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Research Article

Intersection-Soft Filters in R_0-Algebras

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The notions of (strong) intersection-soft filters in R_0-algebras are introduced, and related properties are investigated. Characterizations of a (strong) intersection-soft filter are established, and a new intersection-soft filter from old one is constructed. A condition for an intersection-soft filter to be strong is given, and an extension property of a strong intersection-soft filter is established.

1. Introduction

To solve complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [1]. Maji et al. [2] and Molodtsov [1] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory.

To overcome these difficulties, Molodtsov [1] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [2] described the application of soft set theory to a decision making problem. Maji et al. [3] also studied several operations on the theory of soft sets. Chen et al. [4] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory.

R_0-algebras, which are different from BL-algebras, have been introduced by Wang [5] in order to get an algebraic proof of the completeness theorem of a formal deductive system [6]. The filter theory in R_0-algebras is discussed in [7].

In this paper, we apply the notion of intersection-soft sets to the filter theory in R_0-algebras. We introduced the concept of (strong) intersection-soft filters in R_0-algebras and investigate related properties. We establish characterizations of a (strong) intersection-soft filter and make a new intersection-soft filter from old one. We provide a condition for an intersection-soft filter to be strong and construct an extension property of a strong intersection-soft filter.

2. Preliminaries

2.1. Basic Results on R_0-Algebras

Definition 1 (see [5]). Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\rightarrow$. Then $(L, \wedge, \vee, \neg, \rightarrow)$ is called an R_0-algebra if it satisfies the following axioms:

(R1) $x \rightarrow y = \neg y \rightarrow \neg x$,

(R2) $1 \rightarrow x = x$,

(R3) $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z$,

(R4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(R5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,

(R6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$.

Let L be an R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$ and $x \oplus y = \neg x \to y$. It is proven that $\odot$ and $\oplus$ are commutative, associative, and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice. In the following, let x^n denote $x \odot x \odot \cdots \odot x$ where x appears n times for $n \in \mathbb{N}$.

We refer the reader to the book [8] for further information regarding R_0-algebras.

Lemma 2 (see [7]). *Let L be an R_0-algebra. Then the following properties hold:*

$$(\forall x, y \in L)(x \leq y \Longleftrightarrow x \longrightarrow y = 1), \quad (1)$$

$$(\forall x, y \in L)(x \leq y \longrightarrow x), \quad (2)$$

$$(\forall x \in L)(\neg x = x \longrightarrow 0), \quad (3)$$

$$(\forall x, y \in L)((x \longrightarrow y) \vee (y \longrightarrow x) = 1), \quad (4)$$

$$(\forall x, y \in L) \times (x \leq y \Longrightarrow y \longrightarrow z \leq x \longrightarrow z, z \longrightarrow x \leq z \longrightarrow y), \quad (5)$$

$$(\forall x, y \in L)(((x \longrightarrow y) \longrightarrow y) \longrightarrow y = x \longrightarrow y), \quad (6)$$

$$(\forall x, y \in L) \times (x \vee y = ((x \longrightarrow y) \longrightarrow y) \wedge ((y \longrightarrow x) \longrightarrow x)), \quad (7)$$

$$(\forall x \in L)(x \odot \neg x = 0, x \oplus \neg x = 1), \quad (8)$$

$$(\forall x, y \in L)(x \odot y \leq x \wedge y, x \odot (x \longrightarrow y) \leq x \wedge y), \quad (9)$$

$$(\forall x, y, z \in L)((x \odot y) \longrightarrow z = x \longrightarrow (y \longrightarrow z)), \quad (10)$$

$$(\forall x, y \in L)(x \leq y \longrightarrow (x \odot y)), \quad (11)$$

$$(\forall x, y, z \in L)(x \odot y \leq z \Longleftrightarrow x \leq y \longrightarrow z), \quad (12)$$

$$(\forall x, y, z \in L)(x \leq y \Longrightarrow x \odot z \leq y \odot z), \quad (13)$$

$$(\forall x, y, z \in L)(x \longrightarrow y \leq (y \longrightarrow z) \longrightarrow (x \longrightarrow z)), \quad (14)$$

$$(\forall x, y, z \in L)((x \longrightarrow y) \odot (y \longrightarrow z) \leq x \longrightarrow z). \quad (15)$$

Definition 3 (see [7]). A nonempty subset F of L is called a *filter* of L if it satisfies

(i) $1 \in F$,

(ii) $(\forall x \in F)(\forall y \in L)(x \to y \in F \Rightarrow y \in F)$.

