|_2^2\\ &= -\int_{\mathbb{R}^3}[\Lambda^s,u]\nabla u\Lambda^s u\,dx - \int_{\mathbb{R}^3}[\Lambda^s,u]\nabla\omega\Lambda^s\omega\,dx - \int_{\mathbb{R}^3}[\Lambda^s,u]\nabla b\Lambda^s b\,dx\\ &\quad + \int_{\mathbb{R}^3}\left([\Lambda^s,b]\nabla b\Lambda^s u + [\Lambda^s,b]\nabla u\Lambda^s b\right)dx + -2\chi\int_{\mathbb{R}^3}\Lambda^s(\nabla\times u)\Lambda^s\omega\,dx.\\ &= III_1+III_2+III_3+III_4+III_5.\end{aligned} \quad (44)$$

By Lemma 5, Hölder's inequality, Gagliardo-Nirenberg's inequality

$$\|f\|_{W^{s,4}} \le \|f\|_{W^{s,2}}^{1/4}\|\nabla f\|_{W^{s,2}}^{3/4}, \quad (45)$$

and Young's inequality

$$ab \le \frac{1}{p}a^p+\frac{1}{q}b^q, \qquad \frac{1}{p}+\frac{1}{q}=1, \quad (46)$$

we deduce that

$$\begin{aligned}|III_1| &\le \left\|[\Lambda^s,u]\nabla u\right\|_{4/3}\|\Lambda^s u\|_4\\ &\le C\left(\|\nabla u\|_2\|\nabla u\|_{W^{s-1,4}}+\|u\|_{W^{s,4}}\|\nabla u\|_2\right)\|u\|_{W^{s,4}}\\ &\le C\|\nabla u\|_2\|u\|_{H^s}^{1/2}\|\nabla u\|_{H^s}^{3/2}\\ &\le C\|\nabla u\|_2^4\|u\|_{H^s}^2+\frac{\mu}{4}\|\nabla u\|_{H^s}^2.\end{aligned} \quad (47)$$

Similarly, we estimate III_2, III_3, and III_4 as follows:

$$\begin{aligned}|III_2+III_3+III_4| &\le C\left(\|\nabla u\|_2^4+\|\nabla\omega\|_2^4+\|\nabla b\|_2^4\right)\times\left(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2\right)\\ &\quad + \frac{\mu}{4}\|\nabla u\|_{H^s}^2+\frac{\gamma}{2}\|\nabla\omega\|_{H^s}^2+\frac{\nu}{2}\|\nabla b\|_{H^s}^2.\end{aligned} \quad (48)$$

Finally, we estimate the last term

$$\begin{aligned}|III_5| &\le 2\chi\|\Lambda^s(\nabla\times u)\|_2\|\Lambda^s\omega\|_2\\ &\le \frac{\chi}{2}\|\nabla u\|_{H^s}^2+2\chi\|\omega\|_{H^s}^2.\end{aligned} \quad (49)$$

Summing up (47)–(49) with (44), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2\right)\\
&\qquad+(\mu+\chi)\|\nabla u\|_{H^s}^2+\gamma\|\nabla\omega\|_{H^s}^2+\nu\|\nabla b\|_{H^s}^2\\
&\qquad+\kappa\|\operatorname{div}\omega\|_{H^s}^2\\
&\quad\le C\left(\|u\|_{H^1}^4+\|\omega\|_{H^1}^4+\|b\|_{H^1}^4\right)\\
&\qquad\times\left(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2\right).
\end{aligned}\tag{50}$$

Using Gronwall's inequality we obtain

$$\begin{aligned}
&\left(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2\right)\\
&\qquad+\int_0^t\left((\mu+\chi)\|\nabla u\|_{H^s}^2+\gamma\|\nabla\omega\|_{H^s}^2+\nu\|\nabla b\|_{H^s}^2\right.\\
&\qquad\qquad\left.+\kappa\|\operatorname{div}\omega\|_{H^s}^2\right)(t')\,dt'\\
&\quad\le C\left(\|u_0\|_{H^s}^2+\|\omega_0\|_{H^s}^2+\|b_0\|_{H^s}^2\right)\\
&\qquad\times\exp\left(t\sup_{t'\in[0,t)}\left(\|u\|_{H^1}^4+\|\omega\|_{H^1}^4+\|b\|_{H^1}^4\right)\right).
\end{aligned}\tag{51}$$

Hence by (39) and (51), we can get the H^s regularity at time $t=T$; that is, the smooth solution (u,ω,b) can be extended past time T.

References

[1] B. Yuan, "Regularity of weak solutions to magneto-micropolar fluid equations," *Acta Mathematica Scientia B*, vol. 30, no. 5, pp. 1469–1480, 2010.