Lemma 4 (see [7]). *Let F be a nonempty subset of L. Then F is a filter of L if and only if it satisfies*

(1) $(\forall x \in F)(\forall y \in L)(x \leq y \Rightarrow y \in F)$,

(2) $(\forall x, y \in F)(x \odot y \in F)$.

2.2. Basic Results on Soft Set Theory. Soft set theory was introduced by Molodtsov [1] and Çağman and Enginoğlu [9].

In what follows, let U be an initial universe set, and let E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ (resp., $\mathscr{P}(E)$) denotes the power set of U (resp., E).

By analogy with fuzzy set theory, the notion of soft set is defined as follows.

Definition 5 (see [1, 9]). A soft set of E over U (a soft set of E for short) is any function $f_A : E \to \mathscr{P}(U)$, such that $f_A(x) = \emptyset$ if $x \notin A$, for $A \in \mathscr{P}(E)$, or, equivalently, any set

$$\mathscr{F}_A := \{(x, f_A(x)) \mid x \in E, f_A(x) \in \mathscr{P}(U), f_A(x) = \emptyset \text{ if } x \notin A\}, \quad (16)$$

for $A \in \mathscr{P}(E)$.

Definition 6 (see [9]). Let $\mathscr{F}_A$ and $\mathscr{F}_B$ be soft sets of E. We say that $\mathscr{F}_A$ is a soft subset of $\mathscr{F}_B$, denoted by $\mathscr{F}_A \mathrel{\tilde{\subseteq}} \mathscr{F}_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

3. Intersection-Soft Filters

In what follows, we denote by $S(U, L)$ the set of all soft sets of L over U where L is an R_0-algebra unless otherwise specified.

Definition 7. A soft set $\mathscr{F}_L \in S(U, L)$ is called an *int-soft filter* of L if it satisfies

$$(\forall \gamma \in \mathscr{P}(U))(\mathscr{F}_L^\gamma \neq \emptyset \Longrightarrow \mathscr{F}_L^\gamma \text{ is a filter of } L), \quad (17)$$

where $\mathscr{F}_L^\gamma = \{x \in L \mid \gamma \subseteq f_L(x)\}$ which is called the γ-*inclusive set* of $\mathscr{F}_L$.

If $\mathscr{F}_L$ is an int-soft filter of L, every γ-inclusive set $\mathscr{F}_L^\gamma$ is called an *inclusive filter* of L.

Example 8. Let $L = \{0, a, b, c, 1\}$ be a set with the order $0 < a < b < c < 1$, and the following Cayley tables:

x	$\neg x$
0	1
a	c
b	b
c	a
1	0

$\longrightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

(18)

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra (see [10]) where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Let $\mathscr{F}_L \in S(U, L)$ be given as follows:

$$\mathscr{F}_L = \{(0, \gamma_1), (a, \gamma_1), (b, \gamma_1), (c, \gamma_2), (1, \gamma_2)\}, \quad (19)$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then $\mathscr{F}_L$ is an int-soft filter of L.

We provide characterizations of an int-soft filter.

Theorem 9. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is an int-soft filter of L if and only if the following assertions are valid:*

(1) $(\forall x \in L)(f_L(x) \subseteq f_L(1))$,

(2) $(\forall x, y \in L)(f_L(x \to y) \cap f_L(x) \subseteq f_L(y))$.

Proof. Assume that $\mathscr{F}_L$ is an int-soft filter of L. For any $x \in L$, let $f_L(x) = \gamma$. Then $x \in \mathscr{F}_L^\gamma$. Since $\mathscr{F}_L^\gamma$ is a filter of L, we have $1 \in \mathscr{F}_L^\gamma$ and so $f_L(x) = \gamma \subseteq f_L(1)$. For any $x, y \in L$, let $f_L(x \to y) \cap f_L(x) = \gamma$. Then $x \to y \in \mathscr{F}_L^\gamma$ and $x \in \mathscr{F}_L^\gamma$. Since $\mathscr{F}_L^\gamma$ is a filter of L, it follows that $y \in \mathscr{F}_L^\gamma$. Hence $f_L(x \to y) \cap f_L(x) = \gamma \subseteq f_L(y)$.

Conversely, suppose that $\mathscr{F}_L$ satisfies two conditions (1) and (2). Let $\gamma \in \mathscr{P}(U)$ such that $\mathscr{F}_L^\gamma \neq \emptyset$. Then there exists $a \in \mathscr{F}_L^\gamma$, and so $\gamma \subseteq f_L(a)$. It follows from (1) that $\gamma \subseteq f_L(a) \subseteq f_L(1)$. Thus $1 \in \mathscr{F}_L^\gamma$. Let $x, y \in L$ such that $x \to y \in \mathscr{F}_L^\gamma$ and $x \in \mathscr{F}_L^\gamma$. Then $\gamma \subseteq f_L(x \to y)$ and $\gamma \subseteq f_L(x)$. It follows from (2) that

$$\gamma \subseteq f_L(x \longrightarrow y) \cap f_L(x) \subseteq f_L(y), \tag{20}$$

that is, $y \in \mathscr{F}_L^\gamma$. Thus $\mathscr{F}_L^\gamma (\neq \emptyset)$ is a filter of L, and hence $\mathscr{F}_L$ is an int-soft filter of L. □

Proposition 10. *Let $\mathscr{F}_L \in S(U, L)$ be an int-soft filter of L. Then the following properties are valid.*

(1) *$\mathscr{F}_L$ is order preserving, that is,*

$$(\forall x, y \in L)(x \leq y \Longrightarrow f_L(x) \subseteq f_L(y)). \tag{21}$$

(2) $(\forall x, y \in L)(f_L(x \to y) = f_L(1) \Rightarrow f_L(x) \subseteq f_L(y))$.

(3) $(\forall x, y \in L)(f_L(x \odot y) = f_L(x) \cap f_L(y) = f_L(x \wedge y))$.

(4) $(\forall x \in L)(\forall n \in \mathbb{N})(f_L(x^n) = f_L(x))$.

(5) $(\forall x \in L)(f_L(0) = f_L(x) \cap f_L(\neg x))$.

(6) $(\forall x, y \in L)(f_L(x \to y) \cap f_L(y \to z) \subseteq f_L(x \to z))$.

(7) $(\forall x, y, z \in L)(x \odot y \leq z \Rightarrow f_L(x) \cap f_L(y) \subseteq f_L(z))$.

(8) $(\forall x, y \in L)(f_L(x) \cap f_L(x \to y) = f_L(y) \cap f_L(y \to x) = f_L(x) \cap f_L(y))$.

(9) $(\forall x, y \in L)(f_L(x \odot (x \to y)) = f_L(y \odot (y \to x)) = f_L(x) \cap f_L(y))$.

(10) $(\forall x, y, z \in L)(f_L(x \to (\neg z \to y)) \cap f_L(y \to z) \subseteq f_L(x \to (\neg z \to z)))$.

(11) $(\forall x, y, z \in L)(f_L(x \to (y \to z)) \cap f_L(x \to y) \subseteq f_L(x \to (x \to z)))$.

Proof. (1) Let $x, y \in L$ such that $x \leq y$. Then $x \to y = 1$, and so

$$f_L(x) = f_L(x) \cap f_L(1) = f_L(x) \cap f_L(x \longrightarrow y) \subseteq f_L(y) \tag{22}$$

by (1) and (2) of Theorem 9.

(2) Let $x, y \in L$ such that $f_L(x \to y) = f_L(1)$. Then

$$f_L(x) = f_L(x) \cap f_L(1) = f_L(x) \cap f_L(x \longrightarrow y) \subseteq f_L(y) \tag{23}$$

by (1) and (2) of Theorem 9.

(3) Since $x \odot y \leq x \wedge y$ for all $x, y \in L$, it follows from (1) that $f_L(x \odot y) \subseteq f_L(x) \cap f_L(y)$. Using (11) and (1), we have $f_L(x) \subseteq f_L(y \to (x \odot y))$. It follows from Theorem 9 (2) that $f_L(x) \cap f_L(y) \subseteq f_L(y \to (x \odot y)) \cap f_L(y) \subseteq f_L(x \odot y)$. Therefore $f_L(x \odot y) = f_L(x) \cap f_L(y)$. Since $y \leq x \to y$ and $x \odot (x \to y) \leq x \wedge y$ for all $x, y \in L$, we have $f_L(y) \subseteq f_L(x \to y)$ and $f_L(x) \cap f_L(y) \subseteq f_L(x) \cap f_L(x \to y) = f_L(x \odot (x \to y)) \subseteq f_L(x \wedge y) \subseteq f_L(x) \cap f_L(y)$ by (1). Hence $f_L(x \wedge y) = f_L(x) \cap f_L(y)$ for all $x, y \in L$.

(4) It follows from (3).

(5) Note that $x \odot \neg x =$ for all $x \in L$. Using (3), we have

$$f_L(0) = f_L(x \odot \neg x) = f_L(x) \cap f_L(\neg x) \tag{24}$$

for all $x, y \in L$.

(6) Combining (15), (1), and (3), we have the desired result.

(7) It follows from (1) and (3).

(8) Since $y \leq x \to y$ for all $x, y \in L$, it follows from (1) that

$$f_L(x) \cap f_L(y) \subseteq f_L(x) \cap f_L(x \longrightarrow y). \tag{25}$$

Since $x \odot (x \to y) \leq x \wedge y$ for all $x, y \in L$, we have

$$\begin{aligned} f_L(x) \cap f_L(x \longrightarrow y) &= f_L(x \odot (x \longrightarrow y)) \subseteq f_L(x \wedge y) \\ &= f_L(x) \cap f_L(y) \end{aligned} \tag{26}$$

by (3) and (1). Hence $f_L(x) \cap f_L(y) = f_L(x) \cap f_L(x \to y)$. Similarly, $f_L(y) \cap f_L(y \to x) = f_L(x) \cap f_L(y)$ for all $x, y \in L$.

(9) Using (3), we have

$$\begin{aligned} f_L(x \odot (x \longrightarrow y)) &= f_L(x) \cap f_L(x \longrightarrow y), \\ f_L(y \odot (y \longrightarrow x)) &= f_L(y) \cap f_L(y \longrightarrow x), \end{aligned} \tag{27}$$

for all $x, y \in L$. It follows from (8) that $f_L(x \odot (x \to y)) = f_L(y \odot (y \to x)) = f_L(x) \cap f_L(y)$.

(10) Note that

$$\begin{aligned} &(x \longrightarrow (\neg z \longrightarrow y)) \odot (y \longrightarrow z) \\ &\quad = ((x \odot \neg z) \longrightarrow y) \odot (y \longrightarrow z) \\ &\quad \leq (x \odot \neg z) \longrightarrow z = x \longrightarrow (\neg z \longrightarrow z) \end{aligned} \tag{28}$$

for all $x, y, z \in L$. Using (1) and (3), we have

$$\begin{aligned} &f_L(x \longrightarrow (\neg z \longrightarrow y)) \cap f_L(y \longrightarrow z) \\ &\quad = f_L((x \longrightarrow (\neg z \longrightarrow y)) \odot (y \longrightarrow z)) \\ &\quad \subseteq f_L(x \longrightarrow (\neg z \longrightarrow z)) \end{aligned} \tag{29}$$

for all $x, y, z \in L$.

(11) Note that $(x \to (y \to z)) \odot (x \to y) = (y \to (x \to z)) \odot (x \to y) \leq x \to (x \to z)$ for all $x, y, z \in L$. It follows from (1) and (3) that

$$f_L(x \longrightarrow (y \longrightarrow z)) \cap f_L(x \longrightarrow y) \subseteq f_L(x \longrightarrow (x \longrightarrow z)) \tag{30}$$

for all $x, y, z \in L$. □

Theorem 11. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is an int-soft filter of L if and only if the following assertions are valid:*

(1) *$\mathscr{F}_L$ is order preserving,*

(2) $(\forall x, y \in L)(f_L(x \odot y) = f_L(x) \cap f_L(y))$.

Proof. The necessity follows from (1) and (3) of Proposition 10.

Conversely, suppose that $\mathscr{F}_L$ satisfies two conditions (1) and (2). Let $x, y \in L$. Since $x \leq 1$, we have $f_L(x) \subseteq f_L(1)$ by (1). Note that $x \odot (x \to y) \leq y$. It follows from (2) and (1) that

$$f_L(x) \cap f_L(x \longrightarrow y) = f_L(x \odot (x \longrightarrow y)) \subseteq f_L(y). \quad (31)$$

Therefore $\mathscr{F}_L$ is an int-soft filter of L. □

Proposition 12. *Let $b \in L$ such that $\neg b = b$. If $\mathscr{F}_L \in S(U, L)$ is an int-soft filter of L, then $f_L(b) = f_L(x)$ for all $x \in \{a \in L \mid 0 \leq a \leq b\}$.*

Proof. Suppose that there exists $y \in \{a \in L \mid 0 \leq a \leq b\}$ such that $f_L(b) \neq f_L(y)$. Then $f_L(y) \subsetneq f_L(b)$, and so $b \in \mathscr{F}_L^\gamma$ and $y \notin \mathscr{F}_L^\gamma$ where $\gamma = f_L(b)$. Since $\mathscr{F}_L^\gamma$ is a filter of L, we have $0 = b \odot b \in \mathscr{F}_L^\gamma$. This shows that $\mathscr{F}_L^\gamma = L$, and it is a contradiction. Hence $f_L(b) = f_L(x)$ for all $x \in \{a \in L \mid 0 \leq a \leq b\}$. □

Theorem 13. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is an int-soft filter of L if and only if the following assertion is valid:*

$$(\forall x, y, z \in L) \times (f_L(y) \cap f_L((x \longrightarrow y) \longrightarrow z) \subseteq f_L(x \longrightarrow z)). \quad (32)$$

Proof. Assume that $\mathscr{F}_L$ is an int-soft filter of L, and let $x, y, z \in L$. Since $y \leq x \to y$, it follows from Proposition 10 (1) and Theorem 9 (2) that

$$\begin{aligned} &f_L(y) \cap f_L((x \longrightarrow y) \longrightarrow z) \\ &\quad \subseteq f_L(x \longrightarrow y) \cap f_L((x \longrightarrow y) \longrightarrow z) \\ &\quad \subseteq f_L(z) \subseteq f_L(x \longrightarrow z). \end{aligned} \quad (33)$$

Conversely, suppose that $\mathscr{F}_L$ satisfies the inclusion (32). Let $x, y \in L$. Then

$$\begin{aligned} f_L(x) &= f_L(x) \cap f_L((0 \longrightarrow x) \longrightarrow x) \subseteq f_L(0 \longrightarrow x) \\ &= f_L(1), \\ &f_L(x) \cap f_L(x \longrightarrow y) \\ &\quad = f_L(x) \cap f_L((1 \longrightarrow x) \longrightarrow y) \subseteq f_L(1 \longrightarrow y) \\ &\quad = f_L(y) \end{aligned} \quad (34)$$

by (32). Therefore $\mathscr{F}_L$ is an int-soft filter of L by Theorem 9. □

Theorem 14. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is an int-soft filter of L if and only if the following assertion is valid:*

$$(\forall x, y, z \in L)(x \leq y \longrightarrow z \Longrightarrow f_L(x) \cap f_L(y) \subseteq f_L(z)). \quad (35)$$

Proof. Suppose that $\mathscr{F}_L$ is an int-soft filter of L. Let $x, y, z \in L$ such that $x \leq y \to z$. Then $f_L(x) \subseteq f_L(y \to z)$ by Proposition 10 (1), and so

$$f_L(x) \cap f_L(y) \subseteq f_L(y \longrightarrow z) \cap f_L(y) \subseteq f_L(z) \quad (36)$$

by Theorem 9 (2).

Conversely, assume that $\mathscr{F}_L$ satisfies the condition (35). Let $x, y \in L$. Since $x \leq 1 = x \to 1$, we have $f_L(x) \subseteq f_L(1)$ by (35). Note that $x \to y \leq x \to y$. It follows from (35) that $f_L(x \to y) \cap f_L(x) \subseteq f_L(y)$. Therefore $\mathscr{F}_L$ is an int-soft filter of L by Theorem 9. □

Proposition 15. *Every int-soft filter $\mathscr{F}_L$ of L satisfies.*

$$(\forall x, y, z \in L) \times (f_L((x \longrightarrow y) \longrightarrow z) \subseteq f_L(x \longrightarrow (y \longrightarrow z))). \quad (37)$$

Proof. Let $x, y, z \in L$. Since $1 = y \to (x \to y) \leq ((x \to y) \to z) \to (y \to z)$, we have

$$f_L(1) \subseteq f_L(((x \longrightarrow y) \longrightarrow z) \longrightarrow (y \longrightarrow z)) \quad (38)$$

by Proposition 10 (1). It follows from (2) and Theorem 9 that

$$\begin{aligned} &f_L((x \longrightarrow y) \longrightarrow z) \\ &\quad = f_L((x \longrightarrow y) \longrightarrow z) \cap f_L(1) \\ &\quad \subseteq f_L((x \longrightarrow y) \longrightarrow z) \cap f_L \\ &\qquad \times (((x \longrightarrow y) \longrightarrow z) \longrightarrow (y \longrightarrow z)) \\ &\quad \subseteq f_L(y \longrightarrow z) \subseteq f_L(x \longrightarrow (y \longrightarrow z)). \end{aligned} \quad (39)$$

This completes the proof. □

The following example shows that the converse of Proposition 15 may not be true in general.

Example 16. Let $L = \{0, a, b, c, d, 1\}$ be a set with the order $0 < a < b < c < d < 1$, and the following Cayley tables:

x	$\neg x$
0	1
a	d
b	c
c	b
d	a
1	0

$\longrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

(40)

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0-algebra (see [10]) where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Let $\mathscr{F}_L \in S(U, L)$ be given as follows:

$$\mathscr{F}_L = \{(0, \gamma_1), (a, \gamma_2), (b, \gamma_2), (c, \gamma_2), (d, \gamma_1), (1, \gamma_1)\}, \quad (41)$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then $\mathscr{F}_L$ satisfies the condition (37), but $\mathscr{F}_L$ is not an int-soft filter of L since $f_L(a) \nsubseteq f_L(1)$.

Proposition 17. *For an int-soft filter $\mathscr{F}_L$ of L, the following are equivalent:*

$$(\forall x, y \in L)\left(f_L(y \longrightarrow (y \longrightarrow x)) \subseteq f_L(y \longrightarrow x)\right), \quad (42)$$

$$(\forall x, y, z \in L)\left(f_L(z \longrightarrow (y \longrightarrow x)) \subseteq f_L((z \longrightarrow y) \longrightarrow (z \longrightarrow x))\right). \quad (43)$$

Proof. Assume that (42) is valid, and let $x, y, z \in L$. Using (R4), (5), and (14), we have

$$z \longrightarrow (y \longrightarrow x) \leq z \longrightarrow (z \longrightarrow ((z \longrightarrow y) \longrightarrow x)). \quad (44)$$

It follows from Proposition 10 (1), (42), and (R4) that

$$\begin{aligned} f_L(z \longrightarrow (y \longrightarrow x)) &\subseteq f_L(z \longrightarrow (z \longrightarrow ((z \longrightarrow y) \longrightarrow x))) \\ &\subseteq f_L(z \longrightarrow ((z \longrightarrow y) \longrightarrow x)) \\ &= f_L((z \longrightarrow y) \longrightarrow (z \longrightarrow x)). \end{aligned} \quad (45)$$

Conversely, suppose that (43) holds. If we use z instead of y in (43), then

$$\begin{aligned} f_L(z \longrightarrow (z \longrightarrow x)) &\subseteq f_L((z \longrightarrow z) \longrightarrow (z \longrightarrow x)) \\ &= f_L(1 \longrightarrow (z \longrightarrow x)) = f_L(z \longrightarrow x), \end{aligned} \quad (46)$$

which proves (42). □

We make a new int-soft filter from old one.

Theorem 18. *Let $\mathscr{F}_L \in S(U, L)$. For a subset γ of U, define a soft set $\mathscr{F}_L^*$ of L by*

$$f_L^* : L \longrightarrow \mathscr{P}(U), \qquad x \longmapsto \begin{cases} f_L(x) & \text{if } x \in \mathscr{F}_L^\gamma, \\ \emptyset & \text{otherwise.} \end{cases} \quad (47)$$

If $\mathscr{F}_L$ is an int-soft filter of L, then so is $\mathscr{F}_L^*$.

Proof. Assume that $\mathscr{F}_L$ is an int-soft filter of L. Then $\mathscr{F}_L^\gamma (\neq \emptyset)$ is a filter of L for all $\gamma \in \mathscr{P}(U)$. Hence $1 \in \mathscr{F}_L^\gamma$, and so $f_L^*(1) = f_L(1) \supseteq f_L(x) \supseteq f_L^*(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in \mathscr{F}_L^\gamma$ and $x \rightarrow y \in \mathscr{F}_L^\gamma$, then $y \in \mathscr{F}_L^\gamma$. Hence

$$\begin{aligned} &f_L^*(x) \cap f_L^*(x \longrightarrow y) \\ &\quad = f_L(x) \cap f_L(x \longrightarrow y) \subseteq f_L(y) = f_L^*(y). \end{aligned} \quad (48)$$

If $x \notin \mathscr{F}_L^\gamma$ or $x \rightarrow y \notin \mathscr{F}_L^\gamma$, then $f_L^*(x) = \emptyset$ or $f_L^*(x \rightarrow y) = \emptyset$. Thus

$$f_L^*(x) \cap f_L^*(x \longrightarrow y) = \emptyset \subseteq f_L^*(y). \quad (49)$$

Therefore $\mathscr{F}_L^*$ is an int-soft filter of L. □

Theorem 19. *Any filter of L can be realized as an inclusive filter of some int-soft filter of L.*

Proof. Let F be a filter of L. For a nonempty subset γ of U, let $\mathscr{F}_L$ be a soft set of L defined by

$$f_L : L \longrightarrow \mathscr{P}(U), \qquad x \longmapsto \begin{cases} \gamma & \text{if } x \in \text{F}, \\ \emptyset & \text{otherwise.} \end{cases} \quad (50)$$

Obviously $f_L(x) \subseteq f_L(1)$ for all $x \in L$. For any $x, y \in L$, if $x \in F$ and $x \rightarrow y \in F$, then $y \in F$. Hence $f_L(x) \cap f_L(x \rightarrow y) = \gamma = f_L(y)$. If $x \notin F$ or $x \rightarrow y \notin F$, then $f_L(x) = \emptyset$ or $f_L(x \rightarrow y) = \emptyset$. Thus $f_L(x) \cap f_L(x \rightarrow y) = \emptyset \subseteq f_L(y)$. Therefore $\mathscr{F}_L$ is an int-soft filter of L and clearly $\mathscr{F}_L^\gamma = F$. This completes the proof. □

Definition 20. An int-soft filter $\mathscr{F}_L$ of L is said to be *strong* if the following assertion is valid:

$$(\forall x, y \in L)\left(f_L(y \longrightarrow x) \subseteq f_L(((x \longrightarrow y) \longrightarrow y) \longrightarrow x)\right). \quad (51)$$

Example 21. The int-soft filter $\mathscr{F}_L$ in Example 8 is strong.

Theorem 22. *Let $\mathscr{F}_L \in S(U, L)$. Then $\mathscr{F}_L$ is a strong int-soft filter of L if and only if the following assertions are valid:*

(1) $(\forall x \in L)(f_L(x) \subseteq f_L(1))$,

(2) $(\forall x, y, z \in L)(f_L(z \rightarrow (y \rightarrow x)) \cap f_L(z) \subseteq f_L(((x \rightarrow y) \rightarrow y) \rightarrow x))$.

Proof. Suppose that $\mathscr{F}_L$ is a strong int-soft filter of L. Obviously, (1) is valid. For every $x, y, z \in L$, we have

$$\begin{aligned} f_L(z \longrightarrow (y \longrightarrow x)) \cap f_L(z) &\subseteq f_L(y \longrightarrow x) \\ &\subseteq f_L(((x \longrightarrow y) \longrightarrow y) \longrightarrow x) \end{aligned} \quad (52)$$

by Theorem 9 (2) and (51).

Conversely, assume that $\mathscr{F}_L$ satisfies two conditions (1) and (2). If we take $y = 1$ in (2), then $f_L(z) \cap f_L(z \rightarrow x) \subseteq f_L(x)$ for all $x, z \in L$. Hence $\mathscr{F}_L$ is an int-soft filter of L. Now if we put $z = 1$ in (2), then

$$\begin{aligned} f_L(y \longrightarrow x) &= f_L(1 \longrightarrow (y \longrightarrow x)) \\ &= f_L(1 \longrightarrow (y \longrightarrow x)) \cap f_L(1) \\ &\subseteq f_L(((x \longrightarrow y) \longrightarrow y) \longrightarrow x) \end{aligned} \quad (53)$$

by (R2) and (1). Therefore $\mathscr{F}_L$ is a strong int-soft filter of L. □

Example 23. Let $L = [0, 1]$. For any $a, b \in L$, we define

$$\neg a = 1 - a, \qquad a \wedge b = \min\{a, b\}, \qquad a \vee b = \max\{a, b\}$$

$$a \longrightarrow b = \begin{cases} 1 & a \leq b \\ \neg a \vee b & \text{otherwise.} \end{cases} \quad (54)$$

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0-algebra (see [5]). Let $\mathcal{F}_L \in S(U, L)$ be given as follows:

$$\mathcal{F}_L = \{(1, \gamma_2), (x, \gamma_1) \mid x \in L \setminus \{1\}\}, \tag{55}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then $\mathcal{F}_L$ is an int-soft filter of L. But

$$\begin{aligned} & f_L(1 \longrightarrow (0.3 \longrightarrow 0.8)) \cap f_L(1) \\ &\quad = \gamma_2 \nsubseteq \gamma_1 = f_L(((0.8 \longrightarrow 0.3) \longrightarrow 0.3) \longrightarrow 0.8), \end{aligned} \tag{56}$$

and so $\mathcal{F}_L$ is not a strong int-soft filter of L by Theorem 22.

We provide a condition for an int-soft filter to be strong.

Theorem 24. *Let L be an R_0-algebra satisfying the following inequality:*

$$(\forall x, y \in L)((x \longrightarrow y) \longrightarrow y \leq (y \longrightarrow x) \longrightarrow x). \tag{57}$$

Then every int-soft filter of L is strong.

Proof. Let $\mathcal{F}_L$ be an int-soft filter of L. Using (5), (6), and (57), we have

$$\begin{aligned} y \longrightarrow x &= ((y \longrightarrow x) \longrightarrow x) \longrightarrow x \\ &\leq ((x \longrightarrow y) \longrightarrow y) \longrightarrow x \end{aligned} \tag{58}$$

for all $x, y \in L$. It follows from Proposition 10 (1) that

$$f_L(y \longrightarrow x) \subseteq f_L(((x \longrightarrow y) \longrightarrow y) \longrightarrow x) \tag{59}$$

for all $x, y \in L$. Therefore $\mathcal{F}_L$ is a strong int-soft filter of L. □

We consider an extension property of a strong int-soft filter.

Theorem 25. *Let $\mathcal{F}_L$ and $\mathcal{G}_L$ be two int-soft filters of L such that $\mathcal{F}_L \tilde{\subseteq} \mathcal{G}_L$ and $f_L(1) = g_L(1)$. If $\mathcal{F}_L$ is strong, then so is $\mathcal{G}_L$.*

Proof. Assume that $\mathcal{F}_L$ is a strong int-soft filter of L. For any $x, y \in L$, let $a = y \rightarrow x$. Since $\mathcal{F}_L$ is a strong int-soft filter of L, we have

$$\begin{aligned} g_L(1) &= f_L(1) = f_L(y \longrightarrow (a \longrightarrow x)) \\ &\subseteq f_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow (a \longrightarrow x)) \\ &\subseteq g_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow (a \longrightarrow x)), \end{aligned} \tag{60}$$

by (51) and assumption, and so

$$\begin{aligned} g_L(1) &= g_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow (a \longrightarrow x)) \\ &= g_L(a \longrightarrow ((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x)). \end{aligned} \tag{61}$$

Since $\mathcal{G}_L$ is an int-soft filter of L, it follows that

$$\begin{aligned} g_L(a) &= g_L(a) \cap g_L(1) = g_L(a) \cap g_L \\ &\quad \times (a \longrightarrow ((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x)) \\ &\subseteq g_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x). \end{aligned} \tag{62}$$

Using (R4) and (14), we have

$$\begin{aligned} 1 &= x \longrightarrow (a \longrightarrow x) \leq ((a \longrightarrow x) \longrightarrow y) \longrightarrow (x \longrightarrow y) \\ &\leq ((x \longrightarrow y) \longrightarrow y) \longrightarrow (((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \\ &\leq ((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x) \\ &\quad \longrightarrow (((x \longrightarrow y) \longrightarrow y) \longrightarrow x). \end{aligned} \tag{63}$$

It follows from (62) and Theorem 14 that

$$\begin{aligned} & g_L(y \longrightarrow x) \\ &\quad = g_L(a) \subseteq g_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x) \\ &\quad = g_L(1) \cap g_L((((a \longrightarrow x) \longrightarrow y) \longrightarrow y) \longrightarrow x) \\ &\quad \subseteq g_L(((x \longrightarrow y) \longrightarrow y) \longrightarrow x). \end{aligned} \tag{64}$$

Therefore $\mathcal{G}_L$ is a strong int-soft filter of L. □

4. Conclusion

Using the notion of int-soft sets, we have introduced the concept of (strong) int-soft filters in R_0-algebras and investigated related properties. We have established characterizations of a (strong) int-soft filter and made a new int-soft filter from old one. We have provided a condition for an int-soft filter to be strong and constructed an extension property of a strong int-soft filter.

Work is ongoing. Some important issues for future work are (1) to develop strategies for obtaining more valuable results, (2) to apply these notions and results for studying related notions in other (soft) algebraic structures; and (3) to study the notions of implicative int-soft filters and Boolean int-soft filters.

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