[2] J. Yuan, "Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations," *Mathematical Methods in the Applied Sciences*, vol. 31, no. 9, pp. 1113–1130, 2008.

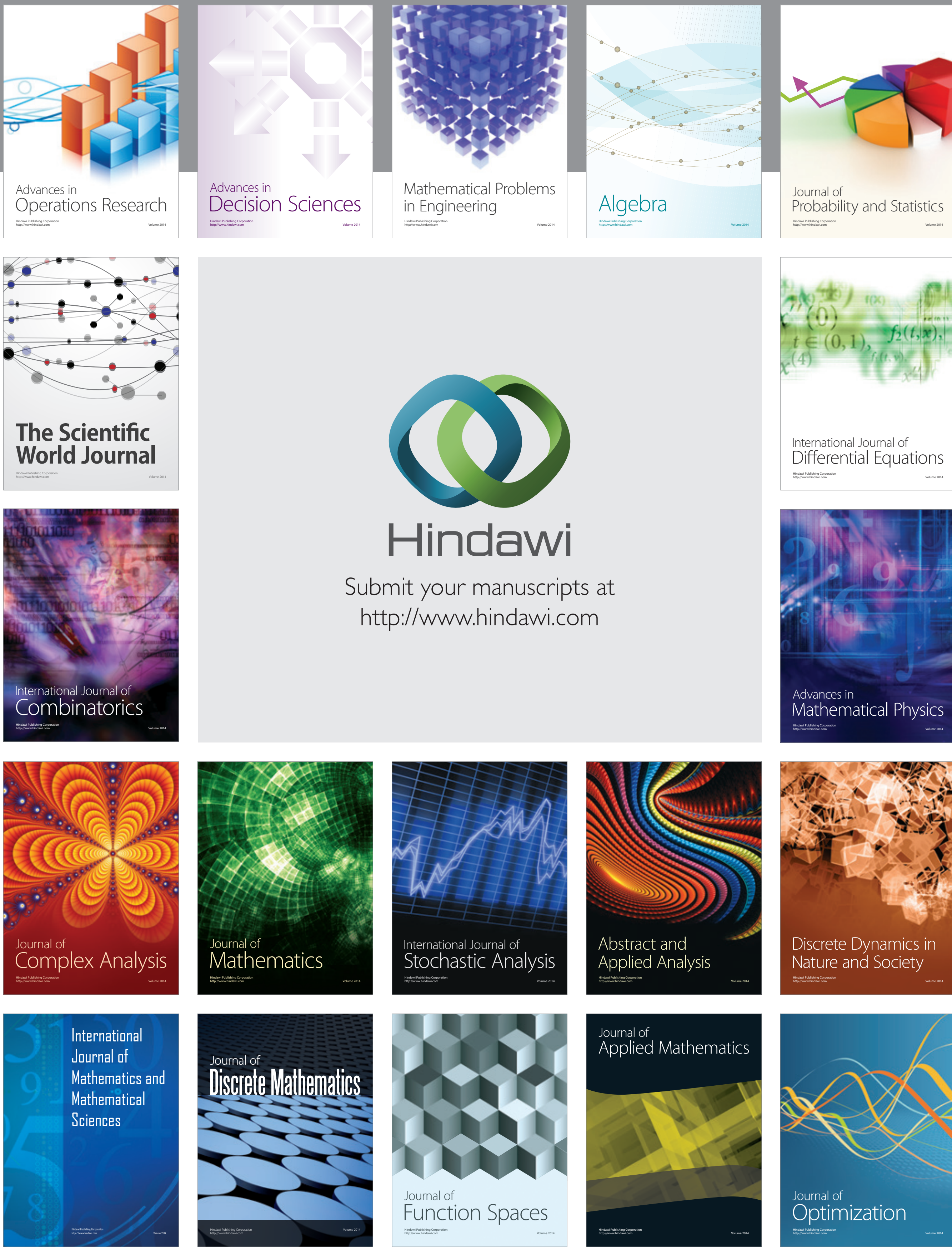
[3] F. Xu, "Regularity criterion of weak solution for the 3D magneto-micropolar fluid equations in Besov spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2426–2433, 2012.

[4] H. Triebel, *Theory of Function Spaces*, vol. 78 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1983.

[5] J.-Y. Chemin, *Perfect Incompressible Fluids*, vol. 14 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, New York, NY, USA, 1998.

[6] J.-Y. Chemin and N. Lerner, "Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes," *Journal of Differential Equations*, vol. 121, no. 2, pp. 314–328, 1995.

[7] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